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Peripheral Distance Energy of Graphs

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Abstract: The peripheral distance matrix of a graph G of order n with k peripheral vertices is a square symmetric matrix of order $k \times k$, denoted as D_p-matrix of G and is defined as $D_p(G) = [d_{ij}]$, where d_{ij} is the distance between two peripheral vertices v_i and v_j in G. The peripheral distance energy of a graph G is the sum of the absolute values of the eigenvalues of D_p-matrix of G. The sum of the distances between all pairs of peripheral vertices is a peripheral Wiener index of a graph G. In this paper, we study some preliminary facts of D_p-matrix of G and give some bounds for peripheral distance energy of a graph G. Specially the bounds are presented for a graph of diameter less than 3. Bounds of peripheral distance energy in terms of peripheral Wiener index are also obtained for graphs of $diam(G) \leq 2$.

Key Words: Distance, peripheral Wiener index, peripheral distance matrix, peripheral distance energy.

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§1. Introduction

Let G be a connected, nontrivial graph with vertex set $V(G)$ and edge set $E(G)$ and let $|V(G)| = n$ and $|E(G)| = m$. Let u and v be two vertices of a graph G. The *distance* $d(u, v|G)$ between the vertices u and v is the length of a shortest path connecting u and v. If $u = v$ then $d(u, v|G) = 0$. The *eccentricity* $e(v)$ of a vertex v in a graph G is the distance between v and a vertex farthest from v in G. The *diameter* $diam(G)$ of G is the maximum eccentricity of G, while the *radius* $rad(G)$ is the smallest eccentricity of G. A vertex v with $e(v) = diam(G)$ is called a *peripheral* vertex of G. The set of peripheral vertices of G is called as periphery and is denoted as $P(G)$.

We claim that the adjacency matrix of a graph is the distance based matrix such that the entries of adjacency matrix are 1 if the distance between two vertices is 1 and 0 otherwise.

The *distance matrix* of a graph G is defined as a square matrix $D = D(G) = [d_{ij}]$, where d_{ij} is the distance between v_i and v_j in G. For the application and the background of the distance matrix on the chemistry, one can refer to [1, 32].

Peripheral distance matrix or D_p-matrix, D_p of a graph G is defined as, $D_p = D_p(G) =$

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$[d_{ij}]$, where d_{ij} is the distance between two peripheral vertices v_i and v_j in G. The eigenvalues $\mu_1, \mu_2, \cdots, \mu_k$ of the D_p-matrix are said to be D_p-eigenvalues of G denoted by $D_p - spec(G)$. Since D_p-matrix of G is symmetric, all of its eigenvalues are real and can be arranged in a non-increasing order as $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_k$. Recalling the definition of peripheral distance matrix, a graph G of order n with k peripheral vertices, the peripheral distance matrix of G is a $(k \times k)$ matrix, whose entries are as follows:

$$D_p(G) = [d_{ij}] = \ [d(v_i, v_j)]; \quad where \ \ v_i, v_j \ \in P(G).$$

The peripheral distance energy (D_p-energy (in short)) of a graph G is defined as the sum of the absolute values of D_p- eigenvalues of D_p-matrix of G. i.e,

$$E_{D_P}(G) = \sum_{i=1}^{k} |\mu_i|. \tag{1}$$

The form of (1) is chosen so as to be fully analogous to the definition of graph energy [5, 6, 9].

$$E = E(G) = \sum_{i=1}^{n} |\lambda_i|, \tag{2}$$

where $\lambda_1, \lambda_2, \cdots, \lambda_n$ are the ordinary eigenvalues [3], i.e the eigenvalues of the adjacency matrix $A(G)$. Observe that the graph energy $E(G)$ in past a few years has been extensively studied and surveyed in Mathematics and Chemistry [8, 11, 14, 18, 19, 20, 21, 22, 25, 26, 27, 29, 30, 31, 33]. Through out the paper $|P(G)| = k$ with labellings $v_1, v_2, \cdots, v_k$, where $2 \leq k \leq n$.

The *characteristic polynomial* of $D_p(G)$ is the $\det(\mu I - D_p(G))$, it is referred to as a characteristic polynomial of G and is denoted by $\psi(G;\mu) = c_0\mu^k + c_1\mu^{k-1} + c_2\mu^{k-2} + \cdots + c_k$. The roots $\mu_1, \mu_2, \cdots, \mu_k$ of the polynomial $\psi(G;\mu)$ are called the *eigenvalues* of $D_p(G)$. The eigenvalues of $D_p(G)$ are said to be the *peripheral distance eigenvalues* (or D_p-eigenvalues (in short)) of G. Since $D_p(G)$ is a real symmetric matrix, the D_p-eigenvalues are real and can be ordered in non-increasing order, $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_k$. Then the *$D_p$-spectrum* of a graph G is the set of eigenvalues of $D_p(G)$, together with the multiplicities of D_p-eigenvalues of $D_p(G)$. If the D_p-eigenvalues of $D_p(G)$ are $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_k$ and their multiplicities are $m(\mu_1), m(\mu_2), \cdots, m(\mu_k)$, then we shall write

$$D_p - spec(G) = \begin{pmatrix} \mu_1 & \mu_2 & \cdots & \mu_k \\ m(\mu_1) & m(\mu_2) & \cdots & m(\mu_k) \end{pmatrix}.$$

For example, let G be a graph as shown in Fig.1. Then

$$D_p(G) = \begin{bmatrix} . & v_1 & v_2 & v_3 \\ v_1 & 0 & 3 & 3 \\ v_2 & 3 & 0 & 2 \\ v_3 & 3 & 2 & 0 \end{bmatrix}$$

Clearly, the characteristic polynomial of G is $\psi(G;\mu) = -\mu^3 + 22\mu + 36$, whose D_p- eigenvalues are $1+\sqrt{19},\ 1-\sqrt{19}$ and -2. Hence E_{D_p}-energy of G is 10.7178.

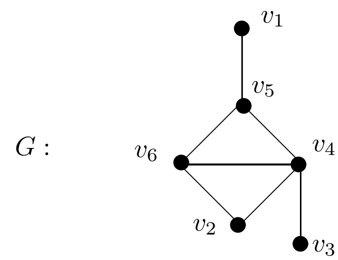

Fig.1 G is a graph of order $n = 6$ with $k = 3$ peripheral vertices.

This paper is organized as follows: In the forthcoming section some preliminary facts of peripheral distance matrix $D_p(G)$ of G are obtained. In section 3 bounds of peripheral distance energy in terms peripheral Wiener index are deduced. In section 4 bounds for the peripheral distance energy are established. In the last section the smallest peripheral distance energy of a graph is obtained thereby posing an open problem for the maximum peripheral distance energy.

§2. Preliminary Results

Lemma 2.1 *Let G be a graph of order n with k peripheral vertices and let $\mu_1, \mu_2, \cdots, \mu_k$ be its peripheral distance eigenvalues. Then,*

(1) $\sum\limits_{i=1}^{k} \mu_i = 0$;

(2) $\sum\limits_{i=1}^{k} {\mu_i}^2 = 2 \sum\limits_{1\le i<j\le k} (d_{ij})^2$.

Proof Since, $\sum\limits_{i=1}^{k} \mu_i = trace[D_p(G)]$ but $d_{ii} = 0$ in $D_p(G)$, therefore, $\sum\limits_{i=1}^{k} \mu_i = 0$.

For $i = 1, 2, \cdots, k$, the $(i,i)^{th}$ entry of $[D_p(G)]^2$ is equal to

$$\sum_{i=1}^{k} d_{ij}, d_{ji} = \sum_{j=1}^{k} (d_{ij})^2$$

since $D_p(G)$ is symmetric. Therefore,

$$\begin{aligned} \sum_{i=1}^{k} {\mu_i}^2 &= trace[D_p(G)]^2 \\ &= \sum_{i=1}^{k}\sum_{j=1}^{k} (d_{ij})^2 = 2\cdot\sum_{i<j}(d_{ij})^2 \\ \Longrightarrow \sum_{i=1}^{k} {\mu_i}^2 &= 2\sum_{i<j}(d_{ij})^2. \end{aligned} \tag{3}$$

□

Lemma 2.2 *Suppose G is a graph of order n and size m with k peripheral vertices having the $diam(G) \leq 2$. Then,*

$$\sum_{i=1}^{k} {\mu_i}^2 = 6\binom{n}{2} + 2\binom{k}{2} - 6m.$$

Proof In the peripheral distance matrix D_p of G there are $x = 2m - 2\{(n-k)k + \frac{(n-k)(n-k-1)}{2}\}$ elements equal to unity, and $y = k(k-1) - x$ elements equal to two. Therefore,

$$\begin{aligned} \sum_{i=1}^{k} {\mu_i}^2 &= trace[D_p(G)]^2 \\ &= \sum_{i=1}^{k}\sum_{j=1}^{k} (d_{ij})^2 = 2\sum_{i<j}(d_{ij})^2 \\ \Longrightarrow \sum_{i=1}^{k} {\mu_i}^2 &= (x).1^2 + (y).2^2 \\ &= (x).1^2 + (k(k-1)-x).2^2 \\ &= 4.k(k-1) - 3x \\ &= 4.k(k-1) - 3\{2m + k(k-1) - n(n-1)\} \\ &= k(k-1) + 3.n(n-1) - 6m \\ \sum_{i=1}^{k} {\mu_i}^2 &= 6\binom{n}{2} + 2\binom{k}{2} - 6m. \end{aligned}$$

§3. Preliminary Results with Respect to Peripheral Wiener Index

Definition 3.1([4, 7]) *The thorn graph of the graph G , with parameters $t_1, t_2, \cdots, t_n$ is obtained by attaching t_i new vertices of degree one to the vertex v_i of the graph $G; i = 1, 2, \cdots, n$. The thorn graph of the graph G will be denoted by G^*, or if the respective parameters need to be specified, by $G^*(t_1, t_2, \cdots, t_n)$.*

Definition 3.2([7, 28]) *The thorn graph of the graph G obtained by attaching t new vertices of* *degree one to all the vertices v_i of the graph G is denoted by G^{+t}.*

If we partition the vertex set $V(G)$ of a graph into two sets, with peripheral vertices in one set and non-peripheral vertices in other. Then the sum of the distances between all pairs of peripheral vertices is the peripheral Wiener index of a graph G. More formally

$$PWI(G) = \sum_{1\le i<j\le k} d(v_i, v_j|G), \tag{4}$$

where G is an (n,m)-graph with k peripheral vertices and $v_i, v_j \in P(G)$.

Theorem 3.3([17]) *Suppose G is a graph of order n and size m with k peripheral vertices having $diam(G) \le 2$. Then,*

$$PWI(G) = \binom{n}{2} + \binom{k}{2} - m. \tag{5}$$

Theorem 3.4 *Suppose G is a graph of order n and size m with k peripheral vertices having the $diam(G) \le 2$. Then, for G^{+t}*

$$\sum_{i=1}^{tk} {\mu_i}^2 = \left\{4k + 14\binom{n}{2} + 18\binom{k}{2} - 14m\right\} t^2 - 4kt.$$

Proof In the peripheral distance matrix $D_p(G^{+t})$ there are $x_1 = kt$ elements equal to 0, $x_2 = k(t^2 - t)$ elements equal to 2, $x_3 = t^2\{2m + 2\binom{k}{2} - 2\binom{n}{2}\}$ elements equal to 3 and $x_4 = t^2\{2\binom{n}{2} - 2m\}$ elements equal to 4. Therefore,

$$\begin{aligned}
\sum_{i=1}^{kt} {\mu_i}^2 &= trace[D_p(G^{+t})]^2 \\
&= \sum_{i=1}^{kt}\sum_{j=1}^{kt} (d_{ij})^2 = 2\sum_{i<j} (d_{ij})^2 \\
\Longrightarrow \sum_{i=1}^{kt} {\mu_i}^2 &= (x_1).0^2 + (x_2).2^2 + (x_3)\dot{3}^2 + (x_4).4^2 \\
&= \{k(t^2-t)\}.2^2 + \left\{t^2\left\{2m + 2\binom{k}{2} - 2\binom{n}{2}\right\}\right\}.3^2 + \left\{t^2\left\{2\binom{n}{2} - 2m\right\}\right\}.4^2 \\
&= \{4k(t^2-t)\} + \left\{9t^2\left\{2m + 2\binom{k}{2} - 2\binom{n}{2}\right\}\right\} + \left\{16t^2\left\{2\binom{n}{2} - 2m\right\}\right\} \\
&= (4kt^2 - 4kt) + \left\{t^2\left\{18m + 18\binom{k}{2} - 18\binom{n}{2}\right\}\right\} + \left\{t^2\left\{32\binom{n}{2} - 32m\right\}\right\} \\
&= 4kt^2 - 4kt + 18mt^2 + 18\binom{k}{2}t^2 - 18\binom{n}{2}t^2 + 32\binom{n}{2}t^2 - 32mt^2 \\
\sum_{i=1}^{kt} {\mu_i}^2 &= \left\{4k + 14\binom{n}{2} + 18\binom{k}{2} - 14m\right\} t^2 - 4kt. \qquad (6)
\end{aligned}$$

□

Corollary 3.5 *Suppose G is a graph of order n and size m with k peripheral vertices having the $diam(G) \leq 2$. Then, for G^{+t}*

$$\sum_{i=1}^{tk} {\mu_i}^2 = \left\{4k + 4\binom{k}{2} + 14PWI(G)\right\} t^2 - 4kt.$$

Proof The proof follows directly from Theorems 3.3 and 3.4. □

Proposition 3.6 *Suppose $G(n,m)$ is a graph with k peripheral vertices and $diam(G) \leq 2$. Then,*

$$\sum_{i=1}^{k} {\mu_i}^2 = 6PWI(G) - 4\binom{k}{2},$$

where PWI(G) is the peripheral Wiener index of G.

Proof From the Lemma 2.2 we have,

$$\begin{aligned}\sum_{i=1}^{k} {\mu_i}^2 &= 6\binom{n}{2} + 2\binom{k}{2} - 6m \\ &= 6\binom{n}{2} + 2\binom{k}{2} - 6m + 4\binom{k}{2} - 4\binom{k}{2} \\ &= 6\left\{\binom{n}{2} + \binom{k}{2} - m\right\} - 4\binom{k}{2} \\ &= 6\left\{PWI(G)\right\} - 4\binom{k}{2}\end{aligned}$$

from Theorem 3.3. □

§4. Bounds for the Peripheral Distance Energy

Theorem 4.1 *Suppose G is a graph with k peripheral vertices. Then*

$$\sqrt{2\sum_{i<j} (d_{ij})^2} \leq E_{D_P}(G) \leq \sqrt{2.k.\sum_{i<j} (d_{ij})^2}. \tag{7}$$

Proof We have from Cauchy-Schwarz inequality

$$\left(\sum_{i=1}^{k} a_i b_i\right)^2 \leq \left(\sum_{i=1}^{k} {a_i}^2\right)\left(\sum_{i=1}^{k} {b_i}^2\right)$$

Put $a_i = 1$ and $b_i = |\mu_i|$ then

$$\begin{aligned}[E_{D_P}(G)]^2 = \left(\sum_{i=1}^{k}|\mu_i|\right)^2 &\leq \left(\sum_{i=1}^{k}1\right)\left(\sum_{i=1}^{k}{\mu_i}^2\right) \\ &= k\left(\sum_{i=1}^{k}{\mu_i}^2\right) \\ &= k\left(2\sum_{i<j}(d_{ij})^2\right)\end{aligned}$$

from Eq.(3) and

$$[E_{D_P}(G)] \leq \sqrt{2\ k\ \sum_{i<j}(d_{ij})^2}. \tag{8}$$

We have from the definition

$$\begin{aligned}[E_{D_P}(G)]^2 &= \left(\sum_{i=1}^{k}|\mu_i|\right)^2 = \sum_{i=1}^{k}{\mu_i}^2 + 2\sum_{i<j}|\mu_i||\mu_j| \\ &= 2\sum_{i<j}(d_{ij})^2 + 2\sum_{i<j}|\mu_i||\mu_j|,\end{aligned}$$

$$[E_{D_P}(G)]^2 = 2\sum_{i<j}(d_{ij})^2 + \sum_{i\neq j}|\mu_i||\mu_j|, \tag{9}$$

$$[E_{D_P}(G)]^2 - 2\sum_{i<j}(d_{ij})^2 = \sum_{i\neq j}|\mu_i||\mu_j|. \tag{10}$$

Also, we know that

$$\begin{aligned}[E_{D_P}(G)]^2 &= \left(\sum_{i=1}^{k}|\mu_i|\right)^2 \geq \sum_{i=1}^{k}|\mu_i|^2 = 2\sum_{i<j}(d_{ij})^2 \\ \Longrightarrow [E_{D_P}(G)]^2 &\geq 2\sum_{i<j}(d_{ij})^2 \\ \Longrightarrow [E_{D_P}(G)] &\geq \sqrt{2\sum_{i<j}(d_{ij})^2}. \end{aligned} \tag{11}$$

Inequations (8) and (11) complete the proof. □

Corollary 4.2 *Suppose G is any graph with k peripheral vertices and $diam(G) = d$. Then,*

$$\sqrt{k(k-1)} \leq E_{D_P}(G) \leq d.k.\sqrt{k-1}.$$

Proof Since $d(v_i, v_j) = d_{ij} \geq 1$, for $i \neq j$ and totally $\binom{k}{2}$ pairs of peripheral vertices in G

form lower bound of Corollary 4.2.

$$\begin{aligned} E_{D_P}(G) &\geq \sqrt{2.\sum_{i<j}(d_{ij})^2} \geq \sqrt{2.[1]^2\binom{k}{2}} \\ &= \sqrt{2.1.\frac{k(k-1)}{2}}, \end{aligned}$$

$$E_{D_P}(G) \geq \sqrt{k(k-1)}. \tag{12}$$

Also, $d(v_j, v_j) = d_{ij} \leq d$, for $i \neq j$ and totally $\binom{k}{2}$ pair of peripheral vertices in G form upper bound of Corollary 4.2.

$$\begin{aligned} E_{D_P}(G) &\leq \sqrt{2.k.\sum_{i<j}(d_{ij})^2} \leq \sqrt{2.k.[d]^2\binom{k}{2}} \\ &= \sqrt{2.k.[d]^2\frac{k(k-1)}{2}} \end{aligned}$$

$$E_{D_P}(G) \leq d.k.\sqrt{k-1}. \tag{13}$$

Inequations (12) and (13) complete the proof. □

Theorem 4.3 *Suppose G is any graph with k peripheral vertices. Then,*

(1) $\sqrt{2\sum_{i<j}(d_{ij})^2 + k(k-1)\Delta^{2/k}} \leq E_{D_P}(G)$;

(2) $E_{D_P}(G) \leq \frac{2}{k}\sum_{i<j}(d_{ij})^2 + \sqrt{(k-1)[2\sum_{i<j}(d_{ij})^2 - (\frac{2}{k}\sum_{i<j}(d_{ij})^2)^2]}$,

where Δ is the absolute value of the determinant of the peripheral distance matrix $D_P(G)$.

Proof We know that, for non-negative numbers the arithmetic mean is not smaller than the geometric mean.

$$\begin{aligned} \frac{1}{k(k-1)}\sum_{i\neq j}|\mu_i||\mu_j| &\geq \left(\prod_{i\neq j}|\mu_i||\mu_j|\right)^{\frac{1}{k(k-1)}} = \left(\prod_{i=1}^{k}|\mu_i|^{2(k-1)}\right)^{\frac{1}{k(k-1)}} \\ &= \left(\prod_{i=1}^{k}|\mu_i|\right)^{2/k} = |\det(D_p(G))|^{2/k} = (\Delta)^{2/k} \end{aligned}$$

$$\begin{aligned} \Longrightarrow \sum_{i\neq j}|\mu_i||\mu_j| &\geq k(k-1).\,(\Delta)^{2/k} \\ \Longrightarrow [E_{D_P}(G)]^2 - 2\sum_{i<j}(d_{ij})^2 &\geq k(k-1).\,(\Delta)^{2/k} \\ [E_{D_P}(G)]^2 &\geq k(k-1).\,(\Delta)^{2/k} + 2\sum_{i<j}(d_{ij})^2, \end{aligned}$$

$$[E_{D_P}(G)] \geq \sqrt{k(k-1).\,(\Delta)^{2/k} + 2\sum_{i<j}(d_{ij})^2}. \tag{14}$$

Therefore, the equation (14) proves lower bound.

To prove the upper bound we follow the ideas of Koolen and Moulton [18, 19], who obtained an analogous upper bound for ordinary graph energy $E(G)$. By applying the Cauchy-Schwartze inequality to the two $(k-1)$ vectors $(1, 1, \cdots, 1)$ and $(|\mu_1|, |\mu_2|, \cdots, |\mu_k|)$ we get.

$$\begin{aligned}
\left(\sum_{i=2}^{k}|\mu_i|\right)^2 &\leq (k-1)\left(\sum_{i=2}^{k}\mu_i^2\right)\\
\left(E_{D_p}(G)-\mu_1\right)^2 &\leq (k-1)\left(2\sum_{i<j}(d_{ij})^2-\mu_1^2\right)\\
E_{D_p}(G) &\leq \mu_1+\sqrt{(k-1)\left(2\sum_{i<j}(d_{ij})^2-\mu_1^2\right)}
\end{aligned}$$

Define the function

$$f(x) = x + \sqrt{(k-1)\left(2\sum_{i<j}(d_{ij})^2 - x^2\right)}$$

we set $x = \mu_1$ and bear in mind that $\mu_1 \geq 1$.

From Equation (3) we get $x^2 = \mu_1^2 \leq 2\sum_{i<j}(d_{ij})^2 \implies x \leq \sqrt{2\sum_{i<j}(d_{ij})^2}$.

Now $f^{'}(x) = 0$ implies, $x = \sqrt{\frac{2}{k}\sum_{i<j}(d_{ij})^2}$. Therefore $f(x)$ is a decreasing function in the interval

$$\sqrt{\frac{2}{k}\sum_{i<j}(d_{ij})^2} \leq x \leq 2\sqrt{\sum_{i<j}(d_{ij})^2}.$$

and

$$\sqrt{\frac{2}{k}\sum_{i<j}(d_{ij})^2} \leq \frac{2}{k}\sum_{i<j}(d_{ij})^2 \leq \mu_1.$$

Hence

$$f(\mu_1) \leq f\left(\frac{2}{k}\sum_{i<j}(d_{ij})^2\right).$$

Hence the proof. □

Theorem 4.4 *Suppose G is a graph of order n and size m with k peripheral vertices having the $diam(G) \leq 2$. Then,*

$$\sqrt{6\binom{n}{2} + 2\binom{k}{2} - 6m} \leq E_{D_P}(G) \leq \sqrt{k\left\{6\binom{n}{2} + 2\binom{k}{2} - 6m\right\}}.$$

Proof From Theorem 4.1 we have

$$\sqrt{2\sum_{i<j}(d_{ij})^2} \leq E_{D_P}(G) \leq \sqrt{2.k.\sum_{i<j}(d_{ij})^2}$$

and next from Lemma 2.2,

$$2\sum_{i<j}(d_{ij})^2 = \sum_{i=1}^{k}{\mu_i}^2 = 6\binom{n}{2} + 2\binom{k}{2} - 6m.$$

By replacing the $2\sum\limits_{i<j}(d_{ij})^2$ by $6\binom{n}{2} + 2\binom{k}{2} - 6m.$ in Ineq.7 gives the proof. □

Corollary 4.5 *Suppose G is a graph with $diam(G) \leq 2.$ having k peripheral vertices. Then,*

$$\sqrt{6PWI(G) - 4\binom{k}{2}} \leq E_{D_P}(G) \leq \sqrt{k.\left\{6PWI(G) - 4\binom{k}{2}\right\}},$$

where $PWI(G)$ is the peripheral Wiener index of a graph G.

Proof The proof follows from Theorem 4.1 and Proposition 3.6. □

Theorem 4.6 *Suppose G is any graph with k peripheral vertices and $diam(G) \leq 2$. Then,*

$$\sqrt{\mathbb{S} + 2\binom{k}{2}\Delta^{2/k}} \leq E_{D_P}(G) \leq \frac{1}{k}\{\mathbb{S}\} + \sqrt{(k-1)[\mathbb{S} - (\frac{1}{k}\{\mathbb{S}\})^2]},$$

where Δ is the absolute value of the determinant of the peripheral distance matrix $D_P(G)$ and $\mathbb{S} = 6\binom{n}{2} + 2\binom{k}{2} - 6m.$

Proof The proof follows from Theorem 4.3 and Lemma 3.4. □

Corollary 4.7 *Suppose G is any graph with k peripheral vertices and $diam(G) \leq 2$. Then,*

$$\sqrt{\mathbb{S} + 2\binom{k}{2}\Delta^{2/k}} \leq E_{D_P}(G) \leq \frac{1}{k}\{\mathbb{S}\} + \sqrt{(k-1)[\mathbb{S} - (\frac{1}{k}\{\mathbb{S}\})^2]},$$

where Δ is the absolute value of the determinant of the peripheral distance matrix $D_P(G)$, $\mathbb{S} = 6PWI(G) - 4\binom{k}{2}$ and $PWI(G)$ is the peripheral Wiener index of a graph G.

Proof The proof follows from Theorem 4.3 and Proposition 3.6. □

Theorem 4.8 *Suppose G is a graph of order n and size m with k peripheral vertices having the $diam(G) \leq 2$. Then,*

$$\sqrt{\mathbb{T}} \leq E_{D_P}(G^{+t}) \leq \sqrt{kt\,\{\mathbb{T}\}},$$

where $\mathbb{T} = \left\{4k + 14\binom{n}{2} + 18\binom{k}{2} - 14m\right\} t^2 - 4kt.$

Proof The proof follows from Theorem 4.1 and Lemma 2.2. □

Corollary 4.9 *Suppose G is a graph of order n and size m with k peripheral vertices having the $diam(G) \leq 2$. Then,*

$$\sqrt{\mathbb{T}} \leq E_{D_P}(G^{+t}) \leq \sqrt{kt\,\{\mathbb{T}\}},$$

where $\mathbb{T} = \left\{4k + 4\binom{k}{2} + 14PWI(G)\right\} t^2 - 4kt$ *and $PWI(G)$ is the peripheral Wiener index of a graph G.*

proof The proof follows from Theorem 4.1 and Corollary 3.5. □

Theorem 4.10 *Suppose G is any graph with k peripheral vertices and $diam(G) \leq 2$. Then,*

$$\sqrt{\mathbb{T} + 2\binom{kt}{2}\Delta^{2/kt}} \leq E_{D_P}(G^{+t}) \leq \frac{1}{kt}\{\mathbb{T}\} + \sqrt{(kt-1)[\mathbb{T} - (\frac{1}{kt}\{\mathbb{T}\})^2]},$$

where Δ is the absolute value of the determinant of the peripheral distance matrix $D_P(G^{+t})$ and $\mathbb{T} = \left\{4k + 14\binom{n}{2} + 18\binom{k}{2} - 14m\right\} t^2 - 4kt.$

proof The proof follows from Theorem 4.3 and Lemma 3.4. □

§5. The Smallest Peripheral Distance Energy of a Graph

By studying the bounds for peripheral distance energy, there arise a common question that, which n vertex graphs with k peripheral vertices have the smallest and greatest peripheral distance energy. Among all n-vertex connected graphs with k peripheral vertices the complete graph is the unique graph with the smallest peripheral distance energy.

Theorem 5.1 *The complete graph $K_{n=k}$ with k peripheral vertices is the graph with smallest peripheral distance energy, which is equal to $2(k-1)$.*

Proof Let G be a graph with k peripheral vertices and K_k be a complete graph on k peripheral vertices. Let A be a peripheral distance matrix of K_k. B be a peripheral distance matrix of G with the D_p-eigenvalues $\mu_1, \mu_2, \cdots, \mu_k$. Clearly A and B are non-negative matrices and obviously $0 \leq A \leq B$. Now, from the fact that if $0 \leq A \leq B$ then $\rho(A) \leq \rho(B)$. And for the complete graph, $\rho(A) = n-1$ and $E_D(K_k) = 2(k-1)$ hence,

$$\begin{aligned} 2(k-1) &= 2\rho(A) \leq 2\rho(B) \\ &\leq \rho(B) + \sum_{i=2}^{k} |\mu_i|. \end{aligned}$$

By using Perron Frobenius theorem, it implies that $\rho(B)$ is a positive eigenvalues. Hence,

$$2(k-1) \quad \leq \quad \sum_{i=1}^{k} |\mu_i| = E_{D_p}(G).$$

But

$$2(k-1) = E_{D_p}(K_k) \leq E_{D_p}(G).$$

Hence, we conclude that the peripheral distance energy of a graph with k peripheral vertices is greater than the peripheral distance energy of a complete graph on k vertices. This proves that among k peripheral vertices graphs complete graph has the smallest peripheral distance energy $= 2(k-1)$. □

Since, distance matrix D of a complete graph is equal to peripheral distance matrix D_p of a complete graph, also distance energy E_D of a complete graph is equal to peripheral distance matrix E_{D_p} of a complete graph, therefore this also settles the conjecture posed by Ramane et al. in [24]. However, in [2], the authors have given the direct reason for the proof of the conjecturer in [24]. Since, we do not have a sufficient stuff to prove graph with greatest peripheral distance energy, but the graph with k peripheral vertices such that all the peripheral vertices are at the distance d $(= diam(G))$ from each other is certainly deserve to be seriously considered graph. In this connection it looks plausible to pose an open problem:

Open Problem *The graph G with k peripheral vertices such that all of its peripheral vertices are at the same distance d $(= diam(G))$ from each other has maximum peripheral distance energy.*

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