International J.Math. Combin. Vol.1(2016), 109-115

Nonholonomic Frames for Finsler Space with (α,β)–Metrics

Brijesh Kumar Tripathi[1], K.B.Pandey[2] and R.B. Tiwari[3]

1. Department of Mathematics, L.E. College, Morbi(Gujrat) 363642, India
2. Department of Mathematics, K.N.I.T., Sultanpur,U.P., 228118, India
3. Department of Mathematics, M.K.R.E.C.I.T., Ambedkar Nagar(U.P.), India

E-mail: brijeshkumartripathi4@gmail.com, kunjbiharipandey05@gmail.com, tiwarirambharat@yahoo.in

Abstract: The purpose of present paper to determine the two special Finsler spaces due to deformations of some special Finsler space with help of (α,β)-metrics. Consequently, we obtain the non-holonomic frame for the (α,β)-metrics, such as (I) $L = \left(\frac{\alpha^2}{\alpha-\beta}\right)\frac{\beta^2}{\alpha} = \frac{\alpha\beta^2}{(\alpha-\beta)}$ i.e. product of Matsumoto metric and Kropina metric and (II) $L = (\alpha+\beta)\frac{\beta^2}{\alpha} = \beta^2 + \frac{\beta^3}{\alpha}$ i.e. product of Randers metric and Kropina metric.

Key Words: Finsler Space, (α,β)–metrics, Randers metric, Kropina metric, Matsumoto metric, GL-metric, Non-holonomic Finsler frame.

AMS(2010): 53C60.

§1. Introduction

In 1982, P.R. Holland [1] and [2] studies a unified formalism that uses a nonholonomic frame on space time arising from consideration of a charged particle moving in an external electromagnetic field. In fact, R.S. Ingarden [3] was the first to point out that the Lorentz force law can be written in this case as geodesic equation on a Finsler space called Randers space. The author R.G. Beil [5], [6] have studied a gauge transformation viewed as a nonholonomic frame on the tangent bundle of a four dimensional base manifold. The geometry that follows from these considerations gives a unified approach to gravitation and gauge symmetries. In the present paper we have used the common Finsler idea to study the existence of a nonholonomic frame on the vertical sub bundle VTM of the tangent bundle of a base manifold M.

In this paper, the fundamental tensor field of a Finsler space might be considered as the deformations of two different special Finsler spaces from the (α,β)-metrics. Further we obtain corresponding frame for each of these two Finsler deformations. Consequently, a nonholonomic frame for a Finsler space with special (α,β)-metrics such as first is the product of Matsumoto metric[11] and kropina metric[11] and second is the product of Randers metric[11] and Kropina metric. This is an extension work of Ioan Bucataru and Radu Miron [10] and also second extension work of S.K. Narasimhamurthy [14].

Consider, $a_{ij}(x)$ the components of a Riemannian metric on the base manifold M, $a(x,y) >$

[1]Received August 12, 2015, Accepted February 26, 2016.

0 and $b\,(x,y) \geq 0$ Two functions on TM and $B\,(x,y) = B_i\,(x,y)\,(dx^i)$ a vertical 1-form on TM. Then

$$g_{ij}(x,y) = a(x,y)a_{ij}(x) + b(x,y)B_i(x,y)B_j(x,y) \tag{1.1}$$

is a generalized Lagrange metric, called the Beil metric. The metric tensor g_{ij} is also known as a Beil deformation of the Riemannian metric a_{ij}. It has been studied and applied by R. Miron and R.K. Tavakol in General Relativity for $a(x,y) = exp(2\sigma(x,y))$ and b = 0. The case a(x, y) = 1 with various choices of b and B_i was introduced and studied by R.G. Beil for constructing a new unified field theory [6].

§2. Preliminaries

An important class of Finsler spaces is the class of Finsler spaces with(α,β)–metrics [11]. The first Finsler spaces with (α,β)–metrics were introduced by the physicist G.Randers in 1940, are called Randers spaces [4]. Recently, R.G. Beil suggested a more general case and considered the class of Lagrange spaces with (α,β)–metric, which was discussed in [12]. A unified formalism which uses a nonholonomic frame on space time, a sort of plastic deformation, arising from consideration of a charged particle moving in an external electromagnetic field in the background space time viewed as a strained mechanism studied by P. R. Holland [1], [2]. If we do not ask for the function L to be homogeneous of order two with respect to the (α,β) variables, then we have a Lagrange space with (α,β)–metric. Next we defined some different Finsler space with (α,β)–metrics.

Definition 2.1 *A Finsler space $F^n = (M, F(x,y))$ is called with (α,β)–metric if there exists a 2-homogeneous function L of two variables such that the Finsler metric $F : TM \to R$ is given by*

$$F^2(x,y) = L(\alpha(x,y), \beta(x,y)), \tag{2.1}$$

where $\alpha^2(x,y) = a_{ij}(x)y^iy^j, \alpha$ is a Riemannian metric on the manifold M, and $\beta(x,y) = b_i(x)y^i$ is a 1-form on M.

Consider $g_{ij} = \dfrac{1}{2}\dfrac{(\partial^2 F^2)}{(\partial y^i \partial y^j)}$ the fundamental tensor of the Randers space(M,F). Taking into account the homogeneity of a and F we have the following formulae:

$$\begin{gathered}
p^i = \frac{1}{a}y^i = a^{ij}\frac{\partial\alpha}{\partial y^j}; \quad p_i = a_{ij}p^j = \frac{\partial\alpha}{\partial y^i};\\
l^i = \frac{1}{L}y^i = g^{ij}\frac{\partial l}{\partial y^i}; l_i = g_{ij}l^j = \frac{\partial L}{\partial y^i} = P_i + b_i,\\
l^i = \frac{1}{L}p^i; l^il_i = p^ip_i = 1; l^ip_i = \frac{\alpha}{L}; p^il_i = \frac{L}{\alpha};\\
b_iP^i = \frac{\beta}{\alpha}; b_il^i = \frac{\beta}{L}
\end{gathered} \tag{2.2}$$

with respect to these notations, the metric tensors (α_{ij}) and (g_{ij}) are related by [13],

$$g_{ij}(x,y) = \frac{L}{\alpha}a_{ij} + b_iP_j + P_ib_j - \frac{\beta}{\alpha}p_ip_j = \frac{L}{\alpha}(a_{ij} - p_ip_j) + l_il_j. \tag{2.3}$$

Theorem 2.1([10]) *For a Finsler space (M,F) consider the matrix with the entries:*

$$Y_i^j = \sqrt{\frac{\alpha}{L}}(\delta_j^i - l^il_j + \sqrt{\frac{\alpha}{L}}p^ip_j) \tag{2.4}$$

defined on TM. Then $Y_j = Y_j^i(\frac{\partial}{\partial y^i}), j \in 1,2,3,\ldots,n$ *is a non holonomic frame.*

Theorem 2.2([7]) *With respect to frame the holonomic components of the Finsler metric tensor* $\alpha_{\alpha\beta}$ *is the Randers metric* g_{ij}*, i.e,*

$$g_{ij} = Y_i^\alpha Y_j^\beta \alpha_{\alpha\beta}. \tag{2.5}$$

Throughout this section we shall rise and lower indices only with the Riemannian metric $\alpha_{ij}(x)$ that is $y_i = \alpha_{ij}y^j, \beta^i = \alpha^{ij}\beta_j$, and so on. For a Finsler space with (α,β)–metric $F^2(x,y) = L(\alpha(x,y), \beta(x,y))$ we have the Finsler invariants [13]

$$\rho_1 = \frac{1}{2\alpha}\frac{\partial L}{\partial\alpha}; \rho_0 = \frac{1}{2}\frac{\partial^2 L}{\partial\beta^2}; \rho_{-1} = \frac{1}{2\alpha}\frac{\partial^2 L}{\partial\alpha\partial\beta}; \rho_{-2} = \frac{1}{2\alpha^2}(\frac{\partial^2 L}{\partial\alpha^2} - \frac{1}{\alpha}\frac{\partial L}{\partial\alpha}) \tag{2.6}$$

where subscripts 1, 0, -1, -2 gives us the degree of homogeneity of these invariants.

For a Finsler space with metric we have,

$$\rho_{-1}\beta + \rho_{-2}\alpha^2 = 0 \tag{2.7}$$

With respect to the notations we have that the metric tensor g_{ij} of a Finsler space with (α,β)–metric is given by [13]

$$g_{ij}(x,y) = \rho\alpha_{ij}(x) + \rho_0 b_i(x) + \rho_{-1}(b_i(x)y_j + b_j(x)y_i) + \rho_{-2}y_iy_j. \tag{2.8}$$

From (2.8) we can see that $g_{ij}$is the result of two Finsler deformations

$$I. \qquad a_{ij} \to h_{ij} = \rho a_{ij} + \frac{1}{\rho_{-2}}(\rho_{-1}b_i + \rho_{-2}y_i)(\rho_{-1}b_j + \rho_{-2}y_j)$$

$$II. \quad h_{ij} \to g_{ij} = h_{ij} + \frac{1}{\rho_{-2}}(\rho_0\rho_{-1} - \rho_{-1}^2)b_ib_j. \tag{2.9}$$

The nonholonomic Finsler frame that corresponding to the I^{st} deformation (2.9) is according to the Theorem 7.9.1 in [10], given by

$$X_j^i = \sqrt{\rho}\delta_j^i - \frac{1}{\beta^2}(\sqrt{\rho} + \sqrt{\rho + \frac{\beta^2}{\rho_{-2}}})(\rho_{-1}b^i + \rho_{-2}y^i)(\rho_{-1}b_j + \rho_{-2}y_j), \tag{2.10}$$

where $B^2 = a_{ij}(\rho_{-1}b^i + \rho_{-2}y^i)(\rho_{-1}b_j + \rho_{-2}y_j) = \rho_{-1}^2 b^2 + \beta\rho_{-1}\rho_{-2}$.

This metric tensor a_{ij} and h_{ij} are related by,

$$h_{ij} = X_i^k X_j^l a_{kl}. \tag{2.11}$$

Again the frame that corresponds to the II_{nd} deformation (2.9) given by,

$$Y_j^i = \delta_j^i - \frac{1}{C^2}\left(1 \pm \sqrt{1 + \left(\frac{\rho_{-2}C^2}{\rho_0\rho_{-2-\rho_{-1}^2}}\right)}\right) b^i b_j, \tag{2.12}$$

where $C^2 = h_{ij}b^i b^j = \rho b^2 + \frac{1}{\rho_{-2}}(\rho_{-2}b^2 + \rho_{-2}\beta)^2$.

The metric tensor h_{ij} and g_{ij} are related by the formula

$$g_{mn} = Y_m^i Y_n^j h_{ij}. \tag{2.13}$$

Theorem 2.3([10]) *Let $F^2(x, y) = L(\alpha(x, y), \beta(x, y))$ be the metric function of a Finsler space with (α,β) metric for which the condition (2.7) is true. Then*

$$V_j^i = X_k^i Y_j^k$$

is a nonholonomic Finsler frame with X_k^i and Y_j^k are given by (2.10) and (2.12) respectively.

§3. Nonholonomic Frames for Finsler Space with (α,β)–Metrics

In this section we consider two cases of non-holonomic Finlser frames with (α,β)-metrics, such a I^{st} Finsler frame product of Matusmoto metric and Kropina metric and II^{nd} Finsler frame product of Randers metric and Kropina metric.

(I) Nonholonomic Frames for $L = \left(\frac{\alpha^2}{\alpha-\beta}\right)\frac{\beta^2}{\alpha} = \frac{\alpha\beta^2}{\alpha-\beta}$

In the first case, for a Finsler space with the fundamental function

$$L = \left(\frac{\alpha^2}{\alpha-\beta}\right)\frac{\beta^2}{\alpha} = \frac{\alpha\beta^2}{\alpha-\beta},$$

the Finsler invariants (2.6) are given by

$$\rho_1 = -\frac{\beta^3}{2\alpha(\alpha-\beta)^2}; \rho_0 = \frac{1}{2}\frac{(2\alpha^3 - \alpha\beta^2)}{(\alpha-\beta)^3};$$

$$\rho_{-1} = \frac{1}{2\alpha}\frac{\beta^2(\beta-3\alpha)}{(\alpha-\beta)^3}; \rho_{-2} = \frac{\beta^3(3\alpha-\beta)}{2\alpha^3(\alpha-\beta)^3};$$

$$B^2 = \frac{\beta^2(1-3\alpha)^2 b^2 + \beta^5(\alpha-\beta)(1-3\alpha)(3\alpha-\beta)}{4\alpha^4(\alpha-\beta)^6}. \tag{3.1}$$

Using (3.1) in (2.10) we have,

$$X^i_j = \sqrt{-\frac{\beta^3}{2\alpha(\alpha-\beta)^2}}\quad \delta^i_j - \frac{\beta^4}{4\alpha^4(\alpha-\beta)^5}\left[\sqrt{\frac{-\beta^3}{2\alpha}} + \sqrt{\frac{4\alpha^4(\alpha-\beta)^5-\beta^4(3\alpha-\beta)}{2\alpha\beta(3\alpha-\beta)}}\right] \times(b^i - \frac{(3\alpha-\beta)}{\alpha^2(\alpha-\beta)}y^i)(b_j - \frac{(3\alpha-\beta)}{\alpha^2(\alpha-\beta)}y_j). \tag{3.2}$$

Again using (3.1) in (2.12) we have,

$$Y^i_j = \delta^i_j - \frac{1}{C^2}\left(1 \pm \sqrt{1+\frac{2(\alpha-\beta)^3C^2}{\alpha^2(2\alpha-3\beta)}}\right) b^i b_j, \tag{3.3}$$

where $C^2 = -\frac{\beta^3}{2\alpha(\alpha-\beta)^2}b^2 + \frac{\beta(3\alpha-\beta)}{2\alpha^3(\alpha-\beta)^3}(\alpha^2b^2-\beta^2)^2.$

Theorem 3.1 *Consider Finsler space* $L = \left(\frac{\alpha^2}{\alpha-\beta}\right)\frac{\beta^2}{\alpha} = \frac{\alpha\beta^2}{\alpha-\beta}$, *for which the condition (2.7) is true. Then*

$$V^i_j = X^i_k Y^k_j$$

is non-holomic Finsler Frame with X^i_k *and* Y^k_j *are given by (3.2) and (3.3) respectively.*

(II) Nonholonomic Frames for $L = (\alpha+\beta)(\frac{\beta^2}{\alpha}) = \beta^2 + \frac{\beta^3}{\alpha}$

In the second case, for a Finsler space with the fundamental function $L = (\alpha+\beta)(\frac{\beta^2}{\alpha})$ the Finsler invariants (2.6) are given by:

$$\rho_1 = -\frac{\beta^3}{2\alpha^3}; \rho_0 = \frac{3\beta+\alpha}{\alpha}; \rho_{-1} = -\frac{3}{2}\frac{\beta^2}{\alpha^3}; \rho_{-2} = \frac{3}{2}\frac{\beta^3}{\alpha^5};$$

$$B^2 = \frac{9}{4}\frac{\beta^4}{\alpha^8}(\alpha^2b^2-\beta^2), \tag{3.4}$$

$$X^i_j = \sqrt{-\frac{\beta^3}{2\alpha^3}}\quad \delta^i_j - \frac{9}{4}\frac{\beta^2}{\alpha^6}\left[\sqrt{-\frac{\beta^3}{2\alpha^3}} + \sqrt{-\frac{\beta^3}{2\alpha^3}+\frac{2}{3}\frac{\alpha^5}{\beta}}\right](b^i - \frac{\beta}{\alpha^2}y^i)(b_j - \frac{\beta}{\alpha^2}y_j). \tag{3.5}$$

Again using (3.4) in (2.12) we have

$$y^i_j = \delta^i_j - \frac{1}{c^2}\left[1 \pm \sqrt{1+(\frac{2\alpha c^2}{2\alpha+3\beta}}\right] b^i b_j, \tag{3.6}$$

where $C^2 = -\frac{\beta^3}{2\alpha^3}b^2 + \frac{3}{2}\frac{\beta}{\alpha^5}[\alpha^2b^2-\beta^2]^2.$

Theorem 3.2 *Consider a Finsler space* $L = (\alpha+\beta)(\frac{\beta^2}{\alpha}) = \beta^2 + \frac{\beta^3}{\alpha}$, *for which the condition 2.7 is true. Then*

$$V^i_j = X^i_k Y^k_j$$

is non-holomic Finsler Frame with X^i_k *and* Y^k_j *are given by (3.5) and (3.6) respectively.*

§4. **Conclusions**

Non-holonomic frame relates a semi-Riemannian metric (the Minkowski or the Lorentz metric) with an induced Finsler metric. Antonelli P.L., Bucataru I. ([7][8]), has been determined such a non-holonomic frame for two important classes of Finsler spaces that are dual in the sense of Randers and Kropina spaces [9]. As Randers and Kropina spaces are members of a bigger class of Finsler spaces, namely the Finsler spaces with(α,β)–metric, it appears a natural question: Does how many Finsler space with(α,β)–metrics have such a nonholonomic frame? The answer is yes, there are many Finsler space with(α,β)–metrics.

In this work, we consider the two special Finsler metrics and we determine the non-holonomic Finsler frames. Each of the frames we found here induces a Finsler connection on TM with torsion and no curvature. But, in Finsler geometry, there are many(α,β)–metrics, in future work we can determine the frames for them also.

References

[1] Holland P.R., Electromagnetism, Particles and Anholonomy, *Physics Letters*, 91 (6)(1982), 275-278.

[2] Holland P.R., Anholonomic deformations in the ether: a significance for the electrodynamic potentials. In: Hiley, B.J. Peat, F.D. (eds.), *Quantum Implications*, Routledge and Kegan Paul, London and New York, 295-311 (1987).

[3] Ingarden R.S., On physical interpretations of Finsler and Kawaguchi spaces, *Tensor N.S.*, 46(1987), 354-360.

[4] Randers G., On asymmetric metric in the four space of general relativity, *Phys. Rev.*, 59 (1941), 195-199.

[5] Beil R.G., Comparison of unified field theories, *Tensor N.S.*, 56(1995), 175-183.

[6] Beil R.G., Equations of motion from Finsler geometric methods, In Antonelli, P.L. (ed), *Finslerian Geometries*, A meeting of minds, Kluwer Academic Publisher, FTPH, No.109(2000), 95-111.

[7] Antonelli P.L., Bucataru I., On Holland's frame for Randers space and its applications in physics, In: Kozma, L. (ed), *Steps in Differential Geometry*, Proceedings of the Colloquium on Differential Geometry, Debrecen, Hungary, July 25-30, 2000, Debrecen: Univ. Debrecen, Institute of Mathematics and Informatics, 39-54, (2001).

[8] Antonelli P.L., Bucataru I., Finsler connections in anholonomic geometry of a Kropina space, *Nonlinear Studies*, 8 (1)(2001), 171-184.

[9] Hrimiuc D., Shimada H., On the L-duality between Lagrange and Hamilton manifolds, *Nonlinear World*, 3 (1996), 613-641.

[10] Ioan Bucataru, Radu Miron, Finsler-Lagrange geometry applications to dynamical systems *CEEX ET 3174/2005-2007*, and *CEEX M III 12595/2007 (2007).*

[11] Matsumoto M., Theory of Finsler spaces with $(\alpha;\beta)$-metric, *Rep. Math. Phys.*, 31, 43-83, (1992).

[12] Bucataru I., Nonholonomic frames on Finsler geometry, *Balkan Journal of Geometry and*

its Applications, 7 (1)(2002), 13-27.

[13] Matsumoto M, *Foundations of Finsler Geometry and Special Finsler Spaces*, Kaishesha Press, Otsu, Japan, 1986.

[14] Narasimhamurthy S.K., Mallikarjun Y.Kumar, Kavyashree A. R., Nonholonomic frames for Finsler space with special $(\alpha;\beta)$-metric, *International Journal of Scientific and Research Publications*, 4(1)(2014), 1-7.