International J.Math. Combin. Vol.1(2017), 22-35

On Transformation and Summation Formulas for Some Basic Hypergeometric Series

D.D.Somashekara[1], S.L.Shalini[2] and K.N.Vidya[1]

1. Department of Studies in Mathematics, University of Mysore, Manasagangotri, Mysuru-570006, India

2. Department of Mathematics, Mysuru Royal Institute of Technology, Srirangapatna-571438, India

E-mail: dsomashekara@yahoo.com, shalinisl.maths@gmail.com, vidyaknagabhushan@gmail.com

Abstract: In this paper, we give an alternate and simple proofs for Sear's three term ${}_3\phi_2$ transformation formula, Jackson's ${}_3\phi_2$ transformation formula and for a nonterminating form of the q-Saalschütz sum by using q-exponential operator techniques. We also give an alternate proof for a nonterminating form of the q-Vandermonde sum. We also obtain some interesting special cases of all the three identities, some of which are analogous to the identities stated by Ramanujan in his lost notebook.

Key Words: Transformation formula, q-series, operator identity.

AMS(2010): 33D15.

§1. Introduction

In 1951 Sears [15] has established the following useful three term transformation formula for ${}_3\phi_2$ series.

Theorem 1.1

$$\sum_{n=0}^{\infty}\frac{(a,b,c)_n}{(q,e,f)_n}\left(\frac{ef}{abc}\right)^n=\frac{(b,e/a,f/a,ef/bc)_\infty}{(e,f,b/a,ef/abc)_\infty}\sum_{n=0}^{\infty}\frac{(a,e/b,f/b)_n}{(q,aq/b,ef/bc)_n}q^n$$
$$+\frac{(a,e/b,f/b,ef/ac)_\infty}{(e,f,a/b,ef/abc)_\infty}\sum_{n=0}^{\infty}\frac{(b,e/a,f/a)_n}{(q,bq/a,ef/ac)_n}q^n, \qquad (1.1)$$

where $|q|<1$, $\left|\dfrac{ef}{abc}\right|<1$ *and as usual*

$$(a)_\infty := (a;q)_\infty := \prod_{n=0}^{\infty}(1-aq^n),$$
$$(a)_n := (a;q)_n := \frac{(a)_\infty}{(aq^n)_\infty}, \qquad n \textit{ is an integer},$$
$$(a_1,a_2,a_3,\cdots,a_m)_n=(a_1)_n(a_2)_n(a_3)_n\cdots(a_m)_n, \qquad n \textit{ is an integer or } \infty.$$

[1]Received February 2, 2016, Accepted February 5, 2017.

Recently, Liu [9] has established (1.1) by parameter augmentation method. This formula was used by Agarwal [1] to deduce an identity of Andrews [2, Thoerem 1] which was instrumental in deriving sixteen partial theta function identities of Ramanujan found in his lost notebook [4], [11].

The main objective of this paper is to give an alternate proof for (1.1) and to give proofs for Jackson's ${}_3\phi_2$ transformation formula and for a nonterminating form of the q-Saalschütz sum found in [5] by using q-exponential operator techniques. And also we give a simple proof for a nonterminating form of the q-Vandermonde sum. Also we obtain a number of interesting applications of these formulas.

We first list some definitions and identities that we use in the remainder of this paper. For any function f, the q-difference operator $D_{q,a}$ is defined by

$$D_{q,a}\{f(a)\} = \frac{f(a) - f(aq)}{a}.$$

The q-shift operator η_a is defined by

$$\eta_a\{f(a)\} = f(aq)$$

and the operator θ_a is given by

$$\theta_a = \eta^{-1} D_{q,a}.$$

The operator identity $T(bD_{q,a})$ [9] is defined by

$$T(bD_{q,a}) = \sum_{n=0}^{\infty} \frac{(bD_{q,a})^n}{(q;q)_n} \tag{1.2}$$

and the basic identity for $T(bD_{q,a})$ operator is

$$T(bD_{q,a})\left\{\frac{1}{(as,at;q)_\infty}\right\} = \frac{(abst;q)_\infty}{(as,at,bs,bt;q)_\infty}. \tag{1.3}$$

The Cauchy operator $T(a,b;D_{q,c})$ [6] is defined by

$$T(a,b;D_{q,c}) := \sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n}(bD_{q,c})^n. \tag{1.4}$$

The two basic identities for the Cauchy operator (1.4) are

$$T(a,b;D_{q,c})\left\{\frac{1}{(ct;q)_\infty}\right\} = \frac{(abt;q)_\infty}{(bt,ct;q)_\infty}, \qquad |bt| < 1, \tag{1.5}$$

$$T(a,b;D_{q,c})\left\{\frac{(cv;q)_\infty}{(cs,ct;q)_\infty}\right\} = \frac{(abs,cv;q)_\infty}{(bs,cs,ct;q)_\infty} \sum_{n=0}^{\infty} \frac{(a,cs,v/t)_n}{(q,cv,abs)_n}(bt)^n. \tag{1.6}$$

The q-exponential operator $R(bD_{q,a})$ [7] is defined by

$$R(bD_{q,a}) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{\binom{n}{2}} b^n}{(q;q)_n} D_{q,a}^n. \tag{1.7}$$

The two basic identities for $R(bD_{q,a})$ are

$$R(bD_{q,a}) \left\{ \frac{1}{(at;q)_\infty} \right\} = \frac{(bt;q)_\infty}{(at;q)_\infty} \tag{1.8}$$

and

$$R(bD_{q,a}) \left\{ \frac{(av;q)_\infty}{(at,as;q)_\infty} \right\} = \frac{(bs;q)_\infty}{(as;q)_\infty} \sum_{n=0}^{\infty} \frac{(v/t,b/a)_n}{(q,bs)_n} (at)^n. \tag{1.9}$$

The q-binomial theorem [5, equation(II.3), p.354] is given by

$$\sum_{n=0}^{\infty} \frac{(a)_n}{(q)_n} z^n = \frac{(az)_\infty}{(z)_\infty}. \tag{1.10}$$

Heine's transformations for ${}_2\phi_1$-series [5, equation(III.1), (III.2), p.359] is given by

$$\sum_{n=0}^{\infty} \frac{(\alpha,\ \beta)_n}{(q,\ \gamma)_n} z^n = \frac{(\beta,\ \alpha z)_\infty}{(\gamma,\ z)_\infty} \sum_{n=0}^{\infty} \frac{(\gamma/\beta, z)_n}{(q,\ \alpha z)_n} \beta^n. \tag{1.11}$$

The Rogers-Fine identity [12, equation(12), p.576] is given by

$$\sum_{n=0}^{\infty} \frac{(\alpha)_n}{(\beta)_n} z^n = \sum_{n=0}^{\infty} \frac{(\alpha, \alpha z q/\beta)_n \beta^n z^n q^{n^2-n} (1-\alpha z q^{2n})}{(\beta)_n (z)_{n+1}}. \tag{1.12}$$

The Sears' transformation for ${}_3\phi_2$-series [5, equation (III.9), p.359] is given by

$$\sum_{n=0}^{\infty} \frac{(\alpha,\beta,\gamma)_n}{(q,\delta,\epsilon)_n} \left(\frac{\delta\epsilon}{\alpha\beta\gamma} \right)^n = \frac{(\epsilon/\alpha, \delta\epsilon/\beta\gamma)_\infty}{(\epsilon, \delta\epsilon/\alpha\beta\gamma)_\infty} \sum_{n=0}^{\infty} \frac{(\alpha, \delta/\beta, \delta/\gamma)_n}{(q,\delta,\delta\epsilon/\beta\gamma)_n} \left(\frac{\epsilon}{\alpha} \right)^n. \tag{1.13}$$

The three-term ${}_2\phi_1$ transformation formula [5, equation (III.31), p.363] is given by

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{(\alpha,\beta)_n}{(q,\gamma)_n} z^n &= \frac{(\alpha\beta z/\gamma,\ q/\gamma)_\infty}{(\alpha z/\gamma,\ q/\alpha)_\infty} \sum_{n=0}^{\infty} \frac{(\gamma/\alpha, \gamma q/\alpha\beta z)_n}{(q, \gamma q/\alpha z)_n} \left(\frac{\beta q}{\gamma} \right)^n \\ &\quad - \frac{(\beta, q/\gamma, \gamma/\alpha, \alpha z/q, q^2/\alpha z)_\infty}{(\gamma/q, \beta q/\gamma, q/\alpha, \alpha z/\gamma, \gamma q/\alpha z)_\infty} \sum_{n=0}^{\infty} \frac{(\alpha q/\gamma, \beta q/\gamma)_n}{(q, q^2/\gamma)_n} z^n. \end{aligned} \tag{1.14}$$

The Jackson's transformation [3, p. 526] is given by

$$\sum_{n=0}^{\infty} \frac{(\alpha,\beta)_n}{(q,\gamma)_n} z^n = \frac{(\alpha z)_\infty}{(z)_\infty} \sum_{n=0}^{\infty} \frac{(\alpha, \gamma/\beta)_n (-\beta z)^n}{(\gamma, \alpha z, q)_n} q^{n(n-1)/2}. \tag{1.15}$$

The Ramanujan's [10, Ch. 16] definition of the theta function is

$$f(a,b)=\sum_{n=-\infty}^{\infty} a^{n(n+1)/2}b^{n(n-1)/2}, \qquad |ab|<1. \tag{1.16}$$

The Jacobi's triple product identity [8] is given by

$$\sum_{n=-\infty}^{\infty} q^{n^2}z^n=(-qz,-q/z,q^2;q^2)_\infty, \qquad z\neq 0. \tag{1.17}$$

If we set $qz=a, q/z=b$ in (1.17), we obtain

$$f(a,b)=(-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty, \tag{1.18}$$

which is the Jacobi's triple product identity in Ramanujan's notation [10, Ch.16, entry 19]. It follows from (1.16) and (1.18) that [10, Ch. 16, entry 22]

$$\varphi(q):=f(q,q)=1+2\sum_{n=1}^{\infty} q^{n^2}=\frac{(-q;q^2)_\infty(q^2;q^2)_\infty}{(q;q^2)_\infty(-q^2;q^2)_\infty}, \tag{1.19}$$

$$\psi(q):=f(q,q^3)=\sum_{n=0}^{\infty} q^{n(n+1)/2}=\frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}, \tag{1.20}$$

$$f(-q):=f(-q,-q^2)=\sum_{n=-\infty}^{\infty}(-1)^n q^{n(3n-1)/2}=(q;q)_\infty \tag{1.21}$$

and

$$\chi(q):=(-q;q^2)_\infty. \tag{1.22}$$

The Ramanujan's functions are given by [4], [11]

$$G_6(q):=(q^3;q^6)^2_\infty(q^6;q^6)_\infty=\sum_{n=-\infty}^{\infty}(-1)^n q^{3n^2}=\varphi(-q^3), \tag{1.23}$$

$$H_6(q):=(q;q^6)_\infty(q^5;q^6)_\infty(q^6;q^6)_\infty=\sum_{n=-\infty}^{\infty}(-1)^n q^{3n^2-2n}=f(-q,-q^5) \tag{1.24}$$

and

$$J_6(q):=(-q;q^3)_\infty(-q^2;q^3)_\infty(q^3;q^3)_\infty=\sum_{n=0}^{\infty} q^{n(3n+1)/2}=f(q,q^2). \tag{1.25}$$

§2. Main Theorems

In this section, we prove the main results.

Proof of Theorem 1.1. Setting $\alpha=b, \beta=a/c, \gamma=qb/c$ and $z=q$ in (1.14), we obtain

$$\sum_{n=0}^{\infty}\frac{(b,a/c)_n}{(q,qb/c)_n}q^n=\frac{(a,c/b)_\infty}{(c,q/b)_\infty}\sum_{n=0}^{\infty}\frac{(q/a)_n}{(q)_n}\left(\frac{a}{b}\right)^n -\frac{(a/c,c/b,b)_\infty}{(b/c,a/b,c)_\infty}\sum_{n=0}^{\infty}\frac{(c,a/b)_n}{(q,qc/b)_n}q^n. \tag{2.1}$$

On using q-binomial theorem for the first series on the right side of (2.1), we obtain

$$\sum_{n=0}^{\infty}\frac{(b,a/c)_n}{(q,qb/c)_n}q^n+\frac{(a/c,c/b,b)_\infty}{(b/c,a/b,c)_\infty}\sum_{n=0}^{\infty}\frac{(c,a/b)_n}{(q,qc/b)_n}q^n=\frac{(a,c/b)_\infty}{(c,a/b)_\infty}. \tag{2.2}$$

Divide the identity (2.2) throughout by $(a/c,c/b,b)_\infty$ to obtain

$$\frac{(a)_\infty}{(b,c,a/b,a/c)_\infty}=\frac{1}{(b,c/b)_\infty}\sum_{n=0}^{\infty}\frac{(b)_nq^n}{(q,qb/c)_n(aq^n/c)_\infty} +\frac{1}{(c,b/c)_\infty}\sum_{n=0}^{\infty}\frac{(c)_nq^n}{(q,qc/b)_n(aq^n/b)_\infty}. \tag{2.3}$$

Applying $T(d,e;D_{q,a})$ to both the sides of the identity (2.3) and using (1.5) and (1.6), we obtain

$$\frac{(a,de/b)_\infty}{(b,c,a/b,a/c,e/b)_\infty}\sum_{n=0}^{\infty}\frac{(d,a/b,c)_n}{(q,de/b,a)_n}\left(\frac{e}{c}\right)^n=\frac{1}{(b,c/b)_\infty}\sum_{n=0}^{\infty}\frac{(b)_n(deq^n/c)_\infty q^n}{(q,qb/c)_n(aq^n/c,eq^n/c)_\infty} +\frac{1}{(c,b/c)_\infty}\sum_{n=0}^{\infty}\frac{(c)_n(deq^n/b)_\infty q^n}{(q,qc/b)_n(aq^n/b,eq^n/b)_\infty}. \tag{2.4}$$

Multiply the identity (2.4) throughout by $(b,c,a/b,a/c,e/b)_\infty/(a,de/b)_\infty$ to obtain

$$\sum_{n=0}^{\infty}\frac{(d,a/b,c)_n}{(q,de/b,a)_n}\left(\frac{e}{c}\right)^n=\frac{(c,a/b,e/b,de/c)_\infty}{(a,c/b,e/c,de/b)_\infty}\sum_{n=0}^{\infty}\frac{(b,a/c,e/c)_n}{(q,qb/c,de/c)_n}q^n +\frac{(b,a/c)_\infty}{(a,b/c)_\infty}\sum_{n=0}^{\infty}\frac{(c,a/b,e/b)_n}{(q,qc/b,de/b)_n}q^n. \tag{2.5}$$

Change a to A, b to C, c to B, d to A/D and e to E in (2.5) to obtain

$$\sum_{n=0}^{\infty}\frac{(B,A/D,A/C)_n}{(q,A,AE/CD)_n}\left(\frac{E}{B}\right)^n =\frac{(B,A/C,E/C,AE/BD)_\infty}{(A,B/C,E/B,AE/CD)_\infty}\sum_{n=0}^{\infty}\frac{(C,A/B,E/B)_n}{(q,Cq/B,AE/BD)_n}q^n +\frac{(C,A/B)_\infty}{(A,C/B)_\infty}\sum_{n=0}^{\infty}\frac{(B,A/C,E/C)_n}{(q,Bq/C,AE/CD)_n}q^n. \tag{2.6}$$

Setting $\alpha = B$, $\beta = A/D$, $\gamma = A/C$, $\delta = A$ and $\epsilon = AE/CD$ in (1.13), using the resulting identity on the left side of (2.6) and then multiplying the resulting identity throughout by $(E/B, AE/CD)_\infty/(E, AE/BCD)_\infty$; change A to e, B to b, C to a, D to c and E to f in the resulting identity, we obtain (1.1). □

Remark 1. The identity (2.3) can be used to prove Lemma 2.1 of Somashekara, Narasimha Murthy and Shalini [13], which played a key role in giving a unified approach to the proofs of the reciprocity theorem of Ramanujan and its generalizations.

Remark 2. The identity (2.3) can also be used to prove Theorem 2.2 of Somashekara, Kim, Kwon and Shalini [14], which played a key role in giving proofs for ten identities of Ramanujan found in his lost notebook [4].

Theorem 2.1([5, equation III.5, p. 359]) *We have*

$$\sum_{n=0}^{\infty} \frac{(a,b)_n}{(q,c)_n} z^n = \frac{(abz/c)_\infty}{(bz/c)_\infty} \sum_{n=0}^{\infty} \frac{(a, c/b, 0)_n}{(q, c, cq/bz)_n} q^n + \frac{(a, bz, c/b)_\infty}{(c, z, c/bz)_\infty} \sum_{n=0}^{\infty} \frac{(z, abz/c, 0)_n}{(q, bz, bzq/c)_n} q^n. \tag{2.7}$$

Proof Applying $R(dD_{q,a})$ to both the sides of the identity (2.3) and using (1.8), (1.9), we obtain

$$\frac{(d/c)_\infty}{(b,c,a/c)_\infty} \sum_{n=0}^{\infty} \frac{(b, d/a)_n}{(q, d/c)_n} \left(\frac{a}{b}\right)^n = \frac{1}{(b, c/b)_\infty} \sum_{n=0}^{\infty} \frac{(b)_n (dq^n/c)_\infty}{(q, bq/c)_n (aq^n/c)_\infty} q^n + \frac{1}{(c, b/c)_\infty} \sum_{n=0}^{\infty} \frac{(c)_n (dq^n/b)_\infty}{(q, cq/b)_n (aq^n/b)_\infty} q^n. \tag{2.8}$$

Multiply the identity (2.8) throughout by $(b, c, a/c)_\infty/(d/c)_\infty$ to obtain

$$\sum_{n=0}^{\infty} \frac{(b, d/a)_n}{(q, d/c)_n} \left(\frac{a}{b}\right)^n = \frac{(c)_\infty}{(c/b)_\infty} \sum_{n=0}^{\infty} \frac{(b, a/c, 0)_n}{(q, bq/c, d/c)_n} q^n + \frac{(b, a/c, d/b)_\infty}{(a/b, b/c, d/c)_\infty} \sum_{n=0}^{\infty} \frac{(c, a/b, 0)_n}{(q, cq/b, d/b)_n} q^n. \tag{2.9}$$

Change a to az, b to a, c to abz/c and d to abz in (2.9) to obtain (2.7). □

Theorem 2.2([5, equation II.23, p. 356]) *We have*

$$\sum_{n=0}^{\infty} \frac{(a,b)_n}{(q,c)_n} q^n + \frac{(q/c, a, b)_\infty}{(c/q, aq/c, bq/c)_\infty} \sum_{n=0}^{\infty} \frac{(aq/c, bq/c)_n}{(q, q^2/c)_n} q^n = \frac{(q/c, abq/c)_\infty}{(aq/c, bq/c)_\infty}. \tag{2.9'}$$

Proof Change lower case letters to upper case letters in (2.2) and then change B to a, A/C to b and Bq/C to c to obtain (2.9'). □

Theorem 2.3([5, equation II.24, p. 356]) *We have*

$$\sum_{n=0}^{\infty}\frac{(a,b,c)_n}{(q,e,f)_n}q^n+\frac{(q/e,a,b,c,qf/e)_\infty}{(e/q,aq/e,bq/e,cq/e,f)_\infty}\sum_{n=0}^{\infty}\frac{(aq/e,bq/e,cq/e)_n}{(q,q^2/e,qf/e)_n}q^n$$
$$=\frac{(q/e,f/a,f/b,f/c)_\infty}{(aq/e,bq/e,cq/e,f)_\infty}, \tag{2.10}$$

where $ef=abcq$.

Proof Divide (2.3) throughout by $(a)_\infty$ to obtain

$$\frac{1}{(b,c,a/b,a/c)_\infty}=\frac{1}{(b,c/b)_\infty}\sum_{n=0}^{\infty}\frac{(b)_nq^n}{(q,qb/c)_n(aq^n/c,a)_\infty}$$
$$+\frac{1}{(c,b/c)_\infty}\sum_{n=0}^{\infty}\frac{(c)_nq^n}{(q,qc/b)_n(aq^n/b,a)_\infty}. \tag{2.11}$$

Applying $T(dD_{q,a})$ to both the sides of the identity (2.11) and using (1.3), we obtain

$$\frac{(ad/bc)_\infty}{(b,c,a/b,a/c,d/b,d/c)_\infty}=\frac{1}{(b,c/b)_\infty}\sum_{n=0}^{\infty}\frac{(b)_n(adq^n/c)_\infty}{(q,bq/c)_n(aq^n/c,a,dq^n/c,d)_\infty}q^n$$
$$+\frac{1}{(c,b/c)_\infty}\sum_{n=0}^{\infty}\frac{(c)_n(adq^n/b)_\infty}{(q,qc/b)_n(aq^n/b,a,dq^n/b,d)_\infty}q^n. \tag{2.12}$$

Multiply the identity (2.12) throughout by $(a,b,d,a/c,c/b,d/c)_\infty/(ad/c)_\infty$ to obtain

$$\sum_{n=0}^{\infty}\frac{(b,a/c,d/c)_nq^n}{(q,bq/c,ad/c)_n}+\frac{(c/b,b,a/c,d/c,ad/b)_\infty}{(b/c,c,a/b,d/b,ad/c)_\infty}\sum_{n=0}^{\infty}\frac{(c,a/b,d/b)_nq^n}{(q,qc/b,ad/b)_n}$$
$$=\frac{(c/b,ad/bc,d,a)_\infty}{(c,a/b,d/b,ad/c)_\infty}. \tag{2.13}$$

Change lower case letters to upper case letters in (2.13) and then change B to a, A/C to b, D/C to c, Bq/C to e and AD/C to f to obtain (2.10). □

§3. Some Applications of Main Results

In this section, we derive some interesting special cases of the main identities. These special cases are found to be analogues to some identities of Ramanujan found in his lost notebook [4], [11].

Setting $a = C, b = B/A, c = D$ and $z = A$ in (2.7), we obtain

$$\sum_{n=0}^{\infty} \frac{(C, B/A)_n}{(q, D)_n} A^n = \frac{(BC/D)_\infty}{(B/D)_\infty} \sum_{n=0}^{\infty} \frac{(C, AD/B)_n}{(q, D, qD/B)_n} q^n + \frac{(B, C, AD/B)_\infty}{(A, D, D/B)_\infty} \sum_{n=0}^{\infty} \frac{(A, BC/D)_n}{(q, B, qB/D)_n} q^n. \tag{3.1}$$

Change B to β, C to τ, D to τq and then let $A \to 0$ in (3.1) to obtain

$$\sum_{n=0}^{\infty} \frac{(-1)^n \beta^n q^{n(n-1)/2}}{(q;q)_n (1-\tau q^n)} = \frac{(\beta/q)_\infty}{(\beta/\tau q)_\infty} \sum_{n=0}^{\infty} \frac{q^n}{(q;q)_n (\tau q^2/\beta)_n (1-\tau q^n)} + \frac{(1-\beta/q)(\beta)_\infty}{(\tau q/\beta)_\infty} \sum_{n=0}^{\infty} \frac{q^n}{(q;q)_n (\beta/\tau)_n (1-\beta q^{n-1})}. \tag{3.2}$$

Change q to q^2 and set $\tau = -1$ and $\beta = -q^3$ in (3.2) to obtain

$$\sum_{n=0}^{\infty} \frac{q^{n^2+2n}}{(q^2;q^2)_n (1+q^{2n})} = \frac{(-q;q^2)_\infty}{(q;q^2)_\infty} \sum_{n=0}^{\infty} \frac{q^{2n}}{(q;q)_{2n-1} (1-q^{4n})} - \frac{(-q;q^2)_\infty}{(q;q^2)_\infty} \sum_{n=0}^{\infty} \frac{q^{2n+1}}{(q;q)_{2n} (1-q^{4n+2})}. \tag{3.3}$$

Use (1.22) to obtain

$$\sum_{n=0}^{\infty} \frac{q^{n^2+2n}}{(q^2;q^2)_{n-1} (1-q^{4n})} = \frac{\chi(q)}{\chi(-q)} \sum_{n=0}^{\infty} \frac{q^{2n}}{(q;q)_{2n-1} (1-q^{4n})} - \frac{\chi(q)}{\chi(-q)} \sum_{n=0}^{\infty} \frac{q^{2n+1}}{(q;q)_{2n} (1-q^{4n+2})}.$$

Setting $\alpha = B/A, \beta = C, \gamma = D$ and $z = A$ in (1.11), we obtain

$$\sum_{n=0}^{\infty} \frac{(B/A, C)_n}{(q, D)_n} A^n = \frac{(B, C)_\infty}{(A, D)_\infty} \sum_{n=0}^{\infty} \frac{(A, D/C)_n}{(q, B)_n} C^n. \tag{3.4}$$

Using (3.4) in (3.1) and then multiplying the resulting identity throughout by $(A, D)_\infty/(B, C)_\infty$, we obtain

$$\sum_{n=0}^{\infty} \frac{(D/C, A)_n}{(q, B)_n} C^n = \frac{(A, D, BC/D)_\infty}{(B, C, B/D)_\infty} \sum_{n=0}^{\infty} \frac{(C, AD/B)_n}{(q, D, Dq/B)_n} q^n + \frac{(AD/B)_\infty}{(B/D)_\infty} \sum_{n=0}^{\infty} \frac{(A, BC/D)_n}{(q, B, qB/D)_n} q^n. \tag{3.5}$$

Change q to q^2 and set $A = t, B = -aq^3, C = -a$ and $D = -aq^2$ in (3.5) and then let

$t \to 0$; divide the resulting identity throughout by $(1+aq)$, we obtain

$$\sum_{n=0}^{\infty} \frac{(-1)^n a^n}{(-aq;q^2)_{n+1}} = \frac{1}{(q;q^2)_\infty} \sum_{n=0}^{\infty} \frac{q^{2n}}{(q;q)_{2n}(1+aq^{2n})} - \frac{1}{(q;q^2)_\infty} \sum_{n=0}^{\infty} \frac{q^{2n+1}}{(q;q)_{2n+1}(1+aq^{2n+1})}. \tag{3.6}$$

In Rogers-Fine identity, change q to q^2, set $\alpha = 0, \beta = -aq^3$ and $z = -a$; multiply the resulting identity throughout by $1/(1+aq)$ to obtain

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{(-1)^n a^n}{(-aq;q^2)_{n+1}} &= \sum_{n=0}^{\infty} \frac{a^{2n} q^{2n^2+n}}{(-a;q^2)_{n+1}(-aq;q^2)_{n+1}} = \sum_{n=0}^{\infty} \frac{a^{2n} q^{2n^2+n}}{(-a;q)_{2n+2}} \\ &= \sum_{n=0}^{\infty} \frac{a^{2n} q^{2n^2+n}((1+aq^{2n+1}) - aq^{2n+1})}{(-a;q)_{2n+2}} \\ &= \sum_{n=0}^{\infty} \frac{a^{2n} q^{2n^2+n}}{(-a;q)_{2n+1}} - \sum_{n=0}^{\infty} \frac{a^{2n+1} q^{2n^2+3n+1}}{(-a;q)_{2n+2}} = \sum_{n=0}^{\infty} \frac{(-1)^n a^n q^{n(n+1)/2}}{(-a;q)_{n+1}}. \end{aligned} \tag{3.7}$$

Use (3.7) in (3.6) and also use (1.21) to obtain

$$\sum_{n=0}^{\infty} \frac{(-1)^n a^n q^{n(n+1)/2}}{(-a;q)_{n+1}} = \frac{f(-q^2)}{f(-q)} \sum_{n=0}^{\infty} \frac{q^{2n}}{(q;q)_{2n}(1+aq^{2n})} - \frac{f(-q^2)}{f(-q)} \sum_{n=0}^{\infty} \frac{q^{2n+1}}{(q;q)_{2n+1}(1+aq^{2n+1})}. \tag{3.8}$$

Change q to q^2, set $A = t, B = aq^3, C = -aq$ and $D = -aq^3$ in (3.5) and let $t \to 0$ in the resulting identity; multiply the resulting identity throughout by $1/(1-aq)$ and also use (1.21) to obtain on some simplifications

$$\sum_{n=0}^{\infty} \frac{(-1)^n a^n q^n}{(aq;q^2)_{n+1}} = \frac{f(-q^2)}{f(-q^4)} \sum_{n=0}^{\infty} \frac{q^{2n}}{(q^4;q^4)_n(1-a^2q^{4n+2})}. \tag{3.9}$$

In Rogers-Fine identity, replace q by q^2, set $\alpha = 0, \beta = aq^3$ and $z = -aq$ and then multiply the resulting identity throughout by $1/(1-aq)$ to obtain

$$\sum_{n=0}^{\infty} \frac{(-1)^n a^n q^n}{(aq;q^2)_{n+1}} = \sum_{n=0}^{\infty} \frac{(-1)^n a^{2n} q^{2n^2+2n}}{(a^2q^2;q^4)_{n+1}}. \tag{3.10}$$

Use (3.10) in (3.9) to obtain

$$\sum_{n=0}^{\infty} \frac{(-1)^n a^{2n} q^{2n^2+2n}}{(a^2q^2;q^4)_{n+1}} = \frac{f(-q^2)}{f(-q^4)} \sum_{n=0}^{\infty} \frac{q^{2n}}{(q^4;q^4)_n(1-a^2q^{4n+2})}. \tag{3.11}$$

Change q to q^2, set $A = t, B = aq^3, C = -aq$ and $D = -aq^3$ in (3.5) and multiply the resulting identity throughout by $1/(1-aq)$ to obtain

$$\sum_{n=0}^{\infty}\frac{(t;q^2)_n(-aq)^n}{(aq;q^2)_{n+1}}=\frac{(t;q^2)_\infty}{2(-q^2;q^2)_\infty}\sum_{n=0}^{\infty}\frac{(-t;q^2)_nq^{2n}}{(q^4;q^4)_n(1+aq^{2n+1})}+\frac{(-t;q^2)_\infty}{2(-q^2;q^2)_\infty}\sum_{n=0}^{\infty}\frac{(t;q^2)_nq^{2n}}{(q^4;q^4)_n(1-aq^{2n+1})}. \tag{3.12}$$

Set $a=-1$ and $t=q$ in (3.12) to obtain

$$\sum_{n=0}^{\infty}\frac{(q;q^2)_nq^n}{(-q;q^2)_{n+1}}=\frac{(q;q^2)_\infty}{2(-q^2;q^2)_\infty}\sum_{n=0}^{\infty}\frac{(-q;q^2)_nq^{2n}}{(q^4;q^4)_n(1-q^{2n+1})}+\frac{(-q;q^2)_\infty}{2(-q^2;q^2)_\infty}\sum_{n=0}^{\infty}\frac{(q;q^2)_nq^{2n}}{(q^4;q^4)_n(1+q^{2n+1})}. \tag{3.13}$$

In Rogers-Fine identity, replace q by q^2, set $\alpha=z=q$ and $\beta=-q^3$; multiply the resulting identity throughout by $1/(1+q)$ to obtain

$$\sum_{n=0}^{\infty}\frac{(q;q^2)_nq^n}{(-q;q^2)_{n+1}}=\sum_{n=0}^{\infty}(-1)^nq^{2n(n+1)}. \tag{3.14}$$

Use (3.14) in (3.13) and also use (1.19), (1.20) and (1.21) to obtain

$$2\sum_{n=0}^{\infty}(-1)^nq^{2n(n+1)}=\frac{f(-q)}{f(-q^4)}\sum_{n=0}^{\infty}\frac{(-q;q^2)_nq^{2n}}{(q^4;q^4)_n(1-q^{2n+1})}+\frac{\varphi(q)}{\psi(q)}\sum_{n=0}^{\infty}\frac{(q;q^2)_nq^{2n}}{(q^4;q^4)_n(1+q^{2n+1})}. \tag{3.15}$$

In (3.5), set $A=q, B=-aq, C=\tau$ and $D=a^2q$ to obtain

$$\sum_{n=0}^{\infty}\frac{(a^2q/\tau)_n}{(-aq)_n}\tau^n=\frac{(-\tau/a,q,a^2q)_\infty}{(-1/a,-aq,\tau)_\infty}\sum_{n=0}^{\infty}\frac{(\tau)_n}{(q,a^2q)_n}q^n+\frac{(-aq)_\infty}{(-1/a)_\infty}\sum_{n=0}^{\infty}\frac{(-\tau/a)_n}{(-aq,-q/a)_n}q^n. \tag{3.16}$$

In Rogers-Fine identity, set $\alpha=a^2q/\tau, \beta=-aq$ and $z=\tau$ to obtain

$$\sum_{n=0}^{\infty}\frac{(a^2q/\tau)_n}{(-aq)_n}\tau^n=\sum_{n=0}^{\infty}\frac{(-1)^n(a^2q/\tau)_na^nq^{n^2}(1-a^2q^{2n+1})}{(\tau)_{n+1}}. \tag{3.17}$$

Use (3.17) in (3.16) and then let $\tau \to 0$ in the resulting identity to obtain

$$\sum_{n=0}^{\infty} a^{3n} q^{n(3n+1)/2}(1-a^2q^{2n+1}) = \frac{(a^2q)_\infty f^2(-q)}{f(aq,1/a)} \sum_{n=0}^{\infty} \frac{q^n}{(q,a^2q)_n} + \frac{(-aq)_\infty}{(-1/a)_\infty} \sum_{n=0}^{\infty} \frac{q^n}{(-aq,-q/a)_n}. \quad (3.18)$$

Set $a = 1$ in (3.18) to obtain

$$\sum_{n=0}^{\infty} q^{n(3n+1)/2}(1-q^{2n+1}) = \frac{f^3(-q)}{f(q,1)} \sum_{n=0}^{\infty} \frac{q^n}{(q)_n^2} + \frac{1}{2} \sum_{n=0}^{\infty} \frac{q^n}{(-q)_n^2}. \quad (3.19)$$

In (1.11), set $\gamma = z = q$ and then $\alpha = 0, \beta = 0$ to obtain

$$\sum_{n=0}^{\infty} \frac{q^n}{(q)_n^2} = \frac{1}{(q)_n^2} \sum_{n=0}^{\infty} (-1)^n q^{n(n+1)/2}. \quad (3.20)$$

Use (3.20) in (3.19) and also use (1.20) to obtain

$$\sum_{n=0}^{\infty} q^{n(3n+1)/2}(1-q^{2n+1}) = \frac{f(-q)\psi(-q)}{f(q,1)} + \frac{1}{2} \sum_{n=0}^{\infty} \frac{q^n}{(-q)_n^2}.$$

In (3.16), let $\tau \to 0$ to obtain

$$\sum_{n=0}^{\infty} \frac{(-1)^n a^{2n} q^{n(n+1)/2}}{(-aq)_n} = \frac{(q,a^2q)_\infty}{(-1/a,-aq)_\infty} \sum_{n=0}^{\infty} \frac{q^n}{(q,a^2q)_n} + \frac{(-aq)_\infty}{(-1/a)_\infty} \sum_{n=0}^{\infty} \frac{q^n}{(-aq,-q/a)_n}. \quad (3.21)$$

Set $a = 1$ in (3.21) to obtain

$$\sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+1)/2}}{(-q)_n} = \frac{f^3(-q)}{f(1,q)} \sum_{n=0}^{\infty} \frac{q^n}{(q)_n^2} + \frac{1}{2} \sum_{n=0}^{\infty} \frac{q^n}{(-q)_n^2}. \quad (3.22)$$

The left side of (3.22) yields

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+1)/2}}{(-q)_n} &= \sum_{n=0}^{\infty} \frac{q^{n(2n+1)}}{(-q;q)_{2n}} - \sum_{n=0}^{\infty} \frac{q^{(n+1)(2n+1)}}{(-q;q)_{2n+1}} \\ &= \sum_{n=0}^{\infty} \frac{q^{n(2n+1)}((1+q^{2n+1})-q^{2n+1})}{(-q;q)_{2n+1}} = \sum_{n=0}^{\infty} \frac{q^{n(2n+1)}}{(-q;q)_{2n+1}}. \quad (3.23)\end{aligned}$$

Use (3.23) in (3.22) to obtain

$$\sum_{n=0}^{\infty} \frac{q^{n(2n+1)}}{(-q;q)_{2n+1}} = \frac{f^3(-q)}{f(1,q)} \sum_{n=0}^{\infty} \frac{q^n}{(q)_n^2} + \frac{1}{2} \sum_{n=0}^{\infty} \frac{q^n}{(-q)_n^2}. \tag{3.24}$$

Use (3.20) in (3.24) to obtain

$$\sum_{n=0}^{\infty} \frac{q^{n(2n+1)}}{(-q;q)_{2n+1}} = \frac{f^3(-q)}{f(1,q)} \sum_{n=0}^{\infty} (-1)^n q^{n(n+1)/2} + \frac{1}{2} \sum_{n=0}^{\infty} \frac{q^n}{(-q)_n^2}. \tag{3.25}$$

Use the definition of ψ to obtain

$$\sum_{n=0}^{\infty} \frac{q^{n(2n+1)}}{(-q;q)_{2n+1}} = \frac{f(-q)\psi(-q)}{f(1,q)} + \frac{1}{2} \sum_{n=0}^{\infty} \frac{q^n}{(-q)_n^2}.$$

In (3.5), replace q by q^2, set $A = q^2, B = -aq^3, C = \tau$ and $D = a^2q^2$; multiply the resulting identity throughout by $1/(1+aq)$ to obtain

$$\sum_{n=0}^{\infty} \frac{(a^2q^2/\tau;q^2)_n}{(-aq;q^2)_n} \tau^n = \frac{(q^2, a^2q^2, -q\tau/a; q^2)_\infty}{(-aq, \tau, -q/a; q^2)_\infty} \sum_{n=0}^{\infty} \frac{(\tau;q^2)_n}{(q^2, a^2q^2; q^2)_n} q^{2n} + \frac{(-aq;q^2)_\infty}{(-q/a;q^2)_\infty} \sum_{n=0}^{\infty} \frac{(-q\tau/a;q^2)_n}{(-q^3/a;q^2)_n(-aq;q^2)_{n+1}} q^{2n}. \tag{3.26}$$

In Rogers-Fine identity, replace q by q^2, set $\alpha = a^2q^2/\tau, \beta = -aq^3, z = \tau$ and then multiply the resulting identity throughout by $1/(1+aq)$ to obtain

$$\sum_{n=0}^{\infty} \frac{(a^2q^2/\tau;q^2)_n}{(-aq;q^2)_n} \tau^n = \sum_{n=0}^{\infty} \frac{(-1)^n (a^2q^2/\tau;q^2)_n \tau^n a^n q^{2n(n+1)} (1 - a^2q^{4n+2})}{(1+aq^{2n+1})(\tau;q^2)_{n+1}}. \tag{3.27}$$

Use (3.27) in (3.26) and then let $\tau \to 0$ to obtain

$$\sum_{n=0}^{\infty} a^{3n} q^{3n^2+2n} (1 - aq^{2n+1}) = \frac{(q^2, a^2q^2; q^2)_\infty}{(-q/a, -aq; q^2)_\infty} \sum_{n=0}^{\infty} \frac{q^{2n}}{(q^2, a^2q^2; q^2)_n} + \frac{(-aq;q^2)_\infty}{(-q/a;q^2)_\infty} \sum_{n=0}^{\infty} \frac{q^{2n}}{(-aq;q^2)_{n+1}(-q^3/a;q^2)_n}. \tag{3.28}$$

In (3.5), replace q to q^2, set $A = q^2, B = -q^3, D = q^2$ and then let $C \to 0$; multiply the resulting identity throughout by $1/(1+q)$ to obtain

$$\sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2+n}}{(-q;q^2)_{n+1}} = \frac{(q^2;q^2)_\infty^2}{(-q;q^2)_\infty^2} \sum_{n=0}^{\infty} \frac{q^{2n}}{(q^2;q^2)_n^2} + (1+q) \sum_{n=0}^{\infty} \frac{q^{2n}}{(-q;q^2)_{n+1}^2}. \tag{3.29}$$

In (2.10), replace q by q^6, set $a = q, b = q^4, c = q^2, e = q^3$and $f = q^7$ to obtain

$$\sum_{n=0}^{\infty} \frac{(q^2, q^4; q^6)_n q^{6n}}{(q^3, q^6; q^6)_n (1 - q^{6n+1})} - \frac{(q^2, q^4; q^6)_\infty}{(q, q^5; q^6)_\infty} \sum_{n=0}^{\infty} \frac{(q; q^6)_{n+1} (q^5; q^6)_n q^{6n+3}}{(q^3; q^6)_{n+1} (q^6; q^6)_n (1 - q^{6n+4})}$$
$$= (1 - q) \frac{(q^3; q^6)_\infty^2 (q^6; q^6)_\infty}{(q; q^6)_\infty^2 (q^4; q^6)_\infty}. \tag{3.30}$$

Use (1.21), (1.23) and (1.24) to obtain on some simplifications

$$\sum_{n=0}^{\infty} \frac{(q^2, q^4; q^6)_n q^{6n}}{(q^3, q^6; q^6)_n (1 - q^{6n+1})} - \frac{f(-q^2)}{H_6(q)} \sum_{n=0}^{\infty} \frac{(q; q^6)_{n+1} (q^5; q^6)_n q^{6n+3}}{(q^3; q^6)_{n+1} (q^6; q^6)_n (1 - q^{6n+4})}$$
$$= (1 - q) \frac{G_6^2(q) H_6^2(q) f(-q^2)}{(q; q^6)_\infty^2 f(-q) f^2(-q^6)}. \tag{3.31}$$

In (2.10), replace q by q^3, set $a = c = -q, b = e = -q^2$ and $f = q^3$ to obtain

$$\sum_{n=0}^{\infty} \frac{(-q; q^3)_n^2}{(q^3; q^3)_n^2} q^{3n} + \frac{(-q; q^3)_\infty^2 (q^2; q^6)_\infty^2}{(q; q)_\infty^2} \sum_{n=0}^{\infty} \frac{(q^2; q^3)_n^2 q^{3n+1}}{(-q; q^3)_{n+1}^2}$$
$$= \frac{(-q; q^3)_\infty^2 (-q^2; q^3)_\infty^2 (q^3; q^3)_\infty^2}{(q^2; q^3)_\infty^2 (q^3; q^3)_\infty^4}. \tag{3.32}$$

Use (1.25) to obtain

$$\sum_{n=0}^{\infty} \frac{(-q; q^3)_n^2}{(q^3; q^3)_n^2} q^{3n} + \frac{(-q; q^3)_\infty^2 (q^2; q^6)_\infty^2}{(q; q)_\infty^2} \sum_{n=0}^{\infty} \frac{(q^2; q^3)_n^2 q^{3n+1}}{(-q; q^3)_{n+1}^2} = \frac{J_6^2(q) (q; q^3)_\infty^2}{(q; q)_\infty^2 (q^3; q^3)_\infty^4}.$$

Acknowledgement

The first author is thankful to University Grants Commission(UGC), India for the financial support under the grant SAP-DRS-1-NO.F.510/2/DRS/2011 and the second author is thankful to UGC for awarding the Rajiv Gandhi National Fellowship, No.F1-17.1/2011-12/RGNF-SC-KAR-2983/(SA-III/Website) and the third author is thankful to UGC for awarding the Basic Science Research Fellowship, No.F.25-1/2014-15(BSR)/No.F.7-349/2012(BSR). The authors are thankful to Prof. Z.G. Liu of East China Normal University for his valuable suggestions.

References

[1] R. P. Agarwal, On the paper, A lost notebook of Ramanujan, *Adv. in Math.* 53(1984), 291-300.

[2] G. E. Andrews, Ramanujan's lost notebook. I. partial θ-functions, *Adv. in Math.* 41(1981), 137-172.

[3] G. E. Andrews, R. Askey and R. Roy, *Special Functions*, Cambridge University Press, Cambridge, 1999.

[4] G. E. Andrews and B. C. Berndt, *Ramanujan's Lost Notebook, Part II,* Springer, New York, 2009.

[5] G. Gasper and M. Rahman, *Basic Hypergeometric Series*, Encyclopedia of Mathematics, Cambridge University press. Cambridge, 2004.

[6] W.Y.C. Chen, and N.S.S. Gu, The Cauchy operator for basic hypergeometric series, *Advances in Applied Mathematics* 41 (2008), 177-196.

[7] Husam L. Saad and Abaas A. Sukhi, The q-Exponential Operator, *Applied Mathematical Sciences* 7(128)(2013), 6369-6380.

[8] C. G. J. Jacobi, *Fundamenta Nova Theoriae Funtionum Ellipticarum,* Sumptibus Fratrum Bornträger, Regiomonti, 1829; reprinted in Gesamelte Werke, 1, Reimer, Berlin, 1881, 49-239; reprinted by Chelsea, New York, 1969.

[9] Z.-G. Liu, Some operator identities and q-series transformation formulas, *Discrete Mathematics* 265(2003), 119-139.

[10] S. Ramanujan, *Notebooks (2 Volumes)*, Tata Institute of Fundamental Research, Bombay, 1957.

[11] S. Ramanujan, *The Lost Notebook and Other Unpublished Papers*, Narosa, New Delhi, 1988.

[12] L. J. Rogers, Second memior on the expansion of certain infinite products, *Proc. London Math. Soc.* 25(1894), 318-343.

[13] D. D. Somashekara, K. Narasimha murthy and S. L. Shalini, On the reciprocity theorem of Ramanujan and its generalizations, *Proc. Jangjeon Math. Soc.* 15(3)(2012), 343-353.

[14] D. D. Somashekara, T. Kim, H. I. Kwon and S. L. Shalini, On some partial theta function identities of Ramanujan found in his lost notebook, *Adv. Studies Contemp. Math.*, **25** (1), 2015.

[15] D.B. Sears, On the transformation theory of basic hypergeometric functions, *Proc. London Math. Soc.*, 53 (1951), 158-180.