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Some Curvature Properties of LP-Sasakian Manifold with Respect to Quarter-Symmetric Metric Connection

Santu Dey and Arindam Bhattacharyya

(Department of Mathematics, Jadavpur University, Kolkata-700032, India)

E-mail: santu.mathju@gmail.com, bhattachar1968@yahoo.co.in

Abstract: The objective of the present paper is to study the curvature tensor of the quarter-symmetric metric connection with respect to Lorentzian Para-Sasakian manifold (briefly, LP-Sasakian manifold). It is shown that if in the manifold M^n, $\tilde{W}_2 = 0$, then the manifold M^n is locally isomorphic to $S^n(1)$, where $\tilde{W}_2$ is the W_2-curvature tensor of the quarter-symmetric metric connection in a LP-Sasakian manifold. Next we study generalized projective ϕ-Recurrent LP-Sasakian manifold with respect to quarter-symmetric metric connection. After that ϕ-pseudo symmetric LP-Sasakian manifold with respect to quarter-symmetric metric connection is studied and we also discuss LP-Sasakian manifold with respect to quarter-symmetric metric connection when it satisfies the condition $\tilde{P}.\tilde{S} = 0$, where $\tilde{P}$ denotes the projective curvature tensor with respect to quarter-symmetric metric connection. Further, we also study ξ-conharmonically flat LP-Sasakian manifold with respect to quarter-symmetric metric connection. Finally, we give an example of LP-Sasakian manifold with respect to quarter-symmetric metric connection.

Key Words: Quarter-symmetric metric connection, W_2-curvature tensor, generalized projective ϕ-recurrent manifold, ϕ-pseudo symmetric LP-Sasakian manifold, projective curvature tensor, ξ-conharmonically flat LP-Sasakian manifold.

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§1. Introduction

The idea of semi-symmetric linear connection on a differentiable manifold was introduced by Friedmann and Schouten ([1]). Further, Hayden ([3]), introduced the idea of metric connection with torsion on a Riemannian manifold. In ([16]), Yano studied some curvature conditions for semi-symmetric connections in Riemannian manifolds.

The quarter-symmetric connection generalizes the semi-symmetric connection. The semi-symmetric metric connection is important in the geometry of Riemannian manifolds having also physical application; for instance, the displacement on the earth surface following a fixed

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point is metric and semi-symmetric.

In 1975, Golab ([2]) defined and studied quarter-symmetric connection in a differentiable manifold.

A linear connection $\tilde{\nabla}$ on an n-dimensional Riemannian manifold (M^n, g) is said to be a *quarter-symmetric connection* [2] if its torsion tensor $\tilde{T}$ defined by

$$\tilde{T}(X,Y) = \tilde{\nabla}_X Y - \tilde{\nabla}_Y X - [X,Y], \tag{1.1}$$

is of the form

$$\tilde{T}(X,Y) = \eta(Y)\phi X - \eta(X)\phi Y, \tag{1.2}$$

where η is a non-zero 1-form and ϕ is a tensor field of type $(1,1)$. In addition, if a quarter-symmetric linear connection $\tilde{\nabla}$ satisfies the condition

$$(\tilde{\nabla}_X g)(Y,Z) = 0 \tag{1.3}$$

for all $X, Y, Z \in \chi(M)$, where $\chi(M)$ is the set of all differentiable vector fields on M, then $\tilde{\nabla}$ is said to be a quarter-symmetric metric connection. In particular, if $\phi X = X$ and $\phi Y = Y$ for all $X, Y \in \chi(M)$, then the quarter-symmetric connection reduces to a semi-symmetric connection [1].

On the other hand Matsumoto ([5]) introduced the notion of LP-Sasakian manifold. Then Mihai and Rosoca([9]) introduced the same notion independently and obtained several results on this manifold. LP-Sasakian manifolds are also studied by Mihai([9]), Singh([15]) and others.

Definition 1.1 *A LP-Sasakian manifold is said to be generalized projective ϕ-recurrent if its curvature tensor R satisfies the condition*

$$\phi^2((\nabla_W P)(X,Y)Z) = A(W)P(X,Y)Z + B(W)[g(Y,Z)X - g(Y,Z)X], \tag{1.4}$$

where A and B are 1-forms, β is non-zero and these are defined by

$$A(W) = g(W,\rho_1), B(W) = g(W,\rho_2),$$

and where ρ_1 and ρ_2 are vector fields associated with 1-forms A and B respectively and P is the projective curvature tensor for an n-dimensional Riemannian manifold M, given by

$$P(X,Y)Z = R(X,Y)Z - \frac{1}{n-1}[S(Y,Z)X - S(X,Z)Y], \tag{1.5}$$

where R and S are the curvature tensor and Ricci tensor of the manifold.

Definition 1.2 *A LP-Sasakian manifold $(M^n, \phi, \xi, \eta, g)(n > 2)$ is said to be ϕ-pseudosymmetric*

([4]) if the curvature tensor R satisfies

$$\begin{aligned}\phi^2((\nabla_W R)(X,Y)Z) &= 2A(W)R(X,Y)Z + A(X)R(W,Y)Z \\ &+ A(Y)R(X,W)Z + A(Z)R(X,Y)W \\ &+ g(R(X,Y)Z,W)\rho \qquad (1.6)\end{aligned}$$

for any vector field X, Y, Z and W, where ρ is the vector field associated to the 1-form A such that $A(X) = g(X,\rho)$. In particular, if $A = 0$ then the manifold is said to be ϕ-symmetric.

After Golab([2]), Rastogi ([13], [14]) continued the systematic study of quarter-symmetric metric connection. In 1980, Mishra and Pandey ([8]) studied quarter-symmetric metric connection in a Riemannian, Kaehlerian and Sasakian manifold. In 1982, Yano and Imai([17]) studied quarter-symmetric metric connection in Hermition and Kaehlerian manifolds. In 1991, Mukhopadhyay et al.([10]) studied quarter-symmetric metric connection on a Riemannian manifold with an almost complex structure ϕ. However these manifolds have been studied by many geometers like K. Matsumoto ([6]), K. Matsumoto and I. Mihai ([8]), I. Mihai and R. Rosca([5]) and they obtained many results on this manifold.

In 1970, Pokhariyal and Mishra ([11]) have introduced new tensor fields, called W_2 and E-tensor fields in a Riemannian manifold and studied their properties. Again, Pokhariyal ([12]) have studied some properties of these tensor fields in a Sasakian manifolds. Recently, Matsumoto, Ianus and Mihai ([6]) have studied P-Sasakian manifolds admitting W_2 and E-tensor fields. The W_2-curvature tensor is defined by

$$W_2(X,Y)Z = R(X,Y)Z + \frac{1}{n-1}\{g(X,Z)QY - g(Y,Z)QX\}, \qquad (1.7)$$

where R and Q are the curvature tensor and Ricci operator and for all $X, Y, Z \in \chi(M)$.

The conharmonic curvature tensor of LP-Sasakian Manifold M^n is given by

$$\begin{aligned}C(X,Y)Z &= R(X,Y)Z - \frac{1}{n-2}[g(Y,Z)QX - g(X,Z)QY \\ &+ S(Y,Z)X - S(X,Z)Y], \qquad (1.8)\end{aligned}$$

where R and S are the curvature tensor and Ricci tensor of the manifold.

Motivated by the above studies, in the present paper, we consider the W_2-curvature tensor of a quarter-symmetric metric connection and study some curvature conditions. Section 2 is devoted to preliminaries. In third section, we find expression for the curvature tensor, Ricci tensor and scalar curvature of LP-Sasakian manifold with respect to quarter-symmetric metric connection and investigate relations between curvature tensor (resp. Ricci tensor) with respect to the semi-symmetric metric connection and curvature tensor (resp. Ricci tensor) with respect to Levi-Civita connection. In section four, W_2 curvature tensor with respect to quarter-symmetric metric connection is studied. In this section, it is seen that if $\tilde{W}_2 = 0$ in M^n, then M^n is locally isomorphic to $S^n(1)$, where $\tilde{W}_2$ is curvature tensor with respect to quarter-symmetric metric connection $\tilde{\nabla}$. Next we have obtained some expression of Ricci tensor when $(\tilde{W}_2(\xi,Z).\tilde{S})(X,Y) = 0$ in LP-Sasakian manifold with respect to quarter-symmetric

metric connection. In section five deals with generalized projective ϕ-Recurrent LP-Sasakian manifold with respect to quarter-symmetric metric connection. In section six, ϕ-pseudo symmetric LP-Sasakian manifold with respect to quarter-symmetric metric connection is studied. In next section, we cultivate LP-Sasakian manifold with respect to quarter-symmetric metric connection satisfying when it satisfies the condition $\tilde{P}.\tilde{S} = 0$, where $\tilde{P}$ denotes the projective curvature tensor with respect to quarter-symmetric metric connection. Finally, We study ξ-conharmonically flat LP-Sasakian manifold with respect to quarter-symmetric metric connection.

§2. **Preliminaries**

A n-dimensional, $(n = 2m + 1)$, differentiable manifold M^n is called Lorentzian para-Sasakian (briefly, LP-Sasakian) manifold ([5], [7]) if it admits a $(1,1)$-tensor field ϕ, a contravariant vector field ξ, a 1-form η and a Lorentzian metric g which satisfy

$$\eta(\xi) = -1, \tag{2.1}$$

$$\phi^2 X = X + \eta(X)\xi, \tag{2.2}$$

$$g(\phi X, \phi Y) = g(X,Y) + \eta(X)\eta(Y), \tag{2.3}$$

$$g(X,\xi) = \eta(X), \tag{2.4}$$

$$\nabla_X \xi = \phi X, \tag{2.5}$$

$$(\nabla_X \phi)(Y) = g(X,Y)\xi + \eta(Y)X + 2\eta(X)\eta(Y)\xi, \tag{2.6}$$

where, ∇ denotes the covariant differentiation with respect to Lorentzian metric g. It can be easily seen that in an LP-Sasakian manifold the following relations hold:

$$\phi\xi = 0, \quad \eta(\phi X) = 0, \tag{2.7}$$

$$rank(\phi) = n - 1. \tag{2.8}$$

If we put

$$\Phi(X,Y) = g(X,\phi Y), \tag{2.9}$$

for any vector field X and Y, then the tensor field $\Phi(X,Y)$ is a symmetric $(0,2)$-tensor field ([5]). Also since the 1-form η is closed in an LP-Sasakian manifold, we have ([5])

$$(\nabla_X \eta)(Y) = \Phi(X,Y), \Phi(X,\xi) = 0 \tag{2.10}$$

for all $X, Y \in \chi(M)$.

Also in an LP-Sasakian manifold, the following relations hold ([7]):

$$g(R(X,Y)Z,\xi) = \eta(R(X,Y)Z) = g(Y,Z)\eta(X) - g(X,Z)\eta(Y), \tag{2.11}$$

$$R(\xi, X)Y = g(X, Y)\xi - \eta(Y)X, \tag{2.12}$$

$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y, \tag{2.13}$$

$$R(\xi, X)\xi = X + \eta(X)\xi, \tag{2.14}$$

$$S(X, \xi) = (n-1)\eta(X), \tag{2.15}$$

$$QX = (n-1)X, r = n(n-1), \tag{2.16}$$

where Q is the Ricci operator, i.e.

$$g(QX, Y) = S(X, Y) \tag{2.17}$$

and r is the scalar curvature of the connection ∇. Also

$$S(\phi X, \phi Y) = S(X, Y) + (n-1)\eta(X)\eta(Y), \tag{2.18}$$

for any vector field X, Y and Z, where R and S are the Riemannian curvature tensor and Ricci tensor of the manifold respectively.

§3. Curvature tensor of LP-Sasakian Manifold with Respect to Quarter-Symmetric Metric Connection

In this section we express $\tilde{R}(X, Y)Z$ the curvature tensor with respect to quarter-symmetric metric connection in terms of $R(X, Y)Z$ the curvature tensor with respect to Riemannian connection.

Let $\tilde{\nabla}$ be the linear connection and ∇ be Riemannian connection of an almost contact metric manifold such that

$$\tilde{\nabla}_X Y = \nabla_X Y + L(X, Y), \tag{3.1}$$

where L is the tensor field of type $(1, 1)$. For $\tilde{\nabla}$ to be a quarter-symmetric metric connection in M^n, we have ([2])

$$L(X, Y) = \frac{1}{2}[\tilde{T}(X, Y) + \tilde{T}'(X, Y) + \tilde{T}'(Y, X)], \tag{3.2}$$

and

$$g(\tilde{T}'(X, Y), Z) = g(\tilde{T}(X, Y), Z). \tag{3.3}$$

From the equation (1.2) and (3.3), we get

$$\tilde{T}'(X, Y) = \eta(X)\phi Y + g(\phi X, Y)\xi. \tag{3.4}$$

Now putting the equations (1.2) and (3.4) in (3.2), we obtain

$$L(X, Y) = \eta(Y)\phi X + g(\phi X, Y)\xi. \tag{3.5}$$

So, a quarter-symmetric metric connection $\tilde{\nabla}$ in an LP-Sasakian manifold is given by

$$\tilde{\nabla}_X Y = \nabla_Y X + \eta(Y)\phi X + g(\phi X, Y)\xi. \tag{3.6}$$

Thus the above equation gives us the relation between quarter-symmetric metric connection and the Levi-Civita connection.

The curvature tensor $\tilde{R}$ of M^n with respect to quarter-symmetric metric connection $\tilde{\nabla}$ is defined by

$$\tilde{R}(X,Y)Z = \tilde{\nabla}_X \tilde{\nabla}_Y Z - \tilde{\nabla}_Y \tilde{\nabla}_X Z - \tilde{\nabla}_{[X,Y]} Z. \tag{3.7}$$

A relation between the curvature tensor of M with respect to the quarter-symmetric metric connection $\tilde{\nabla}$ and the Riemannian connection ∇ is given by

$$\begin{aligned}\tilde{R}(X,Y)Z &= R(X,Y)Z + g(\phi X, Z)\phi Y - g(\phi Y, Z)\phi X \\ &+ \eta(Z)\{\eta(Y)X - \eta(X)Y\} + \{g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\}\xi,\end{aligned} \tag{3.8}$$

where $\tilde{R}$ and R are the Riemannian curvature tensor with respect to $\tilde{\nabla}$ and ∇ respectively.

From the equation (3.8), we get

$$\tilde{S}(Y,Z) = S(Y,Z) + (n-1)\eta(Y)\eta(Z), \tag{3.9}$$

where $\tilde{S}$ and S are the Ricci tensor with respect to $\tilde{\nabla}$ and ∇ respectively. This gives

$$\tilde{Q}Y = QY + (n-1)\eta(Y)\xi. \tag{3.10}$$

Contracting (3.9), we obtain,

$$\tilde{r} = r - (n-1), \tag{3.11}$$

where $\tilde{r}$ and r are the scalar curvature tensor with respect to $\tilde{\nabla}$ and ∇ respectively. Also we have

$$\tilde{R}(X,Y)\xi = 0, \tag{3.12}$$

which gives

$$\eta(\tilde{R}(X,Y)\xi) = 0, \tag{3.13}$$

and

$$\tilde{R}(\xi,Y)Z = 0, \tag{3.14}$$

which gives

$$\eta(\tilde{R}(\xi,Y)Z) = 0. \tag{3.15}$$

§4. W_2-Curvature Tensor of LP-Sasakian Manifold with Respect to Quarter-Symmetric Metric Connection

The W_2-curvature tensor of LP-Sasakian manifold M^n with respect to quarter-symmetric metric connection $\tilde{\nabla}$ is given by

$$\tilde{W}_2(X,Y)Z = \tilde{R}(X,Y)Z + \frac{1}{n-1}\{g(X,Z)\tilde{Q}Y - g(Y,Z)\tilde{Q}X\}. \tag{4.1}$$

Using the equations (3.8) and (3.10) in (4.1), we get

$$\begin{aligned}\tilde{W}_2(X,Y)Z = &= R(X,Y)Z + g(\phi X,Z)\phi Y - g(\phi Y,Z)\phi X \\ &+ \eta(Z)\{\eta(Y)X - \eta(X)Y\} \\ &+ \{g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\}\xi \\ &+ \frac{1}{n-1}[g(X,Z)\{QY + (n-1)\eta(Y)\xi\} \\ &- g(Y,Z)\{QX + (n-1)\eta(X)\xi\}]. \end{aligned} \tag{4.2}$$

Now using the equation (1.7) in (4.2), we obtain

$$\begin{aligned}\tilde{W}_2(X,Y)Z = &= W_2(X,Y)Z + g(\phi X,Z)\phi Y - g(\phi Y,Z)\phi X \\ &+ \eta(Z)\{\eta(Y)X - \eta(X)Y\} \\ &+ \{g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\}\xi \\ &+ \frac{1}{n-1}[g(X,Z)(n-1)\eta(Y)\xi \\ &- g(Y,Z)(n-1)\eta(X)\xi]. \end{aligned} \tag{4.3}$$

Putting $Z = \xi$ in (4.3) and using the equations (2.1), (2.4), (2.7) and (1.7), we get

$$\tilde{W}_2(X,Y)\xi = \eta(X)Y - \eta(Y)X, \tag{4.4}$$

which gives

$$\eta(\tilde{W}_2(X,Y)\xi) = 0. \tag{4.5}$$

Again putting $X = \xi$ in (4.3) and using the equations (2.1), (2.4), (2.7), (2.12) and (1.7), we get

$$\tilde{W}_2(\xi,Y)Z = \eta(Z)Y + \eta(Y)\eta(Z)\xi. \tag{4.6}$$

This gives

$$\eta(\tilde{W}_2(\xi,Y)Z) = 0. \tag{4.7}$$

Theorem 4.1 *In LP-Sasakian Manifold M^n, if the W_2-Curvature tensor of with respect to quarter-symmetric metric connection vanishes, then it is locally isomorphic to $S^n(1)$.*

Proof Let $\tilde{W}_2 = 0$. From the equation (4.2), we have

$$\begin{aligned}R(X,Y)Z &= g(\phi Y,Z)\phi X - g(\phi X,Z)\phi Y + \eta(Z)\{\eta(X)Y - \eta(Y)X\} \\ &+ \{g(X,Z)\eta(Y) - g(Y,Z)\eta(X)\}\xi - \frac{1}{n-1}[g(X,Z)\{QY + (n-1)\eta(Y)\xi\} \\ &- g(Y,Z)\{QX + (n-1)\eta(X)\xi\}]. \end{aligned} \tag{4.8}$$

Taking the inner product of the above equation and using (2.1), (2.4), (2.7), we get

$$\eta(R(X,Y)Z) = \{g(Y,Z)X - g(X,Z)Y\}, \tag{4.9}$$

which gives

$$R(X,Y,Z,U) = \{g(Y,Z)g(X,U) - g(X,Z)g(Y,U)\}. \tag{4.10}$$

This shows that M^n is a space of constant curvature is 1, that is, it is locally isomorphic to $S^n(1)$. □

Suppose let $(\tilde{W}_2(\xi,Z).\tilde{S})(X,Y) = 0$. This gives

$$\tilde{S}(\tilde{W}_2(\xi,Z)X,Y) + \tilde{S}(X,\tilde{W}_2(\xi,Z)Y) = 0. \tag{4.11}$$

Now using the equation (3.9) in (4.11), we get

$$\begin{aligned} &S(\tilde{W}_2(\xi,Z)X,Y) + (n-1)\eta(\tilde{W}_2(\xi,Z)X)\eta(Y) \\ &\qquad S(X,\tilde{W}_2(\xi,Z)Y) + (n-1)\eta(\tilde{W}_2(\xi,Z)Y)\eta(X) = 0. \end{aligned} \tag{4.12}$$

Using the equation (2.15), (4.6) and (4.7) in (4.12), we obtain

$$\begin{aligned} \eta(X)S(Y,Z) &+ (n-1)\eta(X)\eta(Y)\eta(Z) + \eta(Y)S(X,Z) \\ &+ (n-1)\eta(X)\eta(Y)\eta(Z) = 0. \end{aligned} \tag{4.13}$$

Putting $X = \xi$ and using the equation (2.1) and (2.4) in (4.13), we get

$$S(Y,Z) = (1-n)\eta(Y)\eta(Z). \tag{4.14}$$

So, we have the following theorem.

Theorem 4.2 *A LP-Sasakian manifold M^n with respect to quarter-symmetric metric connection $\tilde{\nabla}$ satisfying $(\tilde{W}_2(\xi,Z).\tilde{S})(X,Y) = 0$ is the product of two 1-forms.*

§5. Generalized Projective ϕ-Recurrent LP-Sasakian Manifold with Respect to Quarter-Symmetric Metric Connection

The projective curvature tensor for an n-dimensional Riemannian manifold M with respect to quarter-symmetric metric connection is given by

$$\tilde{P}(X,Y)Z = \tilde{R}(X,Y)Z - \frac{1}{n-1}[\tilde{S}(Y,Z)X - \tilde{S}(X,Z)Y], \tag{5.1}$$

where R and S are the curvature tensor and Ricci tensor of the manifold.

Let us consider generalized projective ϕ-recurrent LP-Sasakian manifold with respect to quarter-symmetric metric connection. By virtue of (1.4) and (2.2), we get

$$\begin{aligned}(\tilde{\nabla}_W \tilde{P})(X,Y)Z &+ \eta((\tilde{\nabla}_W \tilde{P})(X,Y)Z)\xi = A(W)\tilde{P}(X,Y)Z \\ &+ B(W)[g(Y,Z)X - g(X,Z)Y], \qquad (5.2)\end{aligned}$$

from which it follows that

$$\begin{aligned}g((\tilde{\nabla}_W \tilde{P})(X,Y)Z,U) &+ \eta((\tilde{\nabla}_W \tilde{P})(X,Y)Z)\eta(U) = A(W)g(\tilde{P}(X,Y)Z,U) \\ &+ B(W)[g(Y,Z)g(X,U) - g(X,Z)g(Y,U)]. \qquad (5.3)\end{aligned}$$

Let $\{e_i\}$, $i = 1, 2, \cdots, n$ be an orthonormal basis of the tangent space at any point of the manifold. Then putting $X = U = e_i$ in (5.3) and taking summation over i, $1 \le i \le n$, we get

$$\begin{aligned}(\tilde{\nabla}_W \tilde{S})(X,U) &- \frac{\tilde{\nabla}_W \tilde{r}}{n-1} g(X,U) + \frac{(\tilde{\nabla}_W \tilde{S})(X,U)}{n-1} - (\tilde{\nabla}_W \tilde{S})(X,\xi)\eta(U) \\ &+ \frac{\tilde{\nabla}_W \tilde{r}}{n-1}\eta(X)\eta(U) - \frac{(\tilde{\nabla}_W \tilde{S})(X,U)}{n-1}\eta(U) \\ &= A(W)[\frac{n}{n-1}\tilde{S}(X,U) - \frac{\tilde{r}}{n-1} g(X,U)] \\ &+ 2nB(W)g(X,U). \qquad (5.4)\end{aligned}$$

Putting $U = \xi$ in (5.4) and using the equation (3.6), (3.9) an (3.11), we obtain

$$A(W)[1 - \frac{r}{n-1}]\eta(X) + (n-1)B(W)\eta(X) = 0. \qquad (5.5)$$

Putting $X = \xi$ in (5.5), we get

$$B(W) = [\frac{r-n+1}{(n-1)^2}]A(W). \qquad (5.6)$$

Thus we can state the following theorem.

Theorem 5.1 *In a generalized projective ϕ-ecurrent LP-Sasakian manifold M^n $(n > 2)$, the 1-forms A and B are related as* (5.6).

§6. ϕ-Pseudo Symmetric LP-Sasakian Manifold with Respect to Quarter-Symmetric Metric Connection

Definition 6.1 A LP-Sasakian manifold $(M^n, \phi, \xi, \eta, g)(n > 2)$ is said to be ϕ-pseudosymmetric with respect to quarter symmetric metric connection if the curvature tensor $\tilde{R}$ satisfies

$$\begin{aligned}\phi^2((\tilde{\nabla}_W \tilde{R})(X,Y)Z) &= 2A(W)\tilde{R}(X,Y)Z + A(X)\tilde{R}(W,Y)Z \\ &+ A(Y)\tilde{R}(X,W)Z + A(Z)\tilde{R}(X,Y)W + g(\tilde{R}(X,Y)Z,W)\rho \qquad (6.1)\end{aligned}$$

for any vector field X, Y, Z and W, where ρ is the vector field associated to the 1-form A such that $A(X) = g(X, \rho)$. Now using (2.2) in (6.1), we have

$$\begin{aligned}(\tilde{\nabla}_W \tilde{R})(X,Y)Z &+ \eta((\tilde{\nabla}_W \tilde{R})(X,Y)Z)\xi = 2A(W)\tilde{R}(X,Y)Z \\ &+ A(X)\tilde{R}(W,Y)Z + A(Y)\tilde{R}(X,W)Z \\ &+ A(Z)\tilde{R}(X,Y)W + g(\tilde{R}(X,Y)Z,W)\rho. \end{aligned} \tag{6.2}$$

From which it follows that

$$\begin{aligned}g((\tilde{\nabla}_W \tilde{R})(X,Y)Z,U) &+ \eta((\tilde{\nabla}_W \tilde{R})(X,Y)Z)\eta(U) = 2A(W)g(\tilde{R}(X,Y)Z,U) \\ &+ A(X)g(\tilde{R}(W,Y)Z,U) + A(Y)g(\tilde{R}(X,W)Z,U) \\ &+ A(Z)g(\tilde{R}(X,Y)W,U) + g(\tilde{R}(X,Y)Z,W)A(U). \end{aligned} \tag{6.3}$$

Let $\{e_i : i = 1, 2, \cdots, n\}$ be an orthonormal basis of the tangent space at any point of the manifold. Setting $X = U = e_i$ in (6.3) and taking summation over i, $1 \leq i \leq n$, and then using (2.1), (2.4) and (2.7) in (6.3), we obtain

$$\begin{aligned}(\tilde{\nabla}_W \tilde{S})(Y,Z) &+ g((\tilde{\nabla}_W \tilde{R})(\xi,Y)Z,\xi) = 2A(W)\tilde{S}(Y,Z) \\ &+ A(Y)\tilde{S}(W,Z) + A(Z)\tilde{S}(Y,W) \\ &+ A(\tilde{R}(W,Y)Z) + A(\tilde{R}(W,Z)Y). \end{aligned} \tag{6.4}$$

By virtue of (3.14) it follows from (6.4) that

$$\begin{aligned}(\tilde{\nabla}_W \tilde{S})(Y,Z) &= 2A(W)\tilde{S}(Y,Z) + A(Y)\tilde{S}(W,Z) + A(Z)\tilde{S}(Y,W) \\ &+ A(\tilde{R}(W,Y)Z) + A(\tilde{R}(W,Z)Y). \end{aligned} \tag{6.5}$$

So, we have the following theorem:

Theorem 6.1 *A ϕ-pseudo symmetric LP-Sasakian manifold with respect to quarter-symmetric metric connection is pseudo Ricci symmetric with respect to quarter sym- metric non-metric connection if and only if*

$$A(\tilde{R}(W,Y)Z) + A(\tilde{R}(W,Z)Y) = 0.$$

§7. LP-Sasakian Manifold with Respect to Quarter-Symmetric Metric Connection Satisfying $\tilde{P}.\tilde{S} = 0$.

A LP-Sasakian manifold with respect to the quarter-symmetric metric connection satisfying

$$(\tilde{P}(X,Y).\tilde{S})(Z,U) = 0, \tag{7.1}$$

where $\tilde{S}$ is the Ricci tensor with respect to a quarter-symmetric metric connection. Then, we have

$$\tilde{S}(\tilde{P}(X,Y)Z,U)+\tilde{S}(Z,\tilde{P}(X,Y)U)=0. \tag{7.2}$$

Putting $X=\xi$ in the equation (7.2), we have

$$\tilde{S}(\tilde{P}(\xi,Y)Z,U)+\tilde{S}(Z,\tilde{P}(\xi,Y)U)=0. \tag{7.3}$$

In view of the equation (5.1), we have

$$\tilde{P}(\xi,Y)Z=\tilde{R}(\xi,Y)Z-\frac{1}{n-1}[\tilde{S}(Y,Z)\xi-\tilde{S}(\xi,Z)Y] \tag{7.4}$$

for $X,Y,Z\in\chi(M)$.

Using equations (3.9) and (3.14) in the equation (7.4), we get

$$\tilde{P}(\xi,Y)Z=-\frac{1}{n-1}[S(Y,Z)\xi+(n-1)\eta(Y)\eta(Z)\xi]. \tag{7.5}$$

Now using the equation (7.5) and putting $U=\xi$ in the equation (7.3) and using the equations (2.2), (2.15) and (3.9) we get

$$S(Y,Z)+(n-1)\eta(Y)\eta(Z)=0. \tag{7.6}$$

i.e.,

$$S(Y,Z)=-(n-1)\eta(Y)\eta(Z). \tag{7.7}$$

In view of above discussions we can state the following theorem:

Theorem 7.1 *A n-dimensional LP-Sasakian manifold with a quarter-symmetric metric connection satisfying $\tilde{P}.\tilde{S}=0$ is the product of two 1-forms.*

§8. ξ-Conharmonically Flat LP-Sasakian Manifold with Respect to Quarter-Symmetric Metric Connection

The conharmonic curvature tensor of LP-Sasakian manifold M^n with respect to quarter-symmetric metric connection $\tilde{\nabla}$ is given by

$$\begin{aligned}\tilde{C}(X,Y)Z &= \tilde{R}(X,Y)Z-\frac{1}{n-2}[g(Y,Z)\tilde{Q}X-g(X,Z)\tilde{Q}Y\\ &+ \tilde{S}(Y,Z)X-\tilde{S}(X,Z)Y],\end{aligned} \tag{8.1}$$

where $\tilde{R}$ and $\tilde{S}$ are the curvature tensor and Ricci tensor with respect to quarter-symmetric metric connection.

Using (3.8), (3.9) and (3.10) in (8.1), we get

$$\begin{aligned}\tilde{C}(X,Y)Z &= R(X,Y)Z + g(\phi X,Z)\phi Y - g(\phi Y,Z)\phi X\\ &+ \eta(Z)\{\eta(Y)X - \eta(X)Y\}\\ &+ \{g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\}\xi\\ &- \frac{1}{n-1}[g(X,Z)\{QY + (n-1)\eta(Y)\xi\}\\ &- g(Y,Z)\{QX + (n-1)\eta(X)\xi\}]\\ &- \frac{1}{n-2}[g(Y,Z)\{QX + (n-1)\eta(X)\xi\}\\ &- g(X,Z)\{QY + (n-1)\eta(Y)\xi\} + S(Y,Z)X\\ &+ (n-1)\eta(Y)\eta(Z)X - S(X,Z)Y\\ &- (n-1)\eta(X)\eta(Z)Y]. \end{aligned} \tag{8.2}$$

$$\begin{aligned}\tilde{C}(X,Y)Z &= C(X,Y)Z + g(\phi X,Z)\phi Y - g(\phi Y,Z)\phi X\\ &+ \eta(Z)\{\eta(Y)X - \eta(X)Y\} + \{g(Y,Z)\eta(X)\\ &- g(X,Z)\eta(Y)\}\xi - \frac{n-1}{n-2}[g(Y,Z)\eta(X)\xi\\ &- g(X,Z)\eta(Y)\xi + \eta(Y)\eta(Z)X\\ &- \eta(X)\eta(Z)Y], \end{aligned} \tag{8.3}$$

where C is given in (1.8). Putting $Z = \xi$ in (8.3) and using (2.1), (2.4) and (2.7), we obtain

$$\begin{aligned}\tilde{C}(X,Y)\xi &= C(X,Y)\xi - \{\eta(Y)X - \eta(X)Y\}\\ &- \frac{n-1}{n-2}[\eta(X)Y - \eta(Y)X]. \end{aligned} \tag{8.4}$$

Suppose X and Y are orthogonal to ξ, then from (8.4), we obtain

$$\tilde{C}(X,Y)\xi = C(X,Y)\xi. \tag{8.5}$$

So, by the above discussion we can state the following theorem:

Theorem 8.1 *An n-dimensional LP-Sasakian manifold is ξ-conharmonically flat with respect to the quarter-symmetric metric connection if and only if the manifold is also ξ-conharmonically flat with respect to the Levi-Civita connection provided the vector fields X and Y are orthogonal to the associated vector field ξ.*

§9. Example 3-Dimensional LP-Sasakian Manifold with Respect to Quarter-Symmetric Metric Connection

We consider a 3-dimensional manifold $M = \{(x,y,u) \in R^3\}$, where (x,y,u) are the standard coordinates of R^3. Let e_1, e_2, e_3 be the vector fields on M^3 given by

$$e_1 = -e^u \frac{\partial}{\partial x},\ e_2 = -e^{u-x} \frac{\partial}{\partial y},\ e_3 = -\frac{\partial}{\partial u}.$$

Clearly, $\{e_1, e_2, e_3\}$ is a set of linearly independent vectors for each point of M and hence a basis of $\chi(M)$. The Lorentzian metric g is defined by

$$\begin{aligned} &g(e_1, e_2) = g(e_2, e_3) = g(e_1, e_3) = 0, \\ &g(e_1, e_1) = 1,\ g(e_2, e_2) = 1,\ g(e_3, e_3) = -1. \end{aligned}$$

Let η be the 1-form defined by $\eta(Z) = g(Z, e_3)$ for any $Z \in \chi(M)$ and the $(1,1)$ tensor field ϕ is defined by

$$\phi e_1 = -e_1,\ \phi e_2 = -e_2,\ \phi e_3 = 0.$$

From the linearity of ϕ and g, we have

$$\begin{aligned} &\eta(e_3) = -1, \\ &\phi^2 X = X + \eta(X) e_3 \end{aligned}$$

and

$$g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y)$$

for any $X \in \chi(M)$. Then for $e_3 = \xi$, the structure (ϕ, ξ, η, g) defines a Lorentzian paracontact structure on M. Let ∇ be the Levi-Civita connection with respect to the Lorentzian metric g. Then we have

$$[e_1, e_2] = -e^u e_2,\ [e_1, e_3] = -e_1,\ [e_2, e_3] = -e_2.$$

Koszul's formula is defined by

$$\begin{aligned} 2g(\nabla_X Y, Z) \quad = \quad & Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) \\ & -g(X, [Y, Z]) - g(Y, [X, Z]) + g(Z, [X, Y]). \end{aligned}$$

Then from above formula we can calculate followings:

$$\begin{aligned} &\nabla_{e_1} e_1 = e_3,\ \nabla_{e_1} e_2 = 0,\ \nabla_{e_1} e_3 = -e_2, \\ &\nabla_{e_2} e_1 = -e^u e_2,\ \nabla_{e_2} e_2 = -e_3 - e^u e_1,\ \nabla_{e_2} e_3 = -e_2, \\ &\nabla_{e_3} e_1 = 0,\ \nabla_{e_3} e_2 = 0,\ \nabla_{e_3} e_3 = 0. \end{aligned}$$

From the above calculations, we see that the manifold under consideration satisfies $\eta(\xi) = -1$ and $\nabla_X \xi = \phi X$. Hence the structure (ϕ, ξ, η, g) is a LP-Sasakian manifold.

Using (3.6), we find $\tilde{\nabla}$, the quarter-symmetric metric connection on M following:

$$\begin{aligned} &\tilde{\nabla}_{e_1} e_1 = 0,\ \tilde{\nabla}_{e_1} e_2 = 0,\ \tilde{\nabla}_{e_1} e_3 = 0, \\ &\tilde{\nabla}_{e_2} e_1 = -e^u e_2,\ \tilde{\nabla}_{e_2} e_2 = -e^u e_1,\ \tilde{\nabla}_{e_2} e_3 = 0 \end{aligned}$$

and

$$\tilde{\nabla}_{e_3} e_1 = 0,\ \tilde{\nabla}_{e_3} e_2 = 0,\ \tilde{\nabla}_{e_3} e_3 = 0.$$

Using (1.2), the torson tensor T, with respect to quarter-symmetric metric connection $\tilde{\nabla}$ as follows:

$$\tilde{T}(e_i, e_i) = 0,\ \forall i = 1, 2, 3,$$
$$\tilde{T}(e_1, e_2) = 0,\ \tilde{T}(e_1, e_3) = e_3,\ \tilde{T}(e_2, e_3) = e_2.$$

Also,

$$(\tilde{\nabla}_{e_1} g)(e_2, e_3) = 0,\ (\tilde{\nabla}_{e_2} g)(e_3, e_1) = 0,\ (\tilde{\nabla}_{e_3} g)(e_1, e_2) = 0.$$

Thus M is LP-Sasakian manifold with quarter-symmetric metric connection $\tilde{\nabla}$.

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