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On the Second Order Mannheim Partner Curve in E^3

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Abstract: In this study first we worked on the Mannheim curve pair $\{\alpha, \alpha_1\}$ and Mannheim curve pair $\{\alpha_1, \alpha_2\}$ We called α_2 as the second order Mannheim partner curve of the Mannheim curve α. We examined the Frenet apparatus of second order Mannheim partner curve in terms of, Frenet apparatus of Mannheim curve α, with the offset property of second order Mannheim partner α_2. Further we examined third order Mannheim partner α_3 where $\{\alpha_2, \alpha_3\}$ are Mannheim curve pair.

Key Words: Mannheim curve, Frenet apparatus, second Mannheim curve, modified Darboux vector.

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§1. Introduction

Mannheim curve was firstly defined by A. Mannheim in 1878. A curve is called a Mannheim curve if and only if $\dfrac{\kappa}{\kappa^2+\tau^2}$ is a nonzero constant, κ is the curvature and τ is the torsion. Mannheim curve was redefined in [6], if the principal normal vector N of first curve and binormal vector B_1 of second curve are linearly dependent, then first curve is called Mannheim curve, and the second curve is called Mannheim partner curve. As a result they called these new curves as Mannheim partner curves. For more detail see in [6]. Frenet-Serret apparatus of the curve $\alpha : I \to E^3$ are $\{T, N, B, \kappa, \tau\}$. For any unit speed curve α, the Darboux and modified Darboux vectors are, respectively ([2],[4])

$$D(s) = \tau(s)T(s) + \kappa(s)B(s), \tag{1.1}$$

$$\tilde{D}(s) = \frac{\tau}{\kappa}(s)T(s) + B(s). \tag{1.2}$$

In [7] Mannheim curves are studied and Mannheim partner curve of α can be represented

$$\alpha(s_1) = \alpha_1(s_1) + \lambda(s_1)B_1(s_1) \tag{1.3}$$

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for some function λ, since N and B_1 are linearly dependent, equation can be rewritten as

$$\alpha_1(s) = \alpha(s) - \lambda(s)N(s), \tag{1.4}$$

where

$$\lambda(s) = \frac{-\kappa(s)}{\left(\kappa(s)\right)^2 + \left(\tau(s)\right)^2}. \tag{1.5}$$

Frenet-Serret apparatus of Mannheim partner curve α_1 are $\{T_1, N_1, B_1, \kappa_1, \tau_1\}$. The relationship α and α_1 Frenet vectors are as follows

$$\begin{aligned} T_1 &= \cos\theta\ T - \sin\theta\ B \\ N_1 &= \sin\theta\ T + \cos\theta\ B \\ B_1 &= N. \end{aligned} \tag{1.6}$$

where $\angle(T, T_1) = \cos\theta$. The first curvature and the second curvature (torsion) are

$$\kappa_1 = -\frac{d\theta}{ds_1} = \frac{\theta'}{\cos\theta}, \qquad \tau_1 = \frac{\kappa}{\lambda\tau}. \tag{1.7}$$

We use dot $\cdot$ to denote the derivative with respect to the arc length parameter of the curve α. Also

$$\frac{ds}{ds_1} = \frac{1}{\cos\theta} = \frac{-\lambda\tau_1}{\sin\theta}, \tag{1.8}$$

for more detail see in [7], or we can write

$$\frac{ds}{ds_1} = \frac{1}{\sqrt{1+\lambda\tau}}. \tag{1.9}$$

§2. Second Order Mannheim Partner and Frenet Apparatus

Definition 2.1 *Let $\{\alpha\ , \alpha_1\}$ and $\{\alpha_1, \alpha_2\}$ be the Mannheim pairs of α and α_1 respectively. We called as α_2 is a second order Mannheim partner of the curve α. which has the following parametrization ,*

$$\alpha_2 = \alpha + \lambda_1 \sin\theta T - \lambda N + \lambda_1 \cos\theta B, \tag{2.1}$$

where

$$\alpha_1 = \alpha(s) - \lambda N(s) \quad \textit{and} \quad \alpha_2 = \alpha_1(s) - \lambda_1 N_1(s). \tag{2.2}$$

Theorem 2.1 *The Frenet vectors of second order Mannheim partner α_2 of a Mannheim curve*

α, based on the Frenet apparatus of Mannheim curve α are

$$\begin{cases} T_2 = \cos\theta_1 \cos\theta \; T - \sin\theta_1 \; N - \cos\theta_1 \sin\theta \; B \\ N_2 = \sin\theta_1 \cos\theta \; T + \cos\theta_1 \; N - \sin\theta_1 \sin\theta \; B \\ B_2 = \sin\theta \; T + \cos\theta \; B. \end{cases} \tag{2.3}$$

Proof Let α_2 be second order Mannheim partner of a Mannheim curve α. Also α_2 be the Mannheim partner of Mannheim partner α_1. The Frenet vector fields T_1, N_1, B_1 and T_2, N_2, B_2 which are belong to the curves α_1 and α_2, respectively. It is easy to say that Frenet vectors of second order Mannheim partner α_2, based on the Frenet vectors of Mannheim curve α_1 are

$$\begin{cases} T_2 = \cos\theta_1 \; T_1 - \sin\theta_1 \; B_1 \\ N_2 = \sin\theta_1 \; T_1 + \cos\theta_1 \; B_1 \\ B_2 = N_1 \end{cases}$$

where $\angle(T_1, T_2) = \theta_1$. By substituting T_1, N_1, B_1 we have the equalities in terms of the curve α.

$$\begin{aligned} T_2 &= \cos\theta_1 \; (\cos\theta \; T - \sin\theta \; B) - \sin\theta_1 \; N \\ N_2 &= \sin\theta_1 \; (\cos\theta \; T - \sin\theta \; B) + \cos\theta_1 \; N \\ B_2 &= \sin\theta \; T + \cos\theta \; B \end{aligned}$$

This completes the proof. Also the following product give us the same equalities;

$$\begin{bmatrix} T_2 \\ N_2 \\ B_2 \end{bmatrix} = \begin{bmatrix} \cos\theta_1 & 0 & -\sin\theta_1 \\ \sin\theta_1 & 0 & \cos\theta_1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ \sin\theta & 0 & \cos\theta \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}.$$ □

Theorem 2.2 *Let α_2 be second order Mannheim partner of a Mannheim curve α. The curvature and torsion of the second order Mannheim partner α_2 are*

$$\kappa_2 = \frac{-\theta_1'}{\cos\theta} \frac{1}{\cos\theta_1}, \quad \tau_2 = \frac{-\theta'}{\cos\theta} \frac{\lambda\tau}{\lambda_1\kappa}. \tag{2.4}$$

Proof Since $\kappa_1 = \dfrac{-\theta'}{\cos\theta}$ and $\tau_1 = \dfrac{\kappa}{\lambda\tau}$, we have the curvature as in the following way

$$\kappa_2 = -\frac{d\theta_1}{ds_2} = \frac{-\theta_1'}{\cos\theta\cos\theta_1}.$$

Also as in the following way we have the torsion

$$\tau_2 = \frac{\kappa_1}{\lambda_1\tau_1} = \frac{-\theta'}{\cos\theta} \frac{\lambda}{\lambda_1} \frac{\tau}{\kappa}.$$

We use mark to denote the derivative with respect to the parameter of the curve α. Due to this theorem we also get

$$\frac{ds}{ds_2} = \frac{1}{\cos\theta\cos\theta_1}. \tag{2.5}$$

□

Theorem 2.3 *The modified Darboux vector of Mannheim partner α_1 of a Mannheim curve α, is*

$$\tilde{D}_1(s) = \frac{\kappa}{\lambda\tau}\frac{\cos^2\theta}{\theta'}\ T + N - \frac{\kappa}{\lambda\tau}\frac{\cos\theta\sin\theta}{\theta'}\ B \tag{2.6}$$

Proof Similarly from the equation (1.2)

$$\tilde{D}_1(s) = \frac{\tau_1}{\kappa_1}T_1\left(s\right) + B_1\left(s\right). \tag{2.7}$$

Substituting the equation (2.7) into equation (1.6)and (1.7), the proof is complete. □

Theorem 2.4 *The modified Darboux vector of second order Mannheim partner α_2 of a Mannheim curve α, is*

$$\begin{aligned}\tilde{D}_2 &= \left(\frac{\lambda\tau\cos^2\theta_1\cos\theta}{\lambda_1\kappa} + \sin\theta\right)T - \frac{\lambda\tau\cos\theta_1\sin\theta_1}{\lambda_1\kappa}\ N \\ &\quad -\left(\frac{\lambda\tau\cos^2\theta_1\sin\theta}{\lambda_1\kappa} - \cos\theta\right)B.\end{aligned} \tag{2.8}$$

Proof Since

$$\tilde{D}_2(s) = \frac{\tau_2}{\kappa_2}T_2\left(s\right) + B_2\left(s\right). \tag{2.9}$$

Substituting the equation (2.9) into equation (2.3)and (2.4), the proof is complete. □

Theorem 2.5 *The offset property of second order Mannheim partner α_2 can be given if and only if the curvature κ and the torsion τ of α satisfy the following equation*

$$\lambda_1 = \frac{-\theta'\tau\cos\theta}{\theta'^2\tau + \left(\kappa^2+\tau^2\right)^2\cos^2\theta}, \tag{2.10}$$

where $\theta'^2\tau + \left(\kappa^2+\tau^2\right)^2\cos^2\theta \neq 0$.

Proof Notice that $\kappa_1 = \frac{-\theta'}{\cos\theta}$,$\tau_1 = \frac{\kappa}{\lambda\tau}$ with the offset property $-\kappa_1 = \lambda_1\left(\kappa_1^2+\tau_1^2\right)$ and

$$\begin{aligned}\left(\kappa_1^2+\tau_1^2\right) &= \frac{-\kappa_1}{\lambda_1} \\ \lambda_1 &= \frac{-\theta'}{\cos\theta}\ \frac{1}{\dfrac{\theta'^2\tau + \cos^2\theta\left(\kappa^2+\tau^2\right)^2}{\tau\cos^2\theta}} \\ \lambda_1 &= \frac{-\theta'\tau\cos\theta}{\theta'^2\tau + \left(\kappa^2+\tau^2\right)^2\cos^2\theta}.\end{aligned}$$

This completes the proof. □

Theorem 2.6 *The second order Mannheim partner α_2 is not a Mannheim partner curve α.*

Proof Since the definition of Mannheim partner curve,

$$\langle B_2(s), N(s)\rangle = \langle \sin\theta\, T + \cos\theta\, B, N\rangle = 0,$$

hence $N(s)$ and $B_2(s)$ are linear independent. □

Definition 2.2 *Let $\{\alpha, \alpha_1\}$, $\{\alpha_1, \alpha_2\}$ and $\{\alpha_2, \alpha_3\}$ be the Mannheim pairs of α, α_1 and α_2 respectively. We called as α_3 is a third order Mannheim partner of the curve α, which has the following parametrizations,*

$$\begin{aligned} \alpha_3(s) &= \alpha_2(s) - \lambda_2 N_2(s) \\ &= \alpha + (\lambda_1 \sin\theta + \lambda_2 \sin\theta_1 \cos\theta)\, T - (\lambda - \lambda_2 \cos\theta_1)\, N \\ &\quad + (\lambda_1 \cos\theta - \lambda_2 \sin\theta_1 \sin\theta)\, B, \end{aligned} \tag{2.11}$$

where

$$\alpha_2 = \alpha + \lambda_1 \sin\theta T - \lambda N + \lambda_1 \cos\theta B \tag{2.12}$$

and

$$|\lambda + \lambda_1 + \lambda_2|$$

is the distance between the arclengthed curves α and α_3.

Theorem 2.7 *The Frenet vectors of third order Mannheim partner α_3 of a Mannheim curve α, based on the Frenet apparatus of Mannheim curve α are*

$$\begin{cases} T_3 = (\cos\theta_2 \cos\theta_1 \cos\theta - \sin\theta_2 \sin\theta)\, T - \cos\theta_2 \sin\theta_1\, N \\ \quad - (\sin\theta_2 \cos\theta + \cos\theta_2 \cos\theta_1 \sin\theta)\, B \\ \\ N_3 = (\sin\theta_2 \cos\theta_1 \cos\theta + \cos\theta_2 \sin\theta)\, T - \sin\theta_2 \sin\theta_1\, N \\ \quad + (\cos\theta_2 \cos\theta - \sin\theta_2 \cos\theta_1 \sin\theta)\, B \\ \\ B_3 = \sin\theta_1 \cos\theta\, T + \cos\theta_1\, N - \sin\theta_1 \sin\theta\, B \end{cases} \tag{2.13}$$

where $\angle(T_2, T_3) = \cos\theta_2$.

Proof Since

$$\begin{bmatrix} T_3 \\ N_3 \\ B_3 \end{bmatrix} = \begin{bmatrix} \cos\theta_2 & 0 & -\sin\theta_2 \\ \sin\theta_2 & 0 & \cos\theta_2 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \cos\theta_1 & 0 & -\sin\theta_1 \\ \sin\theta_1 & 0 & \cos\theta_1 \\ 0 & 1 & 0 \end{bmatrix} \times \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ \sin\theta & 0 & \cos\theta \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix},$$

we have the proof. □

Corollary 2.1 *The product of Frenet vector fields of third order Mannheim partner α_3 and Mannheim curve α, has the following matrix form*

$$[\mathbf{V}_3]^{\mathbf{T}}[\mathbf{V}] = \begin{bmatrix} \cos\theta_2\cos\theta_1\cos\theta - \sin\theta_2\sin\theta & -\cos\theta_2\sin\theta_1 & -\sin\theta_2\cos\theta - \cos\theta_2\cos\theta_1\sin\theta \\ \sin\theta_2\cos\theta_1\cos\theta + \cos\theta_2\sin\theta & -\sin\theta_2\sin\theta_1 & \cos\theta_2\cos\theta - \sin\theta_2\cos\theta_1\sin\theta \\ \sin\theta_1\cos\theta & \cos\theta_1 & -\sin\theta_1\sin\theta \end{bmatrix} \tag{2.14}$$

where $[\mathbf{V}_3] = [T_3, N_3, B_3]$ *and* $[\mathbf{V}] = [T, N, B]$.

Corollary 2.2 *Let α_3 be third order Mannheim partner of a Mannheim curve α. The curvature and torsion of the third order Mannheim partner α_3 are*

$$\kappa_3 = -\frac{\theta_2'}{\cos\theta\cos\theta_1\cos\theta_2}, \qquad \tau_3 = \frac{\theta_1'\lambda_1\kappa}{\theta'\cos\theta_1\lambda_2\lambda\tau}. \tag{2.15}$$

Proof We can write

$$\kappa_3 = -\frac{d\theta_2}{ds_3} = \frac{-\theta_2'}{\cos\theta\cos\theta_1\cos\theta_2}$$

and

$$\tau_3 = \frac{\kappa_2}{\lambda_2\tau_2} = \frac{\theta_1'\lambda_1\kappa}{\theta'\cos\theta_1\lambda_2\lambda\tau}$$

or also since

$$\cos\theta\cos\theta_1\cos\theta_2 = \frac{-\theta_2'}{\kappa_3}$$

and

$$\cos\theta\cos\theta_1 = \frac{-\theta_1'}{\kappa_2}.$$

□

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