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Fixed Point Results Under Generalized Contraction Involving Rational Expression in Complex Valued Metric Spaces

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Abstract: The aim of this paper is to study common fixed point under generalized contraction involving rational expression in the setting of complex valued metric spaces. The results presented in this paper extend and generalize several results from the existing literature.

Key Words: Common fixed point, generalized contraction involving rational expression, complex valued metric space.

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§1. Introduction

Fixed point theory plays a very crucial role in the development of nonlinear analysis. The Banach [2] fixed point theorem for contraction mapping has been generalized and extended in many directions. This famous theorem can be stated as follows.

Theorem 1.1([2]) *Let (X, d) be a complete metric space and T be a mapping of X into itself satisfying:*

$$d(Tx, Ty) \leq \alpha\, d(x, y),\ \forall\, x, y \in X, \tag{1.1}$$

where α is a constant in $[0, 1)$. Then T has a fixed point $p \in X$.

The Banach contraction principle with rational expressions have been expanded and some fixed point and common fixed point theorems have been obtained in [4, 5].

Recently, Azam et al. [1] introduced the concept of complex valued metric space and established some fixed point results for mappings satisfying a rational inequality. Complex-valued metric space is useful in many branches of mathematics, including algebraic geometry, number theory, applied mathematics; as well as in physics, including hydrodynamics, thermodynamics, mechanical engineering and electrical engineering, for more details, see, [7, 8].

In this paper, we establish common fixed point results for generalized contraction involving rational expression in the framework of complex valued metric spaces.

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§2. Preliminaries

Let $\mathbb{C}$ be the set of complex numbers and $z_1, z_2 \in \mathbb{C}$. Define a partial order $\precsim$ on $\mathbb{C}$ as follows:

$z_1 \precsim z_2$ if and only if $Re(z_1) \le Re(z_2)$, $Im(z_1) \le Im(z_2)$. It follows that $z_1 \precsim z_2$ if one of the following conditions is satisfied:

(i) $Re(z_1) = Re(z_2)$, $Im(z_1) < Im(z_2)$;
(ii) $Re(z_1) < Re(z_2)$, $Im(z_1) = Im(z_2)$;
(iii) $Re(z_1) < Re(z_2)$, $Im(z_1) < Im(z_2)$;
(iv) $Re(z_1) = Re(z_2)$, $Im(z_1) = Im(z_2)$.

In particular, we will write $z_1 \lnsim z_2$ if $z_1 \ne z_2$ and one of (i), (ii), or (iii) is satisfied and we will write $z_1 \prec z_2$ if only (iii) is satisfied. Note that

$$0 \precsim z_1 \lnsim z_2 \Rightarrow |z_1| < |z_2|,$$

$$z_1 \precsim z_2,\ z_2 \prec z_3 \Rightarrow z_1 \prec z_3.$$

The following definition was introduced by Azam et al. in 2011 (see, [1]).

Definition 2.1([1]) *Let X be a nonempty set. Suppose that the mapping $d\colon X \times X \to \mathbb{C}$ satisfies:*

(C_1) $0 \precsim d(x, y)$ *for all* $x, y \in X$ *with* $x \ne y$ *and* $d(x, y) = 0 \Leftrightarrow x = y$*;*
(C_2) $d(x, y) = d(y, x)$ *for all* $x, y \in X$*;*
(C_3) $d(x, y) \precsim d(x, z) + d(z, y)$ *for all* $x, y, z \in X$*.*

Then d is called a complex valued metric on X and (X, d) is called a complex valued metric space.

Example 2.2 Let $X = \mathbb{C}$, where $\mathbb{C}$ is the set of complex numbers. Define a mapping $d\colon X \times X \to \mathbb{C}$ by $d(z_1, z_2) = e^{it}|z_1 - z_2|$ where $z_1 = (x_1, y_1)$, $z_2 = (x_2, y_2)$ and $t \in [0, \frac{\pi}{2}]$. Then (X, d) is a complex valued metric space.

Example 2.3([1]) Let $X = \mathbb{C}$, where $\mathbb{C}$ is the set of complex numbers. Define a mapping $d\colon X \times X \to \mathbb{C}$ by $d(z_1, z_2) = e^{3i}|z_1 - z_2|$ where $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$. Then (X, d) is a complex valued metric space.

Example 2.4 Let $X = \mathbb{C}$. Define a mapping $d\colon X \times X \to \mathbb{C}$ by $d(z_1, z_2) = e^{ia}|z_1 - z_2|$ where $z_1 = (x_1, y_1)$, $z_2 = (x_2, y_2)$ and a is any real constant. Then (X, d) is a complex valued metric space.

Definition 2.5 (i) *A point $x \in X$ is called an interior point of a subset $G \subseteq X$ whenever there exists $0 \prec r \in \mathbb{C}$ such that*

$$B(x, r) = \{y \in X : d(x, y) \prec r\} \subseteq G.$$

(ii) A point $x \in X$ is called a limit of G whenever for every $0 \prec r \in \mathbb{C}$ such that

$$B(x,r) \cap \Big(G - \{x\}\Big) \neq \emptyset.$$

(iii) The set $G \subseteq X$ is called open whenever each element of G is an interior point of G. A subset $H \subseteq X$ is called closed whenever each limit point of H belongs to H.

The family $\mathcal{F} := \{B(x,r) : x \in X, 0 \prec r\}$ is a sub-basis for a Hausdorff topology τ on X.

Definition 2.6([1]) *Let (X,d) be a complex valued metric space. Let $\{x_n\}$ be a sequence in X and $x \in X$. Then*

(i) $\{x_n\}$ is called convergent, if for every $c \in \mathbb{C}$, with $0 \prec c$ there exists $n_0 \in \mathbb{N}$ such that for all $n > n_0$, $d(x_n, x) \prec c$. Also, $\{x_n\}$ converges to x (written as, $x_n \to x$ or $\lim_{n\to\infty} x_n = x$) and x is the limit of $\{x_n\}$.

(ii) $\{x_n\}$ is called a Cauchy sequence in X, if for every $c \in \mathbb{C}$, with $0 \prec c$ there exists $n_0 \in \mathbb{N}$ such that for all $n > n_0$, $d(x_n, x_{n+m}) \prec c$. If every Cauchy sequence converges in X, then X is called a complete complex valued metric space.

Definition 2.7([6]) *Two families of self-mappings $\{T_i\}_{i=1}^m$ and $\{S_i\}_{i=1}^n$ are said to be pairwise commuting if*

(i) $T_iT_j = T_jT_i$, $i,j \in \{1,2,\cdots,m\}$;
(ii) $S_kS_l = S_lS_k$, $k,l \in \{1,2,\cdots,n\}$;
(iii) $T_iS_k = S_kT_i$, $i \in \{1,2,\ldots,m\}$ and $k \in \{1,2,\cdots,n\}$.

Lemma 2.8([1]) *Let (X,d) be a complex valued metric space and let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ converges to x if and only if $\lim_{n\to\infty} |d(x_n,x)| = 0$.*

Lemma 2.9([1]) *Let (X,d) be a complex valued metric space and let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is a Cauchy sequence if and only if $\lim_{n\to\infty} |d(x_n,x_{n+m})| = 0$.*

§3. Main Results

In this section we shall prove some common fixed point results under generalized contraction involving rational expression in the framework of complex valued metric spaces.

Theorem 3.1 *Let (X,d) be a complete complex valued metric space. Suppose that the mappings $S, T\colon X \to X$ satisfy:*

$$\begin{aligned} d(Sx,Ty) \ \precsim\ & \alpha\, d(x,y) + \beta \Big[\frac{d(x,Sx)d(x,Ty) + d(y,Ty)d(y,Sx)}{d(x,Ty)+d(y,Sx)}\Big] \\ & + \gamma\, d(x,Sx) + \delta\, d(y,Ty) \\ & + \lambda\, [d(x,Ty) + d(y,Sx)] \end{aligned} \tag{3.1}$$

for all $x,y \in X$, where α, β, γ, δ, λ are nonnegative reals with $\alpha+\beta+\gamma+\delta+2\lambda < 1$. Then S

and T have a unique common fixed point in X.

Proof Let x_0 be an arbitrary point in X and define

$$x_{2k+1} = Sx_{2k},\ x_{2k+2} = Tx_{2k+1},\ k = 0, 1, 2, \cdots .$$

Then from (3.1), we have

$$\begin{aligned}
d(x_{2k+1}, x_{2k+2}) &= d(Sx_{2k}, Tx_{2k+1}) \\
&\precsim \alpha\, d(x_{2k}, x_{2k+1}) \\
&\quad +\beta\Big[\frac{d(x_{2k}, Sx_{2k})d(x_{2k}, Tx_{2k+1}) + d(x_{2k+1}, Tx_{2k+1})d(x_{2k+1}, Sx_{2k})}{d(x_{2k}, Tx_{2k+1}) + d(x_{2k+1}, Sx_{2k})}\Big] \\
&\quad +\gamma\, d(x_{2k}, Sx_{2k}) + \delta\, d(x_{2k+1}, Tx_{2k+1}) \\
&\quad +\lambda\,[d(x_{2k}, Tx_{2k+1}) + d(x_{2k+1}, Sx_{2k})] \\
&= \alpha\, d(x_{2k}, x_{2k+1}) \\
&\quad +\beta\Big[\frac{d(x_{2k}, x_{2k+1})d(x_{2k}, x_{2k+2}) + d(x_{2k+1}, x_{2k+2})d(x_{2k+1}, x_{2k+1})}{d(x_{2k}, x_{2k+2}) + d(x_{2k+1}, x_{2k+1})}\Big] \\
&\quad +\gamma\, d(x_{2k}, x_{2k+1}) + \delta\, d(x_{2k+1}, x_{2k+2}) \\
&\quad +\lambda\,[d(x_{2k}, x_{2k+2}) + d(x_{2k+1}, x_{2k+1})] \\
&\precsim (\alpha + \beta + \gamma)d(x_{2k}, x_{2k+1}) + \delta\, d(x_{2k+1}, x_{2k+2}) \\
&\quad +\lambda[d(x_{2k}, x_{2k+1}) + d(x_{2k+1}, x_{2k+2})] \\
&= (\alpha + \beta + \gamma + \lambda)d(x_{2k}, x_{2k+1}) + (\delta + \lambda)d(x_{2k+1}, x_{2k+2}). \qquad (3.2)
\end{aligned}$$

This implies that

$$d(x_{2k+1}, x_{2k+2}) \precsim \Big(\frac{\alpha + \beta + \gamma + \lambda}{1 - \delta - \lambda}\Big)\, d(x_{2k}, x_{2k+1}). \qquad (3.3)$$

Similarly, we have

$$\begin{aligned}
d(x_{2k+2}, x_{2k+3}) &= d(Sx_{2k+1}, Tx_{2k+2}) \\
&\precsim \alpha\, d(x_{2k+1}, x_{2k+2}) \\
&\quad +\beta\Big[\frac{d(x_{2k+1}, Sx_{2k+1})d(x_{2k+1}, Tx_{2k+2}) + d(x_{2k+2}, Tx_{2k+2})d(x_{2k+2}, Sx_{2k+1})}{d(x_{2k+1}, Tx_{2k+2}) + d(x_{2k+2}, Sx_{2k+1})}\Big] \\
&\quad +\gamma\, d(x_{2k+1}, Sx_{2k+1}) + \delta\, d(x_{2k+2}, Tx_{2k+2}) \\
&\quad +\lambda\,[d(x_{2k+1}, Tx_{2k+2}) + d(x_{2k+2}, Sx_{2k+1})] \\
&= \alpha\, d(x_{2k+1}, x_{2k+2}) \\
&\quad +\beta\Big[\frac{d(x_{2k+1}, x_{2k+2})d(x_{2k+1}, x_{2k+3}) + d(x_{2k+2}, x_{2k+3})d(x_{2k+2}, x_{2k+2})}{d(x_{2k+1}, x_{2k+3}) + d(x_{2k+2}, x_{2k+2})}\Big] \\
&\quad +\gamma\, d(x_{2k+1}, x_{2k+2}) + \delta\, d(x_{2k+2}, x_{2k+3}) \\
&\quad +\lambda\,[d(x_{2k+1}, x_{2k+3}) + d(x_{2k+2}, x_{2k+2})] \\
&\precsim (\alpha + \beta + \gamma)d(x_{2k+1}, x_{2k+2}) + \delta\, d(x_{2k+2}, x_{2k+3}) \\
&\quad +\lambda[d(x_{2k+1}, x_{2k+2}) + d(x_{2k+2}, x_{2k+3})] \\
&= (\alpha + \beta + \gamma + \lambda)d(x_{2k+1}, x_{2k+2}) + (\delta + \lambda)d(x_{2k+2}, x_{2k+3}). \qquad (3.4)
\end{aligned}$$

This implies that

$$d(x_{2k+2}, x_{2k+3}) \precsim \Big(\frac{\alpha+\beta+\gamma+\lambda}{1-\delta-\lambda}\Big)\, d(x_{2k+1}, x_{2k+2}). \tag{3.5}$$

Putting

$$h = \Big(\frac{\alpha+\beta+\gamma+\lambda}{1-\delta-\lambda}\Big).$$

As $\alpha+\beta+\gamma+\delta+2\lambda < 1$, it follows that $0 < h < 1$, we have

$$d(x_{n+1}, x_{n+2}) \precsim h\, d(x_n, x_{n+1}) \precsim \cdots \precsim h^{n+1} d(x_0, x_1). \tag{3.6}$$

Let $m, n \geq 1$ and $m > n$, we have

$$\begin{aligned} d(x_n, x_m) &\precsim d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_{n+3}) \\ &\quad + \cdots + d(x_{n+m-1}, x_m) \\ &\precsim [h^n + h^{n+1} + h^{n+2} + \cdots + h^{n+m-1}] d(x_1, x_0) \\ &\precsim \Big[\frac{h^n}{1-h}\Big] d(x_1, x_0) \end{aligned}$$

and so

$$|d(x_n, x_m)| \leq \Big[\frac{h^n}{1-h}\Big] |d(x_1, x_0)| \to 0 \text{ as } m, n \to \infty.$$

This implies that $\{x_n\}$ is a Cauchy sequence. Since X is complete, there exists $w \in X$ such that $x_n \to w$ as $n \to \infty$. It follows that $w = Sw$, otherwise $d(w, Sw) = z > 0$ and we would then have

$$\begin{aligned} z &\precsim d(w, x_{2n+2}) + d(x_{2n+2}, Sw) \precsim d(w, x_{2n+2}) + d(Sw, Tx_{2n+1}) \\ &\precsim d(w, x_{2n+2}) + \alpha\, d(w, x_{2n+1}) \\ &\quad +\beta \Big[\frac{d(w, Sw) d(w, Tx_{2n+1}) + d(x_{2n+1}, Tx_{2n+1}) d(x_{2n+1}, Sw)}{d(w, Tx_{2n+1}) + d(x_{2n+1}, Sw)}\Big] \\ &\quad +\gamma\, d(w, Sw) + \delta\, d(x_{2n+1}, Tx_{2n+1}) + \lambda\, [d(w, Tx_{2n+1}) + d(x_{2n+1}, Sw)] \\ &= d(w, x_{2n+2}) + \alpha\, d(w, x_{2n+1}) \\ &\quad +\beta \Big[\frac{d(w, Sw) d(w, x_{2n+2}) + d(x_{2n+1}, x_{2n+2}) d(x_{2n+1}, Sw)}{d(w, x_{2n+2}) + d(x_{2n+1}, Sw)}\Big] \\ &\quad +\gamma\, d(w, Sw) + \delta\, d(x_{2n+1}, x_{2n+2}) + \lambda\, [d(w, x_{2n+2}) + d(x_{2n+1}, Sw)]. \end{aligned}$$

This implies that

$$\begin{aligned} |z| &\leq |d(w, x_{2n+2})| + \alpha\, |d(w, x_{2n+1})| \\ &\quad +\beta \Big[\frac{|z||d(w, x_{2n+2})| + |d(x_{2n+1}, x_{2n+2})||d(x_{2n+1}, Sw)|}{|d(w, x_{2n+2})| + |d(x_{2n+1}, Sw)|}\Big] \\ &\quad +\gamma\, |z| + \delta\, |d(x_{2n+1}, x_{2n+2})| + \lambda\, [|d(w, x_{2n+2})| + |d(x_{2n+1}, Sw)|]. \end{aligned}$$

Letting $n \to \infty$, it follows that

$$|z| \leq (\gamma+\lambda)|z| \leq (\alpha+\beta+\gamma+\delta+2\lambda)|z| < |z|$$

which is a contradiction and so $|z| = 0$, that is, $w = Sw$.

In an exactly the same way, we can prove that $w = Tw$. Hence $Sw = Tw = w$. This shows that w is a common fixed point of S and T.

We now show that S and T have a unique common fixed point. For this, assume that w^* is another common fixed point of S and T, that is, $Sw^* = Tw^* = w^*$ such that $w \neq w^*$. Then

$$\begin{aligned} d(w,w^*) &= d(Sw,Tw^*) \\ &\precsim \alpha\, d(w,w^*) + \beta\Big[\frac{d(w,Sw)d(w,Tw^*)+d(w^*,Tw^*)d(w^*,Sw)}{d(w,Tw^*)+d(w^*,Sw)}\Big] \\ &\quad +\gamma\, d(w,Sw) + \delta\, d(w^*,Tw^*) + \lambda\,[d(w,Tw^*)+d(w^*,Sw)] \\ &= \alpha\, d(w,w^*) + \beta\Big[\frac{d(w,w)d(w,w^*)+d(w^*,w^*)d(w^*,w)}{d(w,w^*)+d(w^*,w)}\Big] \\ &\quad +\gamma\, d(w,w) + \delta\, d(w^*,w^*) + \lambda\,[d(w,w^*)+d(w^*,w)] \\ &= (\alpha+2\lambda)(w,w^*) \end{aligned}$$

So that $|d(w,w^*)| \leq (\alpha+2\lambda)(w,w^*) < |d(w,w^*)|$, since $0 < (\alpha+2\lambda) < 1$, which is a contradiction and hence $d(w,w^*) = 0$. Thus $w = w^*$. This shows that S and T have a unique common fixed point in X. This completes the proof. □

Putting $S = T$ in Theorem 3.1, we have the following result.

Corollary 3.2 *Let (X,d) be a complete complex valued metric space. Suppose that the mapping $T\colon X \to X$ satisfies:*

$$\begin{aligned} d(Tx,Ty) &\precsim \alpha\, d(x,y) + \beta\Big[\frac{d(x,Tx)d(x,Ty)+d(y,Ty)d(y,Tx)}{d(x,Ty)+d(y,Tx)}\Big] \\ &\quad +\gamma\, d(x,Tx) + \delta\, d(y,Ty) + \lambda\,[d(x,Ty)+d(y,Tx)] \end{aligned} \tag{3.7}$$

for all $x,y \in X$, where α, β, γ, δ, λ are nonnegative reals with $\alpha+\beta+\gamma+\delta+2\lambda < 1$. Then T has a unique fixed point in X.

Corollary 3.3 *Let (X,d) be a complete complex valued metric space. Suppose that the mapping $T\colon X \to X$ satisfies (for fixed n):*

$$\begin{aligned} d(T^nx,T^ny) &\precsim \alpha\, d(x,y) + \beta\Big[\frac{d(x,T^nx)d(x,T^ny)+d(y,T^ny)d(y,T^nx)}{d(x,T^ny)+d(y,T^nx)}\Big] \\ &\quad +\gamma\, d(x,T^nx) + \delta\, d(y,T^ny) + \lambda\,[d(x,T^ny)+d(y,T^nx)] \end{aligned} \tag{3.8}$$

for all $x,y \in X$, where α, β, γ, δ, λ are nonnegative reals with $\alpha+\beta+\gamma+\delta+2\lambda < 1$. Then T has a unique fixed point in X.

Proof By Corollary 3.2, there exists $q \in X$ such that $T^n q = q$. Then

$$\begin{aligned} d(Tq, q) &= d(TT^n q, T^n q) = d(T^n Tq, T^n q) \\ &\precsim \alpha\, d(Tq, q) \\ &\quad + \beta \Big[\frac{d(Tq, T^n Tq) d(Tq, T^n q) + d(q, T^n q) d(q, T^n Tq)}{d(Tq, T^n q) + d(q, T^n Tq)} \Big] \\ &\quad + \gamma\, d(Tq, T^n Tq) + \delta\, d(q, T^n q) + \lambda\, [d(Tq, T^n q) + d(q, T^n Tq)] \\ &= \alpha\, d(Tq, q) \\ &\quad + \beta \Big[\frac{d(Tq, TT^n q) d(Tq, T^n q) + d(q, T^n q) d(q, TT^n q)}{d(Tq, T^n q) + d(q, TT^n q)} \Big] \\ &\quad + \gamma\, d(Tq, TT^n q) + \delta\, d(q, T^n q) + \lambda\, [d(Tq, T^n q) + d(q, TT^n q)] \\ &= \alpha\, d(Tq, q) \\ &\quad + \beta \Big[\frac{d(Tq, Tq) d(Tq, q) + d(q, q) d(q, Tq)}{d(Tq, q) + d(q, Tq)} \Big] \\ &\quad + \gamma\, d(Tq, Tq) + \delta\, d(q, q) + \lambda\, [d(Tq, q) + d(q, Tq)] \\ &= (\alpha + 2\lambda)\, d(Tq, q). \end{aligned}$$

So that $|d(Tq, q)| \leq (\alpha + 2\lambda)\, |d(Tq, q)| < |d(Tq, q)|$, since $0 < (\alpha + 2\lambda) < 1$, which is a contradiction and hence $d(Tq, q) = 0$. Thus $Tq = q$. This shows that T has a unique fixed point in X. This completes the proof. □

As an application of Theorem 3.1, we prove the following theorem for two finite families of mappings.

Theorem 3.4 *If $\{T_i\}_{i=1}^m$ and $\{S_i\}_{i=1}^n$ are two finite pairwise commuting finite families of self-mappings defined on a complete complex valued metric space (X, d) such that S and T (with $T = T_1 T_2 \cdots T_m$ and $S = S_1 S_2 \cdots S_n$) satisfy the condition* (3.1), *then the component maps of the two families $\{T_i\}_{i=1}^m$ and $\{S_i\}_{i=1}^n$ have a unique common fixed point.*

Proof In view of Theorem 3.1 one can conclude that T and S have a unique common fixed point g, that is, $T(g) = S(g) = g$. Now we are required to show that g is a common fixed point of all the components maps of both the families. In view of pairwise commutativity of the families $\{T_i\}_{i=1}^m$ and $\{S_i\}_{i=1}^n$, (for every $1 \leq k \leq m$) we can write

$$T_k(g) = T_k S(g) = S T_k(g) \quad \text{and} \quad T_k(g) = T_k T(g) = T T_k(g)$$

which show that $T_k(g)$ (for every k) is also a common fixed point of T and S. By using the uniqueness of common fixed point, we can write $T_k(g) = g$ (for every k) which shows that g is a common fixed point of the family $\{T_i\}_{i=1}^m$. Using the same arguments as above, one can also show that (for every $1 \leq k \leq n$) $S_k(g) = g$. This completes the proof. □

By taking $T_1 = T_2 = \cdots = T_m = G$ and $S_1 = S_2 = \cdots = S_n = F$, in Theorem 3.4, we derive the following result involving iterates of mappings.

Corollary 3.5 *If F and G are two commuting self-mappings defined on a complete complex valued metric space (X, d) satisfying the condition*

$$\begin{aligned} d(F^n x, G^m y) &\precsim \alpha\, d(x,y) + \beta \Big[\frac{d(x,F^n x)d(x,G^m y) + d(y,G^m y)d(y,F^n x)}{d(x,G^m y) + d(y,F^n x)}\Big] \\ &\quad + \gamma\, d(x,F^n x) + \delta\, d(y,G^m y) + \lambda\, [d(x,G^m y) + d(y,F^n x)] \qquad (3.9) \end{aligned}$$

for all $x, y \in X$, where α, β, γ, δ, λ are nonnegative reals with $\alpha+\beta+\gamma+\delta+2\lambda < 1$. Then F and G have a unique common fixed point in X.

By setting $m = n$ and $F = G = T$ in Corollary 3.5, we deduce the following result.

Corollary 3.6 *Let (X, d) be a complete complex valued metric space and let the mapping $T: X \to X$ satisfies (for fixed n)*

$$\begin{aligned} d(T^n x, T^n y) &\precsim \alpha\, d(x,y) + \beta \Big[\frac{d(x,T^n x)d(x,T^n y) + d(y,T^n y)d(y,T^n x)}{d(x,T^n y) + d(y,T^n x)}\Big] \\ &\quad + \gamma\, d(x,T^n x) + \delta\, d(y,T^n y) + \lambda\, [d(x,T^n y) + d(y,T^n x)] \qquad (3.10) \end{aligned}$$

for all $x, y \in X$, where α, β, γ, δ, λ are nonnegative reals with $\alpha+\beta+\gamma+\delta+2\lambda < 1$. Then T has a unique fixed point in X.

Proof By Corollary 3.2, we obtain $p \in X$ such that $T^n p = p$. The rest of the proof is same as that of Corollary 3.3. This completes the proof. □

By taking $\alpha = h$ and $\beta = \gamma = \delta = \lambda = 0$ in Corollary 3.3, we draw following corollary which can be viewed as an extension of Bryant (see, [4]) theorem to complex valued metric space.

Corollary 3.7 *Let (X, d) be a complete complex valued metric space. Suppose that the mapping $T: X \to X$ satisfying the condition*

$$d(T^n x, T^n y) \precsim h\, d(x,y)$$

for all $x, y \in X$ and $h \in [0, 1)$ is a constant. Then T has a unique fixed point in X.

The following example demonstrates the superiority of Bryant (see, [3]) theorem over Banach contraction theorem.

Example 3.8 Let $X = \mathbb{C}$, where $\mathbb{C}$ is the set of complex numbers. Define a mapping $d: \mathbb{C}\times\mathbb{C} \to \mathbb{C}$ by $d(z_1, z_2) = |x_1 - x_2| + i|y_1 - y_2|$ where $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$. Then $(\mathbb{C}, d)$ is a complex valued metric space. Define $T\colon \mathbb{C} \to \mathbb{C}$ as

$$T(x+iy) = \begin{cases} 0, & \text{if } x, y \in Q, \\ i, & \text{if } x, y \in Q^c, \\ 1, & \text{if } x \in Q^c, y \in Q, \\ 1+i, & \text{if } x \in Q, y \in Q^c. \end{cases}$$

Now for $x = \frac{1}{\sqrt{2}}$ and $y = 0$, we get

$$d(T(\frac{1}{\sqrt{2}}), T(0)) = d(1,0) \precsim \lambda d(\frac{1}{\sqrt{2}}, 0) = \frac{\lambda}{\sqrt{2}}.$$

Thus $\lambda \geq \sqrt{2}$ which is a contradiction that $0 \leq \lambda < 1$. However, we notice that $T^2(z) = 0$, so that

$$0 = d(T^2(z_1), T^2(z_2)) \precsim \lambda d(z_1, z_2),$$

which shows that T^2 satisfies the requirement of Bryant theorem and $z = 0$ is a unique fixed point of T.

Finally, we conclude this paper with an illustrative example which satisfied all the conditions of Corollary 3.2.

Example 3.9 Let $X = \{0, \frac{1}{2}, 2\}$ and partial order $'\precsim'$ is defined as $x \precsim y$ iff $x \geq y$. Let the complex valued metric d be given as

$$d(x,y) = |x-y|\sqrt{2}e^{i\frac{\pi}{4}} = |x-y|(1+i) \text{ for } x, y \in X.$$

Let $T\colon X \to X$ be defined as follows:

$$T(0) = 0, T(\frac{1}{2}) = 0, T(2) = \frac{1}{2}.$$

Case 1. Take $x = \frac{1}{2}$, $y = 0$, $T(0) = 0$ and $T(\frac{1}{2}) = 0$ in Corollary 3.2, then we have

$$d(Tx, Ty) = 0 \leq \Big(\frac{1+i}{2}\Big)(\alpha + \beta + \gamma + \lambda).$$

This implies that $\alpha = \beta = \gamma = 0$ and $\delta = \lambda = \frac{1}{2}$ or $\alpha = \beta = \gamma = \frac{1}{9}$ and $\delta = \lambda = \frac{1}{6}$ satisfied all the conditions of Corollary 3.2 and of course 0 is the unique fixed point of T.

Case 2. Take $x = 2$, $y = \frac{1}{2}$, $T(2) = \frac{1}{2}$ and $T(\frac{1}{2}) = 0$ in Corollary 3.2, then we have

$$\begin{aligned} d(Tx, Ty) &= \frac{1+i}{2} \leq \alpha.\Big(\frac{3(1+i)}{2}\Big) + \beta.\Big(\frac{3(1+i)}{2}\Big) + \gamma.\Big(\frac{3(1+i)}{2}\Big) \\ &\quad + \delta.\frac{1+i}{2} + \lambda.2(1+i). \end{aligned}$$

This implies that $\alpha = \beta = \gamma = \delta = \lambda = \frac{1}{13}$ satisfied all the conditions of Corollary 3.2 and of course 0 is the unique fixed point of T.

Case 3. Take $x = 2$, $y = 0$, $T(2) = \frac{1}{2}$ and $T(0) = 0$ in Corollary 3.2, then we have

$$\begin{aligned} d(Tx, Ty) &= \frac{1+i}{2} \leq \alpha.2(1+i) + \beta.\Big(\frac{3(1+i)}{2}\Big) + \gamma.\Big(\frac{3(1+i)}{2}\Big) \\ &\quad + \lambda.\frac{5(1+i)}{2}. \end{aligned}$$

This implies that $\alpha = \beta = \gamma = \lambda = \frac{1}{14}$ and $\delta = 0$ satisfied all the conditions of Corollary 3.2 and of course 0 is the unique fixed point of T.

§4. Conclusion

In this paper, we establish common fixed point theorems using generalized contraction involving rational expression in the setting of complex-valued metric spaces and give an example in support of our result. Our results extend and generalize several results from the current existing literature.

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