International J.Math. Combin. Vol.1(2016), 97-100

Probabilistic Bounds
On Weak and Strong Total Domination in Graphs

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Abstract: A set D of vertices in a graph $G = (V, E)$ is a total dominating set if every vertex of G is adjacent to some vertex in D. A total dominating set D of G is said to be weak if every vertex $v \in V - D$ is adjacent to a vertex $u \in D$ such that $d_G(v) \geq d_G(u)$. The weak total domination number $\gamma_{wt}(G)$ of G is the minimum cardinality of a weak total dominating set of G. A total dominating set D of G is said to be strong if every vertex $v \in V - D$ is adjacent to a vertex $u \in D$ such that $d_G(v) \leq d_G(u)$. The strong total domination number $\gamma_{st}(G)$ of G is the minimum cardinality of a strong total dominating set of G. We present probabilistic upper bounds on weak and strong total domination number of a graph.

Key Words: Weak total domination, strong total domination, pigeonhole property, probability.

AMS(2010): 05C69.

§1. Introduction

We consider finite, undirected, simple graphs. Let G be a graph, with vertex set V and edge set E. The *open neighborhood* of a vertex $v \in V$ is $N(v) = \{u \in V \mid uv \in E\}$ and the *closed neighborhood* is $N[v] = N(v) \cup \{v\}$. For a subset $S \subseteq V$, the *open neighborhood* is $N(S) = \cup_{v \in S} N(v)$ and the *closed neighborhood* is $N[S] = N(S) \cup S$. If v is a vertex of V, then the *degree* of v denoted by $d_G(v)$, is the cardinality of its open neighborhood. By $\Delta(G) = \Delta$ and $\delta(G) = \delta$ we denote the *maximum* and *minimum degree* of a graph G, respectively. A subset $S \subseteq V$ is a *dominating set* of G if every vertex in $V - S$ has a neighbor in S and is a *total dominating set* (td-set) if every vertex in V has a neighbor in S. The *domination number* $\gamma(G)$ (respectively, *total domination number* $\gamma_t(G)$) is the minimum cardinality of a dominating set (respectively, total dominating set) of G. Total domination was introduced by Cockayne, Dawes and Hedetniemi [2].

In [10], Sampathkumar and Pushpa Latha have introduced the concept of weak and strong domination in graphs. A subset $D \subseteq V$ is a *weak dominating set* (wd-set) if every vertex $v \in V - S$ is adjacent to a vertex $u \in D$, where $d_G(v) \geq d_G(u)$. The subset D is a *strong dominating set* (sd-set) if every vertex $v \in V - S$ is adjacent to a vertex $u \in D$, where $d_G(u) \geq$

[1]Received May 5, 2015, Accepted February 24, 2016.

$d_G(v)$. The *weak* (*strong*, respectively) *domination number* $\gamma_w(G)$ ($\gamma_s(G)$, respectively) is the minimum cardinality of a wd-set (an sd-set, respectively) of G. Strong and weak domination have been studied for example in [4, 5, 7, 8, 9]. For more details on domination in graphs and its variations, see [6].

Chellali et al. [3] have introduced the concept of weak total domination in graphs. A total dominating set D of G is said to be *weak* if every vertex $v \in V - D$ is adjacent to a vertex $u \in D$ such that $d_G(v) \geq d_G(u)$. The *weak total domination number* $\gamma_{wt}(G)$ of G is the minimum cardinality of a weak total dominating set of G. The concept of *strong total domination* can be defined analogously. A total dominating set D of G is said to be *strong* if every vertex $v \in V - D$ is adjacent to a vertex $u \in D$ such that $d_G(v) \leq d_G(u)$. The *strong total domination number* $\gamma_{st}(G)$ of G is the minimum cardinality of a strong total dominating set of G.

We obtain probabilistic upper bounds on weak and strong total domination number of a graph.

§2. Results

We adopt the notations of [1]. Let $W = W(G)$ be the set of all vertices $v \in V(G)$ such that $deg(v) < deg(u)$ for every $u \in N(v)$. Note that $W(G)$ may be empty, and if $W(G) \neq \emptyset$, then $W(G)$ is independent and is contained in every weak total dominating set of G. For any vertex $v \in V(G)$ let $\deg_w(v) = \{u \in N(v) | \deg(u) \leq \deg(v)\}$.

Theorem 2.1 *Let G be a graph with $V(G) - N[W] \neq \emptyset$. If $\delta_w = \min\{\deg_w(v) | v \in V(G) - N[W]\}$, then*

$$\gamma_{wt}(G) \leq 2|W| + 2(n - |W|)\left(1 - \frac{\delta_w}{(1+\delta_w)^{1+\frac{1}{\delta_w}}}\right).$$

Proof For each vertex $w \in W$ consider a vertex $w' \in N(w)$, and let $W' = \{w' | w \in W\}$. Clearly $|W'| \leq |W|$. Let A be a set formed by an independent choice of vertices of $G - W$, where each vertex is selected with probability

$$p = 1 - \left(\frac{1}{1+\delta_w}\right)^{\frac{1}{\delta_w}}.$$

Let $B \subseteq V(G) - (A \cup N[W])$ be the set of vertices that have not a weak neighbor in A. Clearly $E(|A|) \leq (n - |W|)p$. Each vertex of B has at least δ_w weak neighbors in $V(G) - W$. It is easy to show that

$$Pr(v \in B) = (1-p)^{1+\deg_w(v)} \leq (1-p)^{1+\delta_w}.$$

Thus $E(|B|) \leq (n - |W|)(1-p)^{\delta_w + 1}$. For each $a \in A$ let $a' \in N(a)$, and let $A' = \{a' | a \in A\}$. Similarly for each $b \in B$ let $b' \in N(b)$, and let $B' = \{b' | b \in B\}$. Then clearly $|A'| \leq |A|$ and $|B'| \leq |B|$. It is obvious that $D = W \cup W' \cup A \cup A' \cup B \cup B'$ is a weak total dominating set for

G. The expectation of $|D|$ is

$$\begin{aligned} E(|D|) &\leq 2E(|W|) + 2E(|A|) + 2E(|B|) \\ &\leq 2|W| + 2(n - |W|)p + 2(n - |W|)(1-p)^{\delta_w + 1} \\ &\leq 2|W| + 2(n - |W|)\left(1 - \frac{\delta_w}{(1+\delta_w)^{1+\frac{1}{\delta_w}}}\right). \end{aligned}$$

By the pigeonhole property of expectation there exists a desired weak total dominating set. □

The proof of Theorem 2.1 implies the following upper bound, which is asymptotically same as the bound of Theorem 2.1.

Corollary 2.2 *Let G be a graph with $V(G) - N[W] \neq \emptyset$. If $\delta_w = \min\{\deg_w(v) | v \in V(G) - N[W]\}$, then*

$$\gamma_{wt}(G) \leq 2|W| + 2(n - |W|)\left(\frac{1 + \ln(\delta_w + 1)}{\delta_w + 1}\right).$$

Proof We use the proof of Theorem 2.1. Using the inequality $1 - p \leq e^{-p}$ we obtain that

$$\begin{aligned} E(|D|) &\leq 2|W| + 2(n - |W|)p + 2(n - |W|)(1-p)^{\delta_w + 1} \\ &\leq 2|W| + 2(n - |W|)p + 2(n - |W|)e^{-p(\delta_w + 1)}. \end{aligned}$$

If we put $p = \dfrac{\ln(1 + \delta_w)}{1 + \delta_w}$ then

$$E(|D|) \leq 2|W| + 2(n - |W|)\left(\frac{1 + \ln(\delta_w + 1)}{\delta_w + 1}\right).$$

By the pigeonhole property of expectation there exists a desired weak total dominating set. □

Next we obtain probabilistic upper bounds for strong total domination number. Let $S = S(G)$ be the set of all vertices $v \in V(G)$ such that $deg(v) > deg(u)$ for every $u \in N(v)$. Note that $S(G)$ may be empty, and if $S(G) \neq \emptyset$, then $S(G)$ is independent and is contained in every strong total dominating set of G. For any vertex $v \in V(G)$ let $\deg_s(v) = \{u \in N(v) | \deg(u) \geq \deg(v)\}$. The following can be proved similar to Theorem 2.1 and Corollary 2.2, and thus we omit the proofs.

Theorem 2.3 *Let G be a graph with $V(G) - N[S] \neq \emptyset$. If $\delta_s = \min\{\deg_s(v) | v \in V(G) - N[S]\}$, then*

$$\gamma_{st}(G) \leq 2|S| + 2(n - |S|)\left(1 - \frac{\delta_s}{(1+\delta_s)^{1+\frac{1}{\delta_s}}}\right).$$

Corollary 2.4 *Let G be a graph with $V(G) - N[S] \neq \emptyset$. If $\delta_s = \min\{\deg_s(v) | v \in V(G) - N[S]\}$,* *then*

$$\gamma_{st}(G) \leq 2|S| + 2(n - |S|)\left(\frac{1 + \ln(\delta_s + 1)}{\delta_s + 1}\right).$$

Acknowledgements

This research is supported by Islamic Azad University, Estahban Branch.

References

[1] N.Alon and J.Spencer, *The Probabilistic Method*, John Wiley, New York, 1992.

[2] E.J.Cockayne, R.M.Dawes, S.T.Hedetniemi, Total domination in graphs, *Networks*, 10 (1980), 211–219.

[3] M.Chellali and N.Jafari Rad, Weak total domination in graphs, *Utilitas Mathematica*, 94 (2014), 221–236.

[4] J.H.Hattingh and M.A.Henning, On strong domination in graphs, *J. Combin. Math. Combin. Comput.*, 26 (1998), 73–92.

[5] J.H.Hattingh and R.C.Laskar, On weak domination in graphs, *Ars Combinatoria*, 49 (1998).

[6] T.W.Haynes, S.T.Hedetniemi, P.J.Slater (Eds.), *Fundamentals of Domination in Graphs*, Marcel Dekker, Inc., New York, 1998.

[7] M.Krzywkowski, On the ratio between 2-domination and total outer-independent domination numbers of trees, *Chinese Annals of Mathematics*, Series B 34 (2013), 765-776.

[8] D.Rautenbach, Bounds on the weak domination number. *Austral. J. Combin.*, 18 (1998), 245–251.

[9] D.Rautenbach, Bounds on the strong domination number, *Discrete Math.*, 215 (2000), 201–212.

[10] E.Sampathkumar and L.Pushpa Latha, Strong, weak domination and domination balance in graphs, *Discrete Math.*, 161 (1996), 235–242.