International J.Math. Combin. Vol.1(2016), 65-75

On Common Fixed Point Theorems With Rational Expressions in Cone b-Metric Spaces

G.S.Saluja

Department of Mathematics, Govt. Nagarjuna P.G. College of Science

Raipur - 492010 (C.G.), India

E-mail: saluja1963@gmail.com

Abstract: In this paper, we establish some common fixed point theorems for rational contraction in the setting of cone b-metric spaces with normal solid cone. Also, as an application of our result, we obtain some results of integral type for such mappings. Our results extend and generalize several known results from the existing literature.

Key Words: Common fixed point, rational expression, cone b- metric space, normal cone.

AMS(2010): 47H10, 54H25.

§1. Introduction and Preliminaries

Fixed point theory plays a very significant role in the development of nonlinear analysis. In this area, the first important result was proved by Banach in 1922 for contraction mapping in complete metric space, known as the Banach contraction principle [2].

In 1989, Bakhtin [3] introduced b-metric spaces as a generalization of metric spaces. He proved the contraction mapping principle in b-metric spaces that generalized the famous contraction principle in metric spaces. Czerwik used the concept of b-metric space and generalized the renowned Banach fixed point theorem in b-metric spaces (see, [5, 6]). In 2007, Huang and Zhang [9] introduced the concept of cone metric spaces as a generalization of metric spaces and establish some fixed point theorems for contractive mappings in normal cone metric spaces. In 2008, Rezapour and Hamlbarani [14] omitted the assumption of normality in cone metric space, which is a milestone in developing fixed point theory in cone metric space.

In 2011, Hussain and Shah [10] introduced the concept of cone b-metric space as a generalization of b-metric space and cone metric spaces. They established some topological properties in such spaces and improved some recent results about KKM mappings in the setting of a cone b-metric space.

In this note, we establish some common fixed point theorems satisfying rational inequality in the framework of cone b-metric spaces with normal solid cone.

Definition 1.1([9]) *Let E be a real Banach space. A subset P of E is called a cone whenever*

[1]Received August 12, 2015, Accepted February 16, 2016.

the following conditions hold:

(C1) P *is closed, nonempty and* $P \neq \{0\}$;

(C2) $a, b \in R$, $a, b \geq 0$ *and* $x, y \in P$ *imply* $ax + by \in P$;

(C3) $P \cap (-P) = \{0\}$.

Given a cone $P \subset E$, *we define a partial ordering* $\leq$ *with respect to* P *by* $x \leq y$ *if and only if* $y - x \in P$. *We shall write* $x < y$ *to indicate that* $x \leq y$ *but* $x \neq y$, *while* $x \ll y$ *will stand for* $y - x \in P^0$, *where* P^0 *stands for the interior of* P. *If* $P^0 \neq \emptyset$ *then* P *is called a solid cone (see [15]).*

There exist two kinds of cones- normal (with the normal constant K) and non-normal ones following ([7]):

Let E be a real Banach space, $P \subset E$ a cone and $\leq$ partial ordering defined by P. Then P is called normal if there is a number $K > 0$ such that for all $x, y \in P$,

$$0 \leq x \leq y \quad \text{imply} \quad \|x\| \leq K \|y\|, \tag{1.1}$$

or equivalently, if $(\forall n)$ $x_n \leq y_n \leq z_n$ and

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} z_n = x \quad \text{imply} \quad \lim_{n\to\infty} y_n = x. \tag{1.2}$$

The least positive number K satisfying (1.1) is called the normal constant of P.

Example 1.2 ([15]) Let $E = C^1_{\mathbb{R}}[0,1]$ with $\|x\| = \|x\|_\infty + \|x'\|_\infty$ on $P = \{x \in E : x(t) \geq 0\}$. This cone is not normal. Consider, for example, $x_n(t) = \frac{t^n}{n}$ and $y_n(t) = \frac{1}{n}$. Then $0 \leq x_n \leq y_n$, and $\lim_{n\to\infty} y_n = 0$, but $\|x_n\| = \max_{t\in[0,1]} |\frac{t^n}{n}| + \max_{t\in[0,1]} |t^{n-1}| = \frac{1}{n} + 1 > 1$; hence x_n does not converge to zero. It follows by (1.2) that P is a non-normal cone.

Definition 1.3([9, 16]) *Let* X *be a nonempty set. Suppose that the mapping* $d\colon X \times X \to E$ *satisfies:*

(CM1) $0 \leq d(x, y)$ *for all* $x, y \in X$ *with* $x \neq y$ *and* $d(x, y) = 0 \Leftrightarrow x = y$;

(CM2) $d(x, y) = d(y, x)$ *for all* $x, y \in X$;

(CM3) $d(x, y) \leq d(x, z) + d(z, y)$ $x, y, z \in X$.

Then d *is called a cone metric on* X *and* (X, d) *is called a cone metric space (CMS).*

The concept of a cone metric space is more general than that of a metric space, because each metric space is a cone metric space where $E = \mathbb{R}$ and $P = [0, +\infty)$.

Example 1.4 ([9]) Let $E = \mathbb{R}^2$, $P = \{(x, y) \in \mathbb{R}^2 : x \geq 0, y \geq 0\}$, $X = \mathbb{R}$ and $d\colon X \times X \to E$ defined by $d(x, y) = (|x - y|, \alpha|x - y|)$, where $\alpha \geq 0$ is a constant. Then (X, d) is a cone metric space with normal cone P where $K = 1$.

Example 1.5 ([13]) Let $E = \ell^2$, $P = \{\{x_n\}_{n\geq1} \in E : x_n \geq 0, \text{for all } n\}$, (X, ρ) a metric space, and $d\colon X \times X \to E$ defined by $d(x, y) = \{\rho(x, y)/2^n\}_{n\geq1}$. Then (X, d) is a cone metric space.

Clearly, the above examples show that class of cone metric spaces contains the class of metric spaces.

Definition 1.6([10]) *Let X be a nonempty set and $s \geq 1$ be a given real number. A mapping $d\colon X \times X \to E$ is said to be cone b-metric if and only if, for all x, y, $z \in X$, the following conditions are satisfied:*

(CbM1) $0 \leq d(x,y)$ *with* $x \neq y$ *and* $d(x,y) = 0 \iff x = y$;

(CbM2) $d(x,y) = d(y,x)$;

(CbM3) $d(x,y) \leq s[d(x,z) + d(z,y)]$.

The pair (X,d) is called a cone b-metric space (CbMS).

Remark 1.7 The class of cone b-metric spaces is larger than the class of cone metric space since any cone metric space must be a cone b-metric space. Therefore, it is obvious that cone b-metric spaces generalize b-metric spaces and cone metric spaces.

We give some examples, which show that introducing a cone b-metric space instead of a cone metric space is meaningful since there exist cone b-metric spaces which are not cone metric spaces.

Example 1.8 ([8]) Let $E = \mathbb{R}^2$, $P = \{(x,y) \in E : x \geq 0, y \geq 0\} \subset E$, $X = \mathbb{R}$ and $d\colon X \times X \to E$ defined by $d(x,y) = (|x-y|^p, \alpha|x-y|^p)$, where $\alpha \geq 0$ and $p > 1$ are two constants. Then (X,d) is a cone b-metric space with the coefficient $s = 2^p > 1$, but not a cone metric space.

Example 1.9 ([8]) Let $X = \ell^p$ with $0 < p < 1$, where $\ell^p = \{\{x_n\} \subset \mathbb{R} : \sum_{n=1}^{\infty} |x_n|^p < \infty\}$. Let $d\colon X \times X \to \mathbb{R}_+$ defined by

$$d(x,y) = \Big(\sum_{n=1}^{\infty} |x_n - y_n|^p\Big)^{\frac{1}{p}},$$

where $x = \{x_n\}$, $y = \{y_n\} \in \ell^p$. Then (X,d) is a cone b-metric space with the coefficient $s = 2^p > 1$, but not a cone metric space.

Example 1.10 ([8]) Let $X = \{1,2,3,4\}$, $E = \mathbb{R}^2$, $P = \{(x,y) \in E : x \geq 0, y \geq 0\}$. Define $d\colon X \times X \to E$ by

$$d(x,y) = \begin{cases} (|x-y|^{-1}, |x-y|^{-1}) & \text{if } x \neq y, \\ 0, & \text{if } x = y. \end{cases}$$

Then (X,d) is a cone b-metric space with the coefficient $s = \frac{6}{5} > 1$. But it is not a cone metric space since the triangle inequality is not satisfied,

$$d(1,2) > d(1,4) + d(4,2), \quad d(3,4) > d(3,1) + d(1,4).$$

Definition 1.11([10]) *Let (X,d) be a cone b-metric space, $x \in X$ and $\{x_n\}$ be a sequence in X. Then*

• $\{x_n\}$ *is a Cauchy sequence whenever, if for every* $c \in E$ *with* $0 \ll c$, *then there is a natural number* N *such that for all* $n, m \geq N$, $d(x_n, x_m) \ll c$;

• $\{x_n\}$ *converges to* x *whenever, for every* $c \in E$ *with* $0 \ll c$, *then there is a natural number* N *such that for all* $n \geq N$, $d(x_n, x) \ll c$. *We denote this by* $\lim_{n\to\infty} x_n = x$ *or* $x_n \to x$ *as* $n \to \infty$.

• (X, d) *is a complete cone b-metric space if every Cauchy sequence is convergent.*

In the following (X, d) will stands for a cone b-metric space with respect to a cone P with $P^0 \neq \emptyset$ in a real Banach space E and $\leq$ is partial ordering in E with respect to P.

§2. Main Results

In this section we shall prove some common fixed point theorems for rational contraction in the framework of cone b-metric spaces with normal solid cone.

Theorem 2.1 *Let* (X, d) *be a complete cone b-metric space (CCbMS) with the coefficient* $s \geq 1$ *and* P *be a normal cone with normal constant* K. *Suppose that the mappings* $S, T \colon X \to X$ *satisfy the rational contraction:*

$$d(Sx, Ty) \leq \alpha\Big[\frac{d(x, Sx)\, d(x, Ty) + [d(x, y)]^2 + d(x, Sx) d(x, y)}{d(x, Sx) + d(x, y) + d(x, Ty)}\Big] \tag{2.1}$$

for all $x, y \in X$, $\alpha \in [0, 1)$ *with* $s\alpha < 1$ *and* $d(x, Sx) + d(x, y) + d(x, Ty) \neq 0$. *Then* S *and* T *have a common fixed point in* X. *Further if* $d(x, Sx) + d(x, y) + d(x, Ty) = 0$ *implies that* $d(Sx, Ty) = 0$, *then* S *and* T *have a unique common fixed point in* X.

Proof Choose $x_0 \in X$. Let $x_1 = S(x_0)$ and $x_2 = T(x_1)$ such that $x_{2n+1} = S(x_{2n})$ and $x_{2n+2} = T(x_{2n+1})$ for all $n \geq 0$. Let $d(x, Sx) + d(x, y) + d(x, Ty) \neq 0$. From (2.1), we have

$$\begin{aligned}
d(x_{2n+1}, x_{2n+2}) &= d(Sx_{2n}, Tx_{2n+1}) \\
&\leq \alpha\Big[\Big(d(x_{2n}, Sx_{2n})\, d(x_{2n}, Tx_{2n+1}) + [d(x_{2n}, x_{2n+1})]^2 \\
&\quad + d(x_{2n}, Sx_{2n})\, d(x_{2n}, x_{2n+1})\Big) \\
&\quad \times \Big(d(x_{2n}, Sx_{2n}) + d(x_{2n}, x_{2n+1}) + d(x_{2n}, Tx_{2n+1})\Big)^{-1}\Big] \\
&= \alpha\Big[\Big(d(x_{2n}, x_{2n+1})\, d(x_{2n}, x_{2n+2}) + [d(x_{2n}, x_{2n+1})]^2 \\
&\quad + d(x_{2n}, x_{2n+1})\, d(x_{2n}, x_{2n+1})\Big) \\
&\quad \times \Big(d(x_{2n}, x_{2n+1}) + d(x_{2n}, x_{2n+1}) + d(x_{2n}, x_{2n+2})\Big)^{-1}\Big] \\
&= \alpha\, d(x_{2n}, x_{2n+1}) \\
&\quad \times \Big[\frac{d(x_{2n}, x_{2n+2}) + d(x_{2n}, x_{2n+1}) + d(x_{2n}, x_{2n+1})}{d(x_{2n}, x_{2n+1}) + d(x_{2n}, x_{2n+1}) + d(x_{2n}, x_{2n+2})}\Big] \\
&= \alpha\, d(x_{2n}, x_{2n+1}).
\end{aligned} \tag{2.2}$$

Similarly, we have

$$\begin{aligned}
d(x_{2n}, x_{2n+1}) &= d(Sx_{2n}, Tx_{2n-1}) \\
&\leq \alpha\Big[\Big(d(x_{2n}, Sx_{2n})\, d(x_{2n}, Tx_{2n-1}) + [d(x_{2n}, x_{2n-1})]^2 \\
&\quad + d(x_{2n}, Sx_{2n})\, d(x_{2n}, x_{2n-1})\Big) \\
&\quad \times\Big(d(x_{2n}, Sx_{2n}) + d(x_{2n}, x_{2n-1}) + d(x_{2n}, Tx_{2n-1})\Big)^{-1}\Big] \\
&= \alpha\Big[\Big(d(x_{2n}, x_{2n+1})\, d(x_{2n}, x_{2n}) + [d(x_{2n}, x_{2n-1})]^2 \\
&\quad + d(x_{2n}, x_{2n+1})\, d(x_{2n}, x_{2n-1})\Big) \\
&\quad \times\Big(d(x_{2n}, x_{2n+1}) + d(x_{2n}, x_{2n-1}) + d(x_{2n}, x_{2n})\Big)^{-1}\Big] \\
&= \alpha\, d(x_{2n}, x_{2n-1}) \\
&\quad \times\Big[\frac{d(x_{2n}, x_{2n-1}) + d(x_{2n}, x_{2n+1})}{d(x_{2n}, x_{2n+1}) + d(x_{2n}, x_{2n-1})}\Big] \\
&= \alpha\, d(x_{2n}, x_{2n-1}). \qquad (2.3)
\end{aligned}$$

By induction, we have

$$\begin{aligned}
d(x_{n+1}, x_n) &\leq \alpha\, d(x_{n-1}, x_n) \leq \alpha^2\, d(x_{n-2}, x_{n-1}) \leq \dots \\
&\leq \alpha^n\, d(x_0, x_1). \qquad (2.4)
\end{aligned}$$

Let $m, n \geq 1$ and $m > n$, we have

$$\begin{aligned}
d(x_n, x_m) &\leq s[d(x_n, x_{n+1}) + d(x_{n+1}, x_m)] \\
&= sd(x_n, x_{n+1}) + sd(x_{n+1}, x_m) \\
&\leq sd(x_n, x_{n+1}) + s^2[d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_m)] \\
&= sd(x_n, x_{n+1}) + s^2 d(x_{n+1}, x_{n+2}) + s^2 d(x_{n+2}, x_m) \\
&\leq sd(x_n, x_{n+1}) + s^2 d(x_{n+1}, x_{n+2}) + s^3 d(x_{n+2}, x_{n+3}) \\
&\quad + \cdots + s^{n+m-1} d(x_{n+m-1}, x_m) \\
&\leq s\alpha^n d(x_1, x_0) + s^2\alpha^{n+1} d(x_1, x_0) + s^3\alpha^{n+2} d(x_1, x_0) \\
&\quad + \cdots + s^m\alpha^{n+m-1} d(x_1, x_0) \\
&= s\alpha^n[1 + s\alpha + s^2\alpha^2 + s^3\alpha^3 + \cdots + (s\alpha)^{m-1}] d(x_1, x_0) \\
&\leq \Big[\frac{s\alpha^n}{1 - s\alpha}\Big] d(x_1, x_0).
\end{aligned}$$

Since P is a normal cone with normal constant K, so we get

$$\|d(x_n, x_m)\| \leq K\frac{s\alpha^n}{1 - s\alpha}\|d(x_1, x_0)\|.$$

This implies $\|d(x_n, x_m)\| \to 0$ as $n, m \to \infty$ since $0 < s\alpha < 1$. Hence $\{x_n\}$ is a Cauchy sequence. Since (X, d) is a complete cone b-metric space, there exists $z \in X$ such that $x_n \to z$ as $n \to \infty$.

Now, since

$$\begin{aligned} d(z,Tz) &\leq s[d(z,x_{2n+1})+d(x_{2n+1},Tz)] \\ &= sd(Sx_{2n},Tz)+sd(z,x_{2n+1}) \\ &\leq s\alpha\Big[\frac{d(x_{2n},Sx_{2n})\,d(x_{2n},Tz)+[d(x_{2n},z)]^2+d(x_{2n},Sx_{2n})d(x_{2n},z)}{d(x_{2n},Sx_{2n})+d(x_{2n},z)+d(x_{2n},Tz)}\Big] \\ &\quad +sd(z,x_{2n+1}) \\ &\leq s\alpha\Big[\frac{d(x_{2n},x_{2n+1})\,d(x_{2n},Tz)+[d(x_{2n},z)]^2+d(x_{2n},x_{2n+1})d(x_{2n},z)}{d(x_{2n},x_{2n+1})+d(x_{2n},z)+d(x_{2n},Tz)}\Big] \\ &\quad +sd(z,x_{2n+1}). \end{aligned}$$

Now using the condition of normal cone, we have

$$\begin{aligned} \|d(z,Tz)\| &\leq K\Big\{s\alpha\Big\|\Big[\frac{d(x_{2n},x_{2n+1})\,d(x_{2n},Tz)+[d(x_{2n},z)]^2+d(x_{2n},x_{2n+1})d(x_{2n},z)}{d(x_{2n},x_{2n+1})+d(x_{2n},z)+d(x_{2n},Tz)}\Big]\Big\| \\ &\quad +s\|d(z,x_{2n+1})\|\Big\}. \end{aligned}$$

As $n\to\infty$, we have

$$\|d(z,Tz)\| \leq 0.$$

Hence $\|d(z,Tz)\|=0$. Thus we get $Tz=z$, that is, z is a fixed point of T.

In an exactly the same fashion we can prove that $Sz=z$. Hence $Sz=Tz=z$. This shows that z is a common fixed point of S and T.

For the uniqueness of z, let us suppose that $d(x,Sx)+d(x,y)+d(x,Ty)=0$ implies $d(Sx,Ty)=0$ and let w be another fixed point of S and T in X such that $z\neq w$. Then

$$d(z,Sz)+d(z,w)+d(z,Tw)=0 \ \Rightarrow\ d(Sz,Tw)=0.$$

Therefore, we get

$$d(z,w)=d(Sz,Tw)=0,$$

which implies that $z=w$. This shows that z is the unique common fixed point of S and T. This completes the proof. □

If S is a map which has a fixed point p, then p is a fixed point of S^n for every $n\in\mathbb{N}$ too. However, the converse need not to be true. Jeong and Rhoades [12] discussed the situation and gave examples for metric spaces, while Abbas and Rhoades [1] examined this for cone metric spaces. If a map satisfies $F(S)=F(S^n)$ for each $n\in\mathbb{N}$ then it is said to have property P. If $F(S^n)\cap F(T^n)=F(S)\cap F(T)$ then we say that S and T have property P^*.

We examine the property P^* for those mappings which satisfy inequality (2.1).

Theorem 2.2 *Let (X,d) be a complete cone b-metric space (CCbMS) with the coefficient $s\geq 1$ and P be a normal cone with normal constant K. Suppose that the mappings $S,T\colon X\to X$ satisfy (2.1). Then S and T have the property P^*.*

Proof By the above theorem, we know that S and T have a common fixed in X. Let $z \in F(S^n) \cap F(T^n)$. Then

$$\begin{aligned}
d(z,Tz) &= d(S^z, T^{n+1}z) = d(S(S^{n-1}z, T(T^n z)) \\
&\leq \alpha\Big[\Big(d(S^{n-1}z, S^n z)\, d(S^{n-1}z, T(T^n z)) + [d(S^{n-1}z, T^n z)]^2 \\
&\quad + d(S^{n-1}z, S^n z)\, d(S^{n-1}z, T^n z)\Big) \\
&\quad \times \Big(d(S^{n-1}z, S^n z) + d(S^{n-1}z, T^n z) + d(S^{n-1}z, T(T^n z))\Big)^{-1}\Big] \\
&= \alpha\Big[\Big(d(S^{n-1}z, z)\, d(S^{n-1}z, Tz) + [d(S^{n-1}z, z)]^2 \\
&\quad + d(S^{n-1}z, z)\, d(S^{n-1}z, z)\Big) \\
&\quad \times \Big(d(S^{n-1}z, z) + d(S^{n-1}z, z) + d(S^{n-1}z, Tz)\Big)^{-1}\Big] \\
&= \alpha\, d(S^{n-1}z, z) \times \Big[\frac{d(S^{n-1}z, Tz) + 2d(S^{n-1}z, z)}{2d(S^{n-1}z, z) + d(S^{n-1}z, Tz)}\Big] \\
&= \alpha\, d(S^{n-1}z, z).
\end{aligned}$$

Similarly

$$\begin{aligned}
d(S^n z, T^{n+1}z) &\leq \alpha\, d(S^{n-1}z, T^n z) = \alpha\, d(S(S^{n-2}z), T(T^{n-1}z)) \\
&\leq \alpha\Big[\Big(d(S^{n-2}z, S^{n-1}z)\, d(S^{n-2}z, T^n z) + [d(S^{n-2}z, T^{n-1}z)]^2 \\
&\quad + d(S^{n-2}z, S^{n-1}z)\, d(S^{n-2}z, T^{n-1}z)\Big) \\
&\quad \times \Big(d(S^{n-2}z, S^{n-1}z) + d(S^{n-2}z, T^{n-1}z) + d(S^{n-2}z, T^n z)\Big)^{-1}\Big] \\
&= \alpha\Big[\Big(d(S^{n-2}z, S^{n-1}z)\, d(S^{n-2}z, T^n z) + [d(S^{n-2}z, S^{n-1}z)]^2 \\
&\quad + d(S^{n-2}z, S^{n-1}z)\, d(S^{n-2}z, S^{n-1}z)\Big) \\
&\quad \times \Big(d(S^{n-2}z, S^{n-1}z) + d(S^{n-2}z, S^{n-1}z) + d(S^{n-2}z, T^n z)\Big)^{-1}\Big] \\
&= \alpha\, d(S^{n-2}z, S^{n-1}z) \times \Big[\frac{d(S^{n-2}z, T^n z) + 2d(S^{n-2}z, S^{n-1}z)}{2d(S^{n-2}z, S^{n-1}z) + d(S^{n-2}z, T^n z)}\Big] \\
&= \alpha\, d(S^{n-2}z, S^{n-1}z).
\end{aligned}$$

Continuing this process, we get that

$$d(S^n z, T^{n+1}z) \leq \alpha\, d(S^{n-1}z, T^n z) \leq \alpha^2\, d(S^{n-2}z, T^{n-1}z) \leq \cdots \leq \alpha^n\, d(z, Tz).$$

That is,

$$d(z,Tz) \leq \alpha^n\, d(z,Tz).$$

Using (1.1), the above inequality implies that

$$\|d(z,Tz)\| \leq K\alpha^n\, \|d(z,Tz)\| \to 0 \text{ as } n \to \infty.$$

Hence $\|d(z, Tz)\| = 0$. Thus we get $Tz = z$, that is, z is a fixed point of T. By using Theorem 2.1, we get $Sz = z$, and consequently, S and T have property P^*. This completes the proof. □

Putting $S = T$, we have the following result.

Corollary 2.3 *Let (X, d) be a complete cone b-metric space (CCbMS) with the coefficient $s \geq 1$ and P be a normal cone with normal constant K. Suppose that the mappings $T : X \to X$ satisfies the rational contraction:*

$$d(Tx, Ty) \leq \alpha\Big[\frac{d(x, Tx)\, d(x, Ty) + [d(x, y)]^2 + d(x, Tx)d(x, y)}{d(x, Tx) + d(x, y) + d(x, Ty)}\Big] \tag{2.5}$$

for all $x, y \in X$, $\alpha \in [0, 1)$ with $s\alpha < 1$ and $d(x, Tx) + d(x, y) + d(x, Ty) \neq 0$. Then T has a fixed point in X. Further if $d(x, Tx) + d(x, y) + d(x, Ty) = 0$ implies that $d(Tx, Ty) = 0$, then T has a unique fixed point in X.

Proof The proof of Corollary 2.3 immediately follows from Theorem 2.1 by taking $S = T$. This completes the proof. □

Theorem 2.4 *Let (X, d) be a complete cone b-metric space (CCbMS) with the coefficient $s \geq 1$ and P be a normal cone with normal constant K. Suppose that the mapping $T : X \to X$ satisfies (2.5) with $s\alpha < 1$, where $\alpha \in [0, 1)$. Then T has the property P.*

Proof Let $v \in F(T^n)$. Then

$$\begin{aligned}
d(v, Tv) &= d(T^n v, T^{n+1} v) = d(T(T^{n-1}v), T(T^n v)) \\
&\leq \alpha\Big[d(T^{n-1}v, T^n v)\, d(T^{n-1}v, T(T^n v)) + [d(T^{n-1}v, T^n v)]^2 \\
&\quad + d(T^{n-1}v, T^n v) d(T^{n-1}v, T^n v) \\
&\quad \times \big\{d(T^{n-1}v, T^n v) + d(T^{n-1}v, T^n v) + d(T^{n-1}v, T(T^n v))\big\}^{-1}\Big] \\
&= \alpha\Big[d(T^{n-1}v, v)\, d(T^{n-1}v, Tv) + [d(T^{n-1}v, v)]^2 \\
&\quad + d(T^{n-1}v, v) d(T^{n-1}v, v) \\
&\quad \times \big\{d(T^{n-1}v, v) + d(T^{n-1}v, v) + d(T^{n-1}v, Tv)\big\}^{-1}\Big] \\
&= \alpha\, d(T^{n-1}v, v) \times \Big[\frac{d(T^{n-1}v, Tv) + 2d(T^{n-1}v, v)}{2d(T^{n-1}v, v) + d(T^{n-1}v, Tv)}\Big] \\
&= \alpha\, d(T^{n-1}v, v).
\end{aligned}$$

That is

$$d(T^n v, T^{n+1} v) \leq \alpha\, d(T^{n-1}v, T^n v).$$

Similarly

$$\begin{aligned}
d(T^{n-1}v, T^n v) &= d(T(T^{n-2}v), T(T^{n-1}v)) \\
&\leq \alpha\Big[d(T^{n-2}v, T^{n-1}v)\, d(T^{n-2}v, T^n v) + [d(T^{n-2}v, T^{n-1}v)]^2 \\
&\quad +d(T^{n-2}v, T^{n-1}v) d(T^{n-2}v, T^{n-1}v) \\
&\quad \times\big\{d(T^{n-2}v, T^{n-1}v) + d(T^{n-2}v, T^{n-1}v) + d(T^{n-2}v, T^n v)\big\}^{-1}\Big] \\
&= \alpha\Big[d(T^{n-2}v, T^{n-1}v)\, d(T^{n-1}v, v) + [d(T^{n-2}v, T^{n-1}v)]^2 \\
&\quad +d(T^{n-2}v, T^{n-1}v) d(T^{n-2}v, T^{n-1}v) \\
&\quad \times\big\{d(T^{n-2}v, T^{n-1}v) + d(T^{n-2}v, T^{n-1}v) + d(T^{n-2}v, v)\big\}^{-1}\Big] \\
&= \alpha\, d(T^{n-2}v, T^{n-1}v) \times \Big[\frac{d(T^{n-1}v, v) + 2d(T^{n-2}v, T^{n-1}v)}{2d(T^{n-2}v, T^{n-1}v) + d(T^{n-1}v, v)}\Big] \\
&= \alpha\, d(T^{n-2}v, T^{n-1}v).
\end{aligned}$$

Continuing this process, we get

$$d(T^n v, T^{n+1}v) \leq \alpha\, d(T^{n-1}v, T^n v) \leq \alpha^2\, d(T^{n-2}v, T^{n-1}v) \leq \cdots \leq \alpha^n\, d(v, Tv).$$

That is,

$$d(v, Tv) \leq \alpha^n\, d(v, Tv).$$

Using (1.1), the above inequality implies that

$$\|d(v, Tv)\| \leq K\alpha^n\, \|d(v, Tv)\| \to 0 \text{ as } n \to \infty.$$

Hence $\|d(v, Tv)\| = 0$. Thus we get $Tv = v$. Thus we conclude that a mapping which satisfies (2.5) has the property P. This completes the proof. □

§3. Applications

The aim of this section is to apply our result to mappings involving contraction of integral type. For this purpose, denote Λ the set of functions $\varphi\colon [0,\infty) \to [0,\infty)$ satisfying the following hypothesis:

(h1) φ is a Lebesgue-integrable mapping on each compact subset of $[0,\infty)$;

(h2) for any $\varepsilon > 0$ we have $\int_0^{\varepsilon} \varphi(t)\, dt > 0$.

Theorem 3.1 *Let (X, d) be a complete cone b-metric space (CCbMS) with the coefficient $s \geq 1$ and P be a normal cone with normal constant K. Suppose that the mappings $S, T\colon X \to X$*

satisfy the contraction of integral type:

$$\int_0^{d(Sx,Ty)} \psi(t)\,dt \quad \le \quad \alpha \int_0^{\left[\frac{d(x,Sx)\,d(x,Ty)+[d(x,y)]^2+d(x,Sx)d(x,y)}{d(x,Sx)+d(x,y)+d(x,Ty)}\right]} \psi(t)\,dt$$

for all $x, y \in X$, $\alpha \in [0,1)$ with $s\alpha < 1$ and $\psi \in \Lambda$. Then S and T have a unique common fixed point in X.

If we put $S = T$ in Theorem 3.1, we have the following result.

Theorem 3.2 *Let (X, d) be a complete cone b-metric space (CCbMS) with the coefficient $s \ge 1$ and P be a normal cone with normal constant K. Suppose that the mapping $T : X \to X$ satisfies the contraction of integral type:*

$$\int_0^{d(Tx,Ty)} \psi(t)\,dt \quad \le \quad \alpha \int_0^{\left[\frac{d(x,Tx)\,d(x,Ty)+[d(x,y)]^2+d(x,Tx)d(x,y)}{d(x,Tx)+d(x,y)+d(x,Ty)}\right]} \psi(t)\,dt$$

for all $x, y \in X$, $\alpha \in [0,1)$ with $s\alpha < 1$ and $\psi \in \Lambda$. Then T has a unique fixed point in X.

§4. **Conclusion**

In this paper, we establish some unique common fixed point theorems for rational contraction in the setting of cone b-metric spaces with normal solid cone. Also, as an application of our result, we obtained some results of integral type for such mappings. Our results extend and generalize several results from the existing literature.

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