International J.Math. Combin. Vol.1(2015), 49-56

Totally Umbilical Hemislant Submanifolds of Lorentzian (α)-Sasakian Manifold

Barnali Laha and Arindam Bhattacharyya

(Department of Mathematics, Jadavpur University, Kolkata, India)

E-mail: barnali.laha87@gmail.com, bhattachar1968@yahoo.co.in

Abstract: This paper is summarized as follows. In the first section we have given a brief history about slant and hemi-slant submanifold of Lorentzian (α)-Sasakian manifold. This section is followed by some preliminaries about Lorentzian (α)-Sasakian manifold. Finally, we have derived some interesting results on the existence of extrinsic sphere for totally umbilical hemi-slant submanifold of Lorentzian (α)-Sasakian manifold.

Key Words: Totally Umbilical, hemi-slant submanifold, extrinsic sphere.

AMS(2010): 53C25, 53C40, 53C42, 53D15

§1. Introduction

Chen in 1990 [2] initiated the study of slant submanifold of an almost Hermitian manifold as a natural generalization of both holomorphic and totally real submanifolds. After this many research papers on slant submanifolds appeared. The notion of slant immersion of a Riemannian manifold into an almost contact metric manifold was introduced by A. Lotta in 1996 [5]. He studied the intrinsic geometry of 3-dimensional non-anti-invariant slant submanifolds of K-contact manifold. Further investigation regarding slant submanifolds of a Sasakian manifold [8] was done by Cabrerizo et al. in 2000. Khan et al. in 2010 defined and studied slant submanifolds in Lorentzian almost paracontact manifolds [14].

The idea of hemislant submanifold was introduced by Carriazo as a particular class of bislant submanifolds, and he called them antislant submanifolds in [9]. Recently, in 2009 totally umbilical slant submanifolds of Kaehler manifold was studied by B.Sahin. Later on, in 2011 Siraj Uddin et.al. studied totally umbilical proper slant and hemislant submanifolds of an LP-cosymplectic manifold [21].

Our present note deals with a special kind of manifold i.e. Lorentzian (α)-Sasakian manifold. At first we give some introduction about the development of such manifold. An almost contact metric structure (ϕ, ξ, η, g) on $\tilde{M}$ is called a trans-Sasakian structure [17] if (MXR, J, G) belongs to the class W_4 [11], where J is the almost complex structure on (MXR) defined by

$$(J, X\frac{d}{dt}) = (\phi X - f, \eta(X)\frac{d}{dt})$$

[1]Received May 23, 2014, Accepted February 20, 2015.

for all vector fields X on M and smooth functions f on $M \times R$, G is the product metric on MXR. This may be expressed by the condition

$$(\tilde{\nabla}_X \phi)Y = \alpha[g(X,Y)\xi + \eta(Y)X] + \beta[g(\phi X, Y) - \eta(Y)\phi X],$$

for some smooth functions α and β on M in [1], and we say that the trans-Sasakian structure is of type (α, β). A trans-Sasakian structure of type (α, β) is α-Sasakian, if $\beta = 0$ and α a nonzero constant [13]. If $\alpha = 1$, then α-Sasakian manifold is a Sasakian manifold. Also in 2008 and 2009 many scientists have extended the study to Lorentzian (α)-Sasakian manifold in [22], [18]. In this paper we have studied some special properties of totally umbilical hemislant submanifolds of Lorentzian (α)-Sasakian manifold.

§2. Preliminaries

An n-dimensional Lorentzian manifold M is a smooth connected paracontact Hausdorff manifold with a Lorentzian metric g, that is, M admits a smooth symmetric tensor field g of type $(0,2)$ such that for each point $p \in M$, the tensor $g_p : T_pM \times T_pM \longmapsto \mathbf{R}$ is a non-degenerate inner product of signature $(-,+,+,\cdots,+)$, where T_pM denotes the tangent vector space of M at p and $\mathbf{R}$ is the real number space. A non-zero vector $v \in T_pM$ is said to be timelike if it satisfies $g_p(v,v) < 0$ [16]. Let $\tilde{M}$ be an n-dimensional differentiable manifold. An almost paracontact structure $(\phi, \xi, \eta, \tilde{g})$, where ϕ is a tensor of type $(1,1)$, ξ is a vector field, η is a 1-form and g is Lorentzian metric, satisfying following properties :

$$\phi^2 X = X + \eta(X)\xi, \quad \eta \circ \phi = 0, \quad \phi\xi = 0, \quad \eta(\xi) = -1, \tag{2.1}$$

$$g(\phi X, \phi Y) = g(X,Y) + \eta(X)\eta(Y), g(X,\xi) = \eta(X). \tag{2.2}$$

for all vector fields X, Y on $\tilde{M}$. On $\tilde{M}$ if the following additional condition hold for any $X, Y \in T\tilde{M}$,

$$(\tilde{\nabla}_X \phi)Y = \alpha[g(X,Y)\xi + \eta(Y)X], \tag{2.3}$$

$$\tilde{\nabla}_X \xi = \alpha\phi X, \tag{2.4}$$

where $\tilde{\nabla}$ is the Levi-Civita connection on $\tilde{M}$, then $\tilde{M}$ is said to be an Lorentzian α-Sasakian manifold (Matsumoto, 1989 [15], [22]).

Let M be a submanifold of $\tilde{M}$ with Lorentzian almost paracontact structure (ϕ, ξ, η, g) with induced metric g and let ∇ is the induced connection on the tangent bundle TM and $\nabla^\perp$ is the induced connection on the normal bundle $T^\perp M$ of M.

The Gauss and Weingarten formulae are characterized by

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X,Y), \tag{2.5}$$

$$\tilde{\nabla}_X N = -A_N X + \nabla^\perp_X N, \tag{2.6}$$

for any $X, Y \in TM$, $N \in T^\perp M$, h is the second fundamental form and A_N is the Weingarten map associated with N via

$$g(A_N X, Y) = g(h(X, Y), N). \tag{2.7}$$

For any $X \in \Gamma(TM)$ we can write,

$$\phi X = TX + FX, \tag{2.8}$$

where TX is the tangential component and FX is the normal component of ϕX. Similarly for any $N \in \Gamma(T^\perp M)$ we can put

$$\phi V = tV + fV, \tag{2.9}$$

where tV denote the tangential component and fV denote the normal component of ϕV. The covariant derivatives of the tensor fields T and F are defined as

$$(\tilde{\nabla}_X \phi)Y = \tilde{\nabla}_X \phi Y - \phi\tilde{\nabla}_X Y \qquad \forall \; X, Y \in T\tilde{M}, \tag{2.10}$$

$$(\tilde{\nabla}_X T)Y = \nabla_X TY - T\nabla_X Y \qquad \forall \; X, Y \in TM, \tag{2.11}$$

$$(\tilde{\nabla}_X F)Y = \nabla^\perp_X FY - F\nabla_X Y, \quad \forall \; X, Y \in TM. \tag{2.12}$$

From equation (2.3), (2.5), (2.8), (2.9), (2.11) and (2.12) we can calculate

$$(\tilde{\nabla}_X T)Y = \alpha[g(X, Y)\xi + \eta(Y)X] + A_{FY}X + th(X, Y), \tag{2.13}$$

$$(\tilde{\nabla}_X F)Y = -h(X, TY) + fh(X, Y). \tag{2.14}$$

A submanifold M is said to be invariant if F is identically zero, i.e., $\phi X \in \Gamma(TM)$ for any $X \in \Gamma(TM)$. On the other hand, M is said to be anti-invariant if T is identically zero, i.e., $\phi X \in \Gamma(T^\perp M)$ for any $X \in \Gamma(TM)$.

A submanifold M of $\tilde{M}$ is called totally umbilical if

$$h(X, Y) = g(X, Y)H, \tag{2.15}$$

for any $X, Y \in \Gamma(TM)$. The mean curvature vector H is denoted by $H = \sum_{i=1}^{k} h(e_i, e_i)$, where k is the dimension of M and $\{e_1, e_2, e_3, \cdots, e_k\}$ is the local orthonormal frame on M. A submanifold M is said to be totally geodesic if $h(X, Y) = 0$ for each $X, Y \in \Gamma(TM)$ and is minimal if $H = 0$ on M.

§3. Slant Submanifolds of a Lorentzian (alpha)-Sasakian Manifold

Here, we consider M as a proper slant submanifold of a Lorentzian (α)-Sasakian manifold $\tilde{M}$. We always consider such submanifold tangent to the structure vector field ξ.

Definition 3.1 *A submanifold M of $\tilde{M}$ is said to be slant submanifold if for any $x \in M$ and $X \in T_x M \setminus \xi$, the angle between ϕX and $T_x M$ is constant. The constant angle $\theta \in [0, \pi/2]$ is then called slant angle of M in $\tilde{M}$. If $\theta = 0$ the submanifold is invariant submanifold, if $\theta = \pi/2$*

then it is anti-invariant submanifold and if $\theta \neq 0, \pi/2$ then it is proper slant submanifold.

From [20] we have

Theorem 3.1 *Let M be a submanifold of an Lorentzian (α)-Sasakian manifold $\tilde{M}$ such that $\xi \in TM$. Then M is slant submanifold if and only if there exists a constant $\lambda \in [0, 1]$ such that*

$$T^2 = \lambda(I + \eta \otimes \xi). \tag{3.1}$$

Again, if θ is slant angle of M, then $\lambda = \cos^2 \theta$.

From [20], for any X, Y tangent to M, we can easily draw the following results for an Lorentzian (α)-Sasakian manifold $\tilde{M}$,

$$g(TX, TY) = cos^2\theta\{g(X, Y) + \eta(X)\eta(Y)\}, \quad g(FX, FY) = sin^2\theta\{g(X, Y) + \eta(X)\eta(Y)\}.$$

Definition 3.2 *A submanifold M of $\tilde{M}$ is said to be hemi-slant submanifold of a Lorentzian (α)-Sasakian manifold $\tilde{M}$ if there exists two orthogonal distribution D_1 and D_2 on M such that*

(*a*) $TM = D_1 \oplus D_2 \oplus < \xi >$;

(*b*) *The distribution D_1 is anti-invariant i.e., $\phi D_1 \subseteq T^{\perp}M$;*

(*c*) *The distribution D_2 is slant with slant angle $\theta \neq \pi/2$.*

If μ is invariant subspace under ϕ of the normal bundle $T^{\perp}M$, then in the case of hemi-slant submanifold, the normal bundle $T^{\perp}M$ decomposes as

$$T^{\perp}M = < \mu > \oplus \phi D^{\perp} \oplus FD_{\theta}.$$

The curvature tensor of an Lorentzian (α)-Sasakian manifold is defined as [4]

$$\tilde{R}(X, Y)Z = \tilde{\nabla}_X \tilde{\nabla}_Y Z - \tilde{\nabla}_Y \tilde{\nabla}_X Z - \tilde{\nabla}_{[X,Y]} Z. \tag{3.2}$$

For the curvature tensor we can compute by using the equations (2.10) and (3.2) the relation

$$\begin{aligned} \tilde{R}(X, Y)\phi Z &= \phi\tilde{R}(X, Y)Z + \alpha^2 g(Y, Z)\phi X - \alpha^2 g(X, Z)\phi Y \\ &-\alpha^2 g([X, Y], Z)\phi X + \alpha g(X, \tilde{\nabla}_Y Z)\xi + \alpha\eta(\tilde{\nabla}_Y Z)X \\ &-\alpha g(Y, \tilde{\nabla}_X Z)\xi - \alpha\eta(\tilde{\nabla}_X Z)Y - \alpha\eta(Z)\tilde{\nabla}_X Y \\ &+\alpha\eta(Z)\tilde{\nabla}_Y X - \alpha\eta(Z)[X, Y] + \alpha g(\tilde{\nabla}_X Y, Z)\xi + \alpha g(\tilde{\nabla}_Y X, Z)\xi. \end{aligned} \tag{3.3}$$

Definition 3.3 *A submanifold of an arbitrary Lorentzian (α)-Sasakian manifold which is totally umbilical and has a nonzero parallel mean curvature vector [10] is called an Extrinsic sphere.*

§4. **Main Results**

This section mainly deals with a special class of hemi-slant submanifolds which are totally umbilical. Throughout this section we have considered M as a totally umbilical hemi-slant submanifold of Lorentzian (α)-Sasakian manifold. We derive the following.

Theorem 4.1 *Let M be a totally umbilical hemi-slant submanifold of a Lorentzian (α)-Sasakian manifold $\tilde{M}$ such that the mean curvature vector $H \in <\mu>$. Then one of the following is true:*

(i) M is totally geodesic;

(ii) M is semi-invariant submanifold.

Proof For $V \in \phi D^{\perp}$ and $X \in D_{\theta}$, we have from (2.3), (2.5),(2.6) and (2.10)

$$\alpha[g(X,V)\xi + \eta(V)X] = \nabla_X \phi V + g(X,\phi V)H + \phi A_V X - \phi \nabla^{\perp}_X V. \tag{4.1}$$

Since the distributions are orthogonal and from the assumption that $H \in \mu$, above equation can be written as

$$g(\nabla^{\perp}_X V, H) = g(V, \nabla^{\perp}_X H) = 0. \tag{4.2}$$

This implies $\nabla^{\perp}_X H \in \mu \oplus FD_{\theta}$. Now for any $X \in D_{\theta}$, we obtain on using the Gauss and Weingarten equations

$$\alpha[g(X,H)\xi + \eta(H)X] = \nabla^{\perp}_X \phi H - A_{\phi H} X + \phi A_H X - \phi \nabla^{\perp}_X H. \tag{4.3}$$

Now, using the assumption that , M is totally umbilical we have

$$\alpha\eta(H)X = \nabla^{\perp}_X \phi H - Xg(H,\phi H) + \phi Xg(H,H) - \phi \nabla^{\perp}_X H. \tag{4.4}$$

On using equation (2.8) we calculate

$$\alpha\eta(H)X = \nabla^{\perp}_X \phi H + TXg(H,H) + FXg(H,H) - \phi \nabla^{\perp}_X H. \tag{4.5}$$

Taking inner product with $FX \in FD_{\theta}$,

$$\alpha\eta(H)g(X,FX) = g(\nabla^{\perp}_X \phi H, FX) + g(FX,FX)g(H,H) - g(\phi \nabla^{\perp}_X H, FX). \tag{4.6}$$

From Theorem 3.1 the equation becomes

$$\alpha\eta(H)g(X,FX) - g(\nabla^{\perp}_X \phi H, FX) - sin^2\theta ||H||^2 ||X||^2 + g(\phi \nabla^{\perp}_X H, FX) = 0. \tag{4.7}$$

If either $H \neq 0$ then $D_{\theta} = \{0\}$, i.e. M is totally real submanifold, and if $D_{\theta} \neq \{0\}$, M is totally geodesic submanifold or M is semi-invariant submanifold. For any $Z \in D^{\perp}$ from (2.13) we get

$$\nabla_Z TZ - T\nabla_Z Z = \alpha[g(Z,Z)\xi + \eta(Z)Z] + A_{FZ}Z + th(Z,Z). \tag{4.8}$$

Taking inner product with $W \in D^{\perp}$ the above equation takes the form

$$\begin{aligned} g(\nabla_Z TZ, W) - g(T\nabla_Z Z, W) &= \alpha[g(Z,Z)g(\xi,W) + \eta(Z)g(Z,W)] \\ &\quad + g(A_{FZ}Z, W) + g(th(Z,Z), W). \end{aligned} \tag{4.9}$$

As M is totally umbilical hemi-slant submanifold and using (2.7) we can write

$$g(\nabla_Z TZ, W) - g(T\nabla_Z Z, Z) = \alpha g(Z,W)g(H,FZ) + g(tH,W)||Z||^2. \tag{4.10}$$

The above equation has a solution if either $H \in \mu$ or dim $D^{\perp} = 1$. □

If however, H does not belong to μ then we give the next theorem.

Theorem 4.2 *Let M be a totally umbilical hemi-slant submanifold of a Lorentzian (α)-Sasakian manifold $\tilde{M}$ such that the dimension of slant distribution $D_\theta \geq 4$ and F is parallel to the submanifold, then M is either extrinsic sphere or anti-invariant submanifold.*

Proof Since the dimension of slant distribution $D_\theta \geq 4$, therefore we can select a set of orthogonal vectors $X, Y \in D_\theta$, such that $g(X,Y) = 0$. Now by replacing Z by TY in (3.4) we have for any $X, Y, Z \in D_\theta$,

$$\begin{aligned} \tilde{R}(X,Y)\phi TY &= \phi\tilde{R}(X,Y)TY + \alpha^2 g(Y,TY)\phi X \\ &\quad - \alpha^2 g(X,TY)\phi Y - \alpha^2 g([X,Y],TY) \\ &\quad + \alpha g(X, \tilde{\nabla}_Y TY)\xi + \alpha\eta(\tilde{\nabla}_Y TY)X \\ &\quad - \alpha g(Y, \tilde{\nabla}_X TY)\xi - \alpha\eta(\tilde{\nabla}_X TY)Y. \end{aligned} \tag{4.11}$$

Now using equation (2.3) and (3.1) we obtain on calculation

$$\begin{aligned} \tilde{R}(X,Y)FTY + cos^2\theta\tilde{R}(X,Y)Y &= \phi\tilde{R}(X,Y)TY + \alpha^2 g(Y,TY)\phi X \\ &\quad - \alpha^2 g(X,TY)\phi Y - \alpha^2 g([X,Y],TY) \\ &\quad + \alpha g(X, \tilde{\nabla}_Y TY)\xi + \alpha\eta(\tilde{\nabla}_Y TY)X \\ &\quad - \alpha g(Y, \tilde{\nabla}_X TY)\xi - \alpha\eta(\tilde{\nabla}_X TY)Y. \end{aligned} \tag{4.12}$$

Again if F is parallel, then above equation can be written as

$$\begin{aligned} F\tilde{R}(X,Y)TY + cos^2\theta\tilde{R}(X,Y)Y &= \phi\tilde{R}(X,Y)TY + \alpha^2 g(Y,TY)\phi X \\ &\quad - \alpha^2 g(X,TY)\phi Y - \alpha^2 g([X,Y],TY) \\ &\quad + \alpha g(X, \tilde{\nabla}_Y TY)\xi + \alpha\eta(\tilde{\nabla}_Y TY)X \\ &\quad - \alpha g(Y, \tilde{\nabla}_X TY)\xi - \alpha\eta(\tilde{\nabla}_X TY)Y. \end{aligned} \tag{4.13}$$

Taking inner product with $N \in T^{\perp}M$, we obtain on using (3.3) and the orthogonality of X and Y vectors,

$$cos^2\theta||Y||^2 g(\nabla_X^{\perp} H, N) = 0$$

The above equation has a solution if either $\theta = \pi/2$ i.e. M is anti-invariant or $\nabla^{\perp}_{X}H = 0 \ \forall \ X \in D_{\theta}$. Similarly for any $X \in D^{\perp} \oplus <\xi>$ we can obtain $\nabla^{\perp}_{X}H = 0$, therefore $\nabla^{\perp}_{X}H = 0 \ \forall \ X \in TM$ i.e. the mean curvature vector H is parallel to submanifold, i.e., M is extrinsic sphere. Hence the theorem is proved. □

Now we are in a position to draw our main conclusions following.

Theorem 4.3 *Let M be a totally umbilical hemi-slant submanifold of a Lorentzian (α)-Sasakian manifold $\tilde{M}$. then M is either totally geodesic, or semi-invariant, or dim $D^{\perp} = 1$, or Extrinsic sphere, and the case (iv) holds if F is parallel and dim $M \geq 5$.*

Proof The proof follows immediately from Theorems 4.1 and 4.2. □

References

[1] D.E.Blair and J.A.Oubina, Conformal and related changes of metric on the product of two almost contact metric manifolds, *Publications Matematiques*, Vol. 34, 1990, pp. 99-207.

[2] B.Y.Chen, Slant immersions, *Bulletin of the Australian Mathematical Society*, vol. 41, no.1, pp. 135-147, 1990.

[3] B.Y.Chen, *Geometry of slant Submnaifolds*, Katholieke Universiteit Leuven, Leuven, Belgium, 1990.

[4] U.C.De, A.A.Shaikh, *Complex Manifolds and Contact Manifolds*, Narosa Publishing House Pvt. Ltd.

[5] A.Lotta, Slant submanifolds in contact geometry, *Bull.Math.Soc. Roumanie*, 39(1996), 183-198.

[6] N.Papaghiuc, *Semi-slant submanifolds of Kaehlerian manifold*, An.Stiint.Univ.Al. I. Cuza Iasi, Ser.Noua, mat. 40(1)(1994), 55-61.

[7] J.L.Cabrerizo, A.Carriazo, L.M.Fernandez, M.Fernandez, Semi-slant submanifolds of a Sasakian manifold, *Geom.Dedicata*, 78(1999), 183-199.

[8] J.L.Cabrerizo, A.Carriazo, L.M.Fernandez, M.Fernandez, Slant submanifolds in Sasakian manifolds, *Glasg. Math. J.*, 42(2000).

[9] A.Carriazo, Bi-slant immersions, *in Proceedings of the Integrated Car Rental and Accounts Management System*, pp 88-97, Kharagpur, West Bengal, India, 2000.

[10] S.Deshmukh and S.I.Hussain, Totally umbilical CR-submanifolds of a Kaehler manifold, *Kodai Math. J.*, 9(3)(1986), 425-429.

[11] A.Gray and L.M.Hervella, The sixteen classes of almost Hermitian manifolds and their linear invariants, *Annali di Matematica Pura ed Applicata*, Vol. 123, No. 4, 1980, pp. 35-58. doi:10.1007/BF01796539.

[12] S.K.Hui, S.Uddin, C.Ozel, A.A.Mustafa, Warped product submanifolds of LP-Sasakian manifold, Hindawi Publishing Corporation, *Discrete Dynamics in nature and Society*, Vol.2012, Article ID 868549.

[13] D.Janssens and L.Vanhecke, Almost contact structures and curvature tensors, *Kodai Mathematical Journal*, Vol. 4, No. 1, 1981, pp.1-27. doi:10.2996/kmj/1138036310.

[14] M.A.Khan, K.Singh and V.A.Khan, Slant submanifolds of LP-contact manifolds, *Differential Geometry Dynamical Systems*, vol.12, pp. 102-108, 2010.

[15] K.Matsumoto, On Lorentzian paracontact manifolds, *Bull.Yamagata Univ. Nat. Sci.*, 12, pp. 151-156.

[16] B.O'Neill, *Semi Riemannian Geometry with Applications to Relativity*, Academy Press, Inc.1953.

[17] J.A.Oubina, New classes of contact metric structures, *Publicationes Mathematicae Debrecen*, Vol.32, No. 4, 1985, pp. 187-193.

[18] D.G.Prakasha, C.S.Bagewadi and N.S.Basavarajappa, On pseudosymmetric Lorentzian α-Sasakian manifolds, *IJPAM*, Vol. 48, No. 1, 2008, 57-65.

[19] B.Sahin, Every totally umbilical proper slant submanifold of a Kaehler manifold is totally geodesic, *Results in Mathematics*, vol.54, no.1-2, pp. 167-172, 2009.

[20] Khushwant Singh, Siraj Uddin, Cenap Ozel, M.A.Khan, A class of totally umbilical slant submanifolds of Lorentzian para-saskian manifolds, *International Journal of Physical Science*, vol. 7(10), pp. 1526-1529, 2012.

[21] S.Uddin, M.A.Khan, K.Singh, Totally Umbilical Proper slant and hemislant submanifolds of an LP-cosymplectic manifold, Hindawi Publishing Corporation, *Mathematical Problems in Engineering*, vol.2011, Article ID 516238.

[22] A.Yildiz, M.Turan and B.F.Acet, On three dimensional Lorentzian α-Sasakian manifolds, *Bulletin of mathematical Analysis and Applications*, Vol. 1, Issue 3(2009), pp. 90-98.