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A Study on Cayley Graphs over Dihedral Groups

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Abstract: Let G be the dihedral group D_n and $Cay(G,S)$ is the Cayley graph of G with respect to S , and let $C_G(x)$ is the centralizer of an element x in G and $\bar{x}$ is the orbit of x in G. In this paper, we prove that if G act on G by conjugation, the vertex induced subgraph with vertex set $C_G(x)$ of the Cayley graph $Cay(G,\bar{x})$ is either Hamiltonian or it contain Hamiltonian decompositions. But if n is prime, it is always Hamiltonian.

Key Words: Smarandache-Cayley graph, Cayley graph, dihedral group, Hamiltonian cycle, complete graph.

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§1. Introduction

Let $(G;\cdot)$ be a finite group. A *Smarandache-Cayley graph* of G respect to a pair $\{S,T\}$ of non-empty subsets $S \subset G$, $T \subset G \setminus S$ is the graph with vertex set G and edge set consisting of pairs (x,y) such that $s\cdot x = t\cdot y$, where $s \in S$ and $t \in T$. Particularly, let $T = \{1_G\}$. Then such a Smarandache-Cayley graph is the usual Cayley graph $Cay(G,S)$, whose vertex set is G and edges are the pairs (x,y) such that $s\cdot x = y$ for some $s \in S$ and $x \neq y$. Arthur Cayley (1878) introduced the Cayley graphs of groups and it has received much attention in the literature. Brian Alspach et al. (2010) proved that every connected Cayley graphs of valency at least three on a generalized dihedral group, whose order is divisible by four is Hamilton-connected, unless it is bipartite. Recently Adrian Pastine and Daniel Jaume (2012) proved that given a dihedral group D_H and a generating subset S, if $S \cap H \neq \phi$, then the Cayley digraph $Cay(D_H,S)$ is Hamiltonian. In this paper, we denote a group $(G;\cdot)$ by G for convenience.

§2. Main Results

In this section we deals with some basic definitions and terminologies of group theory and graph theory which are needed in sequel. For details see Fraleigh (2003), Gallian (2009) and Diestel (2010).

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Definition 2.1 *Let G be a group. The orbit of an element x under G is usually denoted as $\bar{x}$ and is defined as $\bar{x} = \{gx/g \in G\}$.*

Definition 2.2 *Let x be a fixed element in a group G . The centralizer of an element x in G , $C_G(x)$ is the set of all element in G that commute with x. In symbols, $C_G(x) = \{g \in G/gx = xg\}$.*

Definition 2.3 *A group G act on G by conjugation means $gx = gxg^{-1}$ for all $x \in G$.*

Definition 2.4 *An element x in a group G is called an involution if $x^2 = e$.*

Definition 2.5 *The n^{th} dihedral group D_n is the group of symmetries of the regular n-gon and $D_n \subset S_n$, where S_n is the symmetric group of n letters for $n \geq 3$ with $|D_n| = 2n$.*

The structure of D_n is $\{g, g^2, g^3, \cdots, g^n, y, yg, yg^2, yg^3, \cdots, yg^{n-1}\}$, where g denote rotation by $\frac{2\pi}{n}$ and y be any one of reflections (reflections along perpendicular bisector of sides or along diagonal flips). D_n can be represented as $G_1 \cup G_2$ where $G_1 =< g >$ and $G_2 = \{y, yg, yg^2, yg^3, \cdots, yg^{n-1}\}$. We say g and y are generators of D_n, and the equations $g^n = y^2 = e$, the identity and $yg = g^{n-1}y$ are relations for these generators. Generally all reflections are involutions and rotations may or may not. If n is odd, e is the only involution in G_1 and G_2 consist of reflections along perpendicular bisector of sides only. Except for e , generally G_1 and G_2 never commute and G_2 is non-abelian , but if n is even, $g^{\frac{n}{2}}$ is the only involution in G_1 which commute G_2.

Definition 2.6 *A subgraph (U, F) of a graph (V, E) is said to be vertex induced subgraph if F consist of all the edges of (V, E) joining pairs of vertices of U.*

Definition 2.7 *A Hamiltonian path is a path in (V, E) which goes through all the vertices in (V, E) exactly ones. A Hamiltonian cycle is a closed Hamiltonian path. A graph (V, E) is said to be Hamiltonian, if it contains a Hamiltonian cycle.*

Theorem 2.8 *Let G be the dihedral group D_p, p is prime and G act on G by conjugation. Then for every element $x \in G_1$ with $x \neq e$, the induced subgraph with vertex set $C_G(x)$ of the Cayley graph $Cay(G, \bar{x})$ is Hamiltonian.*

Proof Given $G = D_p$, so $G = \{g, g^2, g^3, \cdots, g^p, y, yg, yg^2, \cdots, yg^{p-1}\}$. Since $x \in G_1$, we have $C_G(x) = \{x, x^2, x^3, \cdots, x^p\}$. Let $u \in C_G(x)$. Then $ux = xu$ for $x \in G$. $\overline{x}$ is the orbit of $x \in G$ with $x^2 \neq e$ and G act on G by conjugation, we have $\overline{x} = \{x, x^{p-1}\}$, since $C_G(x)$ is abelian and $yx = x^{n-1}y$. We can choose an element $s \in \overline{x}$ such that $s = (ux)x(ux)^{-1}$. Now $su = (ux)x(ux)^{-1}u = (ux)x(x^{-1}u^{-1})u = (ux)(xx^{-1})(u^{-1}u) = ((ux)e)e = (ux)$, then there is an edge from u to ux. Again $s(ux) = (ux)x(ux)^{-1}ux = ((ux)x) = ux^2$, then there is an edge from ux to ux^2 and consequently a path from u to ux^2. Continuing in this way, we get a finite path $u \to ux \to ux^2 \to ux^3 \to \cdots \to ux^p = ue = u$ in the induced subgraph with vertex set $C_G(x)$ of the Cayley graph $Cay(G, \overline{x})$, which is Hamiltonian. In particular for $u = e$, we get a Hamiltonian cycle $e \to x \to x^2 \to x^3 \to \cdots \to x^p = e$. □

Definition 2.9 *A graph (V, E) is said to be complete if for eah pair of arbitrary vertices in*

(V,E) can be joined by an edge. A complete graph of n vertices is denoted as K_n.

Theorem 2.10 *Let G be the dihedral group D_{2n+1} and G act on G by conjugation. Then for every element $x \in G_2$, the induced subgraph with vertex set $C_G(x)$ of the Cayley graph $Cay(G,\overline{x})$ is K_2.*

Proof Given $G = D_{2n+1}$, so we have $G = \{g, g^2, g^3, \cdots, g^{2n+1}, y, yg, yg^2, \cdots, yg^{2n}\}$. Since $x \in G_2$, which is non-abelian, we have $C_G(x) = \{x, e\}$. Let $u \in C_G(x)$. Then $ux = xu$ for $x \in G$. Since $\overline{x}$ is the orbit of $x \in G_2$ and G act on G by conjugation, we have $x \in \overline{x}$. We can choose the element $s = x \in \overline{x}$ such that $s = (ux)x(ux)^{-1}$. Now $su = (ux)x(ux)^{-1}u = (ux)x(x^{-1}u^{-1})u = (ux)(xx^{-1})(u^{-1}u) = ((ux)e) = (ux)$, then there is an edge from u to ux. Again $s(ux) = (ux)x(ux)^{-1}ux = ((ux)x)e = ux^2$, so there exist an edge from ux to ux^2 and consequently a path from u to ux^2. Since $x^2 = e$, we get a Hamiltonian cycle $u \to ux \to ux^2 = ue = u$ in the induced subgraph with vertex set $C_G(x)$ of the Cayley graph $Cay(G,\overline{x})$, which is K_2. □

Corollary 2.11 *Let G be the dihedral group D_p, where p is prime and G act on G by conjugation. Then for every element $x \in G_2$, the induced subgraph with vertex set $C_G(x)$ of the Cayley graph $Cay(G,\overline{x})$ is Hamiltonian.*

Theorem 2.12 *Let G be the dihedral group D_p ,where p is prime and G act on G by conjugation. Then for $x \in G$ with $x \neq e$, the induced subgraph with vertex set $C_G(x)$ of the Cayley graph $Cay(G,\overline{x})$ is Hamiltonian.*

Proof Since $|G| = 2p$, we have an element $x \in G$ such that either $x^p = e$ or $x^2 = e$. So there exists a Hamiltonian cycle $u \to ux \to ux^2 \to ux^3 \cdots \to ux^p = u$ by Theorem 2.8 or a Hamiltonian cycle $u \to ux \to ux^2 = ue = u$ by Theorem 2.10 in the induced subgraph with vertex set $C_G(x)$ of the Cayley graph $Cay(G,\overline{x})$. □

Definition 2.13 *A graph (V,E) is called bipartite if $V = V_1 \cup V_2$ with $V_1 \cap V_2 = \phi$, and every edge of (V,E) is of the form $\{a,b\}$ with $a \in V_1$ and $b \in V_2$.*

Theorem 2.14 *Let G be the dihedral group D_n and G act on G by conjugation. Then for every element $x \in G_1$ with $x \neq e$ and $C_G(x) = G$, the induced subgraph with vertex set $C_G(x)$ of the Cayley graph $Cay(G,\overline{x})$ is a bipartite graph on n vertices.*

Proof Given $G = D_n$, so we have $G = \{g, g^2, g^3, \cdots, g^n, y, yg, yg^2, yg^3, \cdots, yg^{n-1}\}$. Since $x \in G_1$ with $C_G(x) = G$, we have either $x = e$ or $x = g^{\frac{n}{2}}$. But $x \neq e$. Let $u \in C_G(x)$. Then $ux = xu$ for all $u \in G$. $\bar{x}$ is the orbit of $x \in G_1$ and G act on G by conjugation ,we have $\bar{x} = \{x\}$. We can choose the element $s = x \in \bar{x}$ such that $s = (ux)x(ux)^{-1}$. Now $su = (ux)x(ux)^{-1}u = (ux)x(x^{-1}u^{-1})u = (ux)(xx^{-1})(u^{-1}u) = ((ux)e) = (ux)$, then there is an edge from u to ux. Again $s(ux) = (ux)x(ux)^{-1}ux = ((ux)x)e = ux^2$, then there is an edge from ux to ux^2 and consequently a path from u to ux^2. Since $x = g^{\frac{n}{2}}$, we have $x^2 = e$. Thus we get a complete graph $u \to ux \to u$ in the induced subgraph with vertex set $C_G(x)$ of the Cayley graph $Cay(G,\overline{x})$.

Let us consider the following cases.

Case 1. If $u = g^i$, $i = 1, 2, 3, \cdots, n$, we get $\frac{n}{2}$ distinct complete graph of two vertices with one end vertex in $\{g, g^2, g^3, \cdots, g^{\frac{n}{2}}\}$ and other in $\{g^{\frac{n}{2}+1}, g^{\frac{n}{2}+2}, \cdots, g^n\}$ as shown below.

$g \to g^{\frac{n}{2}+1} \to g$, $g^2 \to g^{\frac{n}{2}+2} \to g^2$ $,\cdots,$ $g^{\frac{n}{2}} \to g^n \to g^{\frac{n}{2}}$, $g^{\frac{n}{2}+1} \to g \to g^{\frac{n}{2}+1}$ $,\cdots,$ $g^n \to g^{\frac{n}{2}} \to g^n$.

Case 2. If $u = yg^i$,$i = 1, 2, 3, \cdots, n$, we get another $\frac{n}{2}$ distinct complete graph of two vertices with one end vertex in $\{yg, yg^2, yg^3, \cdots, yg^{\frac{n}{2}}\}$ and other in $\{yg^{\frac{n}{2}+1}, yg^{\frac{n}{2}+2}, \cdots, yg^n\}$ as shown below.

$yg \to yg^{\frac{n}{2}+1} \to yg$, $yg^2 \to yg^{\frac{n}{2}+2} \to yg^2$, $\cdots$, $yg^{\frac{n}{2}} \to y \to yg^{\frac{n}{2}}$, $yg^{\frac{n}{2}+1} \to yg \to yg^{\frac{n}{2}+1}$,$\cdots$, $yg^n \to yg^{\frac{n}{2}} \to yg^n$.

Thus the induced subgraph with vertex set $C_G(x)$ of the Cayley graph $Cay(G, \overline{x})$ is a bipartite graph on n vertices. □

Remark 2.15 By Theorem 2.14 ,the graphs in case.2 have been completely characterized. If $n = pq$ with p and q are distinct primes, the induced subgraph with vertex set $C_G(x)$ of the Caylay graph $Cay(G, \overline{x})$ has $\frac{n}{2}$ distinct complete graphs on two vertices with one end vertex in $\{y, yg^2, \cdots, yg^{n-2}\}$ and other in $\{yg, yg^3, \cdots, yg^{n-1}\}$.

If $n \neq pq$, we get $\frac{n}{2}$ distinct complete graph on two vertices. Out of which $\frac{n}{4}$ graphs have one end vertex in $\{y, yg^2, yg^4, \cdots, yg^{\frac{n}{2}-2}\}$ and other in $\{yg^{\frac{n}{2}}, yg^{\frac{n}{2}+2}, \cdots, yg^{n-2}\}$ and the remaining $\frac{n}{4}$ graphs have one end vertex in $\{yg, yg^3, yg^5, \cdots, yg^{\frac{n}{2}-1}\}$ and others in $\{yg^{\frac{n}{2}+1}, yg^{\frac{n}{2}+3}, \cdots, yg^{n-1}\}$.

Corollary 2.16 *Let G be the dihedral group D_n, where n is even and G act on G by conjugation. Then for the element $x = g^{\frac{n}{2}} \in G$, the induced subgraph with vertex set $C_G(x)$ of the Cayley graph $Cay(G, \overline{x})$ is a bipartite graph on n vertices.*

Theorem 2.17 *Let G be the dihedral group D_{4n} and G act on G by conjugation. Then for every involuted element $x \in G$ with $C_G(x) \neq G$, the induced subgraph with vertex set $C_G(x)$ of the Cayley graph $Cay(G, \overline{x})$ is Hamiltonian.*

Proof Let $G = D_{4n}$. So we have $G = \{g, g^2, \cdots, g^{4n}, y, yg, yg^2, \cdots, yg^{4n-1}\}$. Since $x \in G$ with $x^2 = e$ and $C_G(x) \neq G$, we have $x \neq e$ and $x \neq g^{\frac{n}{2}}$. Thus $x \in G_2$ and $C_G(x) = \{x, e, xg^{2n}, g^{2n}\}$. We decompose G_1 as $G_1^{'} \cup G_1^{''}$ where $G_1^{'} = \{g^2, g^4, \cdots, g^{4n}\}$ and $G_1^{''} = \{g, g^3, \cdots, g^{4n-1}\}$. Similarly G_2 an be decomposed as $G_2^{'} \cup G_2^{''}$, where $G_2^{'} = \{y, yg^2, yg^4, \cdots, yg^{4n-2}\}$ and $G_2^{''} = \{yg, yg^3, \cdots, yg^{4n-1}\}$. Since $x \in G_2$, we have either $x \in G_2^{'}$ or $x \in G_2^{''}$. If $x \in G_2^{'}$, it implies that $xg^{2n} \in G_2^{'}$. From the composition table and also from the relation $yg = g^{4n-1}y$, we get $G_1^{'}G_2^{'}(G_1^{'})^{-1} = G_1^{''}G_2^{'}(G_1^{''})^{-1} = G_2^{'}G_2^{'}(G_2^{'})^{-1} = G_2^{''}G_2^{'}(G_2^{''})^{-1} = G_2^{'}$. Thus $\bar{x} = G_2^{'}$. Similarly if $x \in G_2^{''}$,implies that $xg^{2n} \in G_2^{''}$. From the composition table it follows that $G_1^{'}G_2^{''}(G_1^{'})^{-1} = G_1^{''}G_2^{''}(G_1^{''})^{-1} = G_2^{'}G_2^{''}(G_2^{'})^{-1} = G_2^{''}G_2^{''}(G_2^{''})^{-1} = G_2^{''}$ and hence $\bar{x} = G_2^{''}$.

Let $u \in C_G(x)$. Then $ux = xu$ for $x \in G$. We can choose two involutions s_1 and s_2 in $\bar{x}$ such that $s_1 = (ux)x(ux)^{-1}$ and $s_2 = (uxg^{2n})xg^{2n}(uxg^{2n})^{-1}$. Now $s_1u = (ux)x(ux)^{-1}u = (ux)x(x^{-1}u^{-1})u = (ux)(xx^{-1})(u^{-1}u) = ((ux)e) = (ux)$, then there is an edge from u to ux.

Again $s_2(ux) = (uxg^{2n})xg^{2n}(uxg^{2n})^{-1}(ux) = (uxg^{2n})xg^{2n}(g^{2n})^{-1}x^{-1}u^{-1}ux = (uxg^{2n})x = (ux)g^{2n}x = u(xg^{2n})x = u(g^{2n}x)x = (ug^{2n})x^2 = ug^{2n}$, then there is an edge from ux to ug^{2n} and consequently a path from u to ug^{2n}. Again $s_1(ug^{2n}) = (ux)x(ux)^{-1}(ug^{2n}) = (ux)xx^{-1}u^{-1}ug^{2n} = uxg^{2n}$, then there is an edge from ug^{2n} to uxg^{2n} and consequently a path from u to uxg^{2n}. Again $s_2(uxg^{2n}) = (uxg^{2n})xg^{2n}(uxg^{2n})^{-1}(uxg^{2n}) = (uxg^{2n})xg^{2n} = (uxg^{2n})g^{2n}x = uxg^{4n}x = ux^2 = ue = u$. Thus we get a Hamiltonian cycle $u \to ux \to ug^{2n} \to uxg^{2n} \to u$ in the induced subgraph with vertex set $C_G(x)$ of the Cayley graph $Cay(G, \overline{x})$. In particular for $u = e$, we get a Hamiltonian cycle $e \to x \to g^{2n} \to xg^{2n} \to e$. □

Corollary 2.18 *Let G be the dihedral group D_{4n} and G act on G by conjugation. Then for every $x \in G_2$, the induced subgraph with vertex set $C_G(x)$ of the Cayley graph $Cay(G, \overline{x})$ is Hamiltonian.*

Theorem 2.19 *Let G be the dihedral group D_{4n} and G act on G by conjugation. Then for every involuted element $x \in G$ with $C_G(x) \neq G$, the induced subgraph with vertex set $C_G(x)$ of the Cayley graph $Cay(G, \overline{x} \cup g^{2n})$ is K_4.*

Proof Since $x \in D_{4n}$ with $x^2 = e$ and $C_G(x) \neq G$ by Theorem 2.17, we get a Hamiltonian cycle $u \to ux \to ug^{2n} \to uxg^{2n} \to u$ in the induced subgraph with vertex set $C_G(x)$ of the Cayley graph $Cay(G, \overline{x})$. To prove that this graph is K_4, it is enough to show that there exist edges from $u \to ug^{2n}$ and $ux \to uxg^{2n}$. We can choose $s = g^{2n}$ as $ug^{2n}u^{-1}$. Now $su = (ug^{2n}u^{-1})u = ug^{2n}$, then there is an edge from u to ug^{2n}. Similarly we get an edge from ux to uxg^{2n}, since $s(ux) = (ug^{2n}u^{-1})ux = ug^{2n}x = uxg^{2n}$. □

Corollary 2.20 *Let G be the dihedral group D_{4n} and G act on G by conjugation. Then for every $x \in G_2$, the induced subgraph with vertex set $C_G(x)$ of the Cayley graph $Cay(G, \overline{x} \cup g^{2n})$ is K_4.*

Theorem 2.21 *Let G be the dihedral group D_{4n+2} and G act on G by conjugation. Then for every involuted element $x \in G$ with $C_G(x) \neq G$, the induced subgraph with vertex set $C_G(x)$ of the Cayley graph $Cay(G, \overline{x})$ is a bipartite graph on four vertices.*

Proof Given $G = D_{4n+2}$, so we have $G = \{g, g^2, g^3, \cdots, g^{4n+2}, y, yg, yg^2, \cdots, yg^{4n+1}\}$. Since $x \in G$ with $x^2 = e$ and $C_G(x) \neq G$, clearly $x \in G_2$ and hence $C_G(x) = \{x, e, g^{2n+1}, xg^{2n+1}\}$. Since $x \in G_2$, either $x \in G_2'$ or $x \in G_2''$, where $G_2' = \{y, yg^2, \cdots, yg^{4n}\}$ and $G_2'' = \{yg, yg^3, \cdots, yg^{4n+1}\}$. If $x \in G_2'$, then $xg^{2n+1} \in G_2''$. Since $\bar{x}$ is the orbit of an element x in G_2' and G act on G by conjugation, we get $\bar{x} = G_2'$. Similarly if $x \in G_2''$, we have $xg^{2n+1} \in G_2'$ and $\bar{x} = G_2''$. Thus there exist exactly one involution in $\bar{x} \cap C_G(x)$. We can choose that $s \in \bar{x}$ such that $s = (ux)x(ux)^{-1}$.

Let $u \in C_G(x)$. Then $ux = xu$ for $x \in G$. Now $su = (ux)x(ux)^{-1}u = (ux)x(x^{-1}u^{-1})u = (ux)(xx^{-1})(u^{-1}u) = ((ux)e) = (ux)$, then there is an edge from u to ux. Again $s(ux) = (ux)x(ux)^{-1}ux = ((ux)x)e = ux^2 = ue = u$. Thus we get a Hamiltonian cycle $u \to ux \to u$ in the induced subgraph with vertex set $C_G(x)$ of the Cayley graph $Cay(G, \overline{x})$. Since $|C_G(x)| = 4$, there exist an element other than u and ux in $C_G(x)$. Since ug^{2n+1} commute with all reflections, we have $ug^{2n+1} \in C_G(x)$. Again $s(ug^{2n+1}) = (ux)x(ux)^{-1}(ug^{2n+1}) = (ux)(xx^{-1})(u^{-1}u)g^{2n+1} = uxg^{2n+1}$ and on the other hand $s(uxg^{2n+1}) = (ux)x(ux)^{-1}(uxg^{2n+1}) = (ux)(xx^{-1})(u^{-1}u)xg^{2n+1} = ux^2g^{2n+1} = ug^{2n+1}$. Thus we get another cycle $ug^{2n+1} \to uxg^{2n+1} \to ug^{2n+1}$ in the induced subgraph with vertex set $C_G(x)$ of the Cayley graph $Cay(G, \overline{x})$. Thus the induced subgraph with vertex set $C_G(x)$ of the Cayley graph $Cay(G, \overline{x})$ is a bipartite graph on four vertices. □

Corollary 2.22 *Let G be the dihedral group D_{4n+2} and G act on G by conjugation. Then for every $x \in G_2$, the induced subgraph with vertex set $C_G(x)$ of the Cayley graph $Cay(G, \overline{x})$ is a bipartite graph on four vertices.*

Theorem 2.23 *Let G be the dihedral group D_{4n+2} and G act on G by conjugation. Then for every involuted element $x \in G$ with $C_G(x) \neq G$, the induced subgraph with vertex set $C_G(x)$ of the Cayley graph $Cay(G, \overline{x} \cup g^{2n+1})$ is Hamiltonian.*

Proof Since $G = D_{4n+2}$ and G act on G by conjugation, by Theorem 2.21, for every $x \in G$ with $C_G(x) \neq G$, the induced subgraph with vertex set $C_G(x)$ of the Cayley graph $Cay(G, \overline{x})$ is a bipartite graph on 4 vertices with one cycle $u \to ux \to u$ and another cycle $ug^{2n+1} \to uxg^{2n+1} \to ug^{2n+1}$. If we add an element g^{2n+1} in $\bar{x}$, then we get an edge from u to ug^{2n+1} and ux to uxg^{2n+1} , since $g^{2n+1}u = ug^{2n+1}$ and $g^{2n+1}(ux) = (ux)g^{2n+1}$. Thus we get a Hamiltonian cycle $ug^{2n+1} \to u \to ux \to uxg^{2n+1} \to ug^{2n+1}$ in the induced subgraph with vertex set $C_G(x)$ of the Cayley graph $Cay(G, \overline{x} \cup g^{2n+1})$. □

Corollary 2.24 *Let G be the dihedral group D_{4n+2} and G act on G by conjugation. Then for every $x \in G_2$, the induced subgraph with vertex set $C_G(x)$ of the Cayley graph $Cay(G, \overline{x} \cup g^{2n+1})$ is Hamiltonian.*

Theorem 2.25 *Let G be the dihedral group D_n, n is even and G act on G by conjugation. Then for every $x \in G_2$, the induced subgraph with vertex set $C_G(x)$ of the Cayley graph $Cay(G, \overline{x} \cup g^{\frac{n}{2}})$ is Hamiltonian.*

Proof Suppose $G = D_{4n}$ and G act on G by conjugation. Then by Corollary 2.20, for every $x \in G_2$, the induced subgraph with vertex set $C_G(x)$ of the Cayley graph $Cay(G, \overline{x} \cup g^{2n})$ is K_4. Also we have if $G = D_{4n+2}$ and G act on G by conjugation, by Corollary 2.24, for every $x \in G_2$, the induced subgraph with vertex set $C_G(x)$ of the Cayley graph $Cay(G, \overline{x} \cup g^{2n+1})$ is Hamiltonian. Thus if $G = D_n$, n is even ,we get for every $x \in G_2$, the induced subgraph with vertex set $C_G(x)$ of the Cayley graph $Cay(G, \overline{x} \cup g^{\frac{n}{2}})$ is Hamiltonian. □

Theorem 2.26 *Let G be the dihedral group D_n and G act on G by conjugation. Then for $x \in G$ with $x = g^m$, the induced subgraph with vertex set $C_G(x)$ of the Cayley graph $Cay(G, \bar{x})$ is Hamiltonian if $gcd(m, n) = 1$.*

Proof Given $G = D_n$, so we have $G = \{g, g^2, g^3, \cdots, g^n, y, yg, yg^2, \cdots, yg^{n-1}\}$. Since $x \in G$ with $x = g^m$ and $gcd(m, n) = 1$, we get $C_G(x) = \{x, x^2, x^3, \cdots, x^n\}$ and $\bar{x} = \{x, x^{n-1}\}$. As in the proof Theorem 2.8, we get a Hamiltonian cycle $u \to ux \to ux^2 \to ux^3 \to \cdots \to ux^n = ue = u$ in the induced subgraph with vertex set $C_G(x)$ of the Cayley graph $Cay(G, \overline{x})$. □

Theorem 2.27 *Let G be the dihedral group D_n and G act on G by conjugation. Then for every element $x \in G$ with $x = g^m$ with $C_G(x) \neq G$, induced subgraph with vertex set $C_G(x)$ of the Cayley graph $Cay(G, \bar{x})$ has d Hamiltonian decompositions on $\frac{n}{d}$ vertices if $gcd(m, n) = d$.*

Proof Given $G = D_n$, so we have $G = \{g, g^2, g^3, \cdots, g^n, y, yg, yg^2, \cdots, yg^{n-1}\}$. Since $x \in G$ with $x = g^m$ and $C_G(x) \neq G$, we have $m \neq \frac{n}{2}$ and n. Thus $x \in G_1$ other than $g^{\frac{n}{2}}$ and g^n and hence $C_G(x) = \{x, x^2, x^3, \cdots, x^{m-1}, x^m, x^{m+1}, \cdots, x^n\}$. Let $u \in C_G(x)$. Then $ux = xu$ for $x \in G$. $\bar{x}$ is the orbit of $x \in G$ and G act on G by conjugation, we have $\bar{x} = \{x, x^{n-1}\}$. Choose an element $s = x \in \bar{x}$ such that $s = (ux)x(ux)^{-1}$. Now $su = (ux)x(ux)^{-1}u = (ux)$, then there is an edge from u to ux. Again $s(ux) = (ux)x(ux)^{-1}ux = ux^2$, then there is an edge from ux to ux^2 and consequently a path from u to ux^2. Continuing in this way, we get a cycle $u \to ux \to ux^2 \to \cdots \to ux^{\frac{n}{d}} = u$ in the induced subgraph with vertex set $C_G(x)$ of the Cayley graph $Cay(G, \bar{x})$. In particular, for $u = g^i$,$i = 1, 2, \cdots, n$, we get d Hamiltonian decompositions on $\frac{n}{d}$ vertices as $g \to g^{1+m} \to g^{1+2m} \to \cdots \to g^{1+\frac{mn}{d}} = g$, $g^2 \to g^{2+m} \to g^{2+2m} \to \cdots \to g^{2+\frac{mn}{d}} = g^2, \cdots, g^d \to g^{d+m} \to g^{d+2m} \to \cdots \to g^{d+\frac{mn}{d}} = g^d$, $g^{d+1} \to g^{d+1+m} \to g^{d+1+2m} \to \cdots \to g^{d+1}, \ldots, g^n \to g^m \to \cdots \to g^n$ of which the decompositions when $u = g^i$ and $u = g^{i+d}$ are same. □

Theorem 2.28 *Let G be the dihedral group D_n and G act on G by conjugation. Then for every element $x \in G_1$ with $x = g^m$ and $x \neq e$, induced subgraph with vertex set $C_G(x)$ of the Cayley graph $Cay(G, \bar{x})$ has d Hamiltonian decompositions on $\frac{n}{d}$ vertices if $gcd(m, n) = d$.*

Proof Given $G = D_n$ and G act on G by conjugation.Then by Theorem 2.27, for every element $x \in G$ with $x = g^m$ with $C_G(x) \neq G$, induced subgraph with vertex set $C_G(x)$ of the Cayley graph $Cay(G, \bar{x})$ has d Hamiltonian decompositions on $\frac{n}{d}$ vertices if $gcd(m, n) = d$. Also we have ,by Theorem 2.14, for every $x \in G_1$ with $x \neq e$ and $C_G(x) = G$, the induced subgraph with vertex set $C_G(x)$ of the Cayley graph $Cay(G, \overline{x})$ is a bipartite graph on n vertices. Thus if $G = D_n$ and G act on G by conjugation, for every element $x \in G_1$ with $x = g^m$ and $x \neq e$, induced subgraph with vertex set $C_G(x)$ of the Cayley graph $Cay(G, \bar{x})$ has d Hamiltonian decompositions on $\frac{n}{d}$ vertices if $gcd(m, n) = d$. □

References

[1] Adrian Pastine and Danial Jaume, On Hamiltonian circuits in Cayley graph over generalized dihedral groups, *Union Mathematica Argentina*, 53(2012),79-87.

[2] Brian Alspach, C.C.Chen and Matthew Deans, Hamiltonian paths in Cayley graph on generalized dihedral groups, *ARS Mathematica Contemporanea*, 3(2010), 29-47.

[3] Diestel.R, *Graph Theory*, Graduate Texts in Mathematics, Newyork, 1997.

[4] John B.Fraleigh, *A First Course in Abstrat Algebra*, Seventh Edition, Pearson Education,Inc, 2003.

[5] Joseph A.Gallian, *Contemporary Abstract Algebra*, Narosa Publications, Seventh Edition, 2004.

[6] Riyas A and K.Geetha, A study on Hamiltonian property of Cayley graphs over non-abelian

groups, *International Journal of Mathematical Combinatorics*, Vol.3, (2016), 142-146.

[7] Riyas A and K.Geetha, A study on Cayley graphs of non-abelian groups, *International Journal of Mathematical Combinatorics*, Vol.4, (2016), 79-87.