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The Forcing Vertex Monophonic Number of a Graph

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Abstract: For any vertex x in a connected graph G of order $p \geq 2$, a set $S_x \subseteq V(G)$ is an *x-monophonic set* of G if each vertex $v \in V(G)$ lies on an $x-y$ monophonic path for some element y in S_x. The minimum cardinality of an x-monophonic set of G is the *x-monophonic number* of G and is denoted by $m_x(G)$. A subset T_x of a minimum x-monophonic set S_x of G is an *x-forcing subset* for S_x if S_x is the unique minimum x-monophonic set containing T_x. An x-forcing subset for S_x of minimum cardinality is a *minimum x-forcing subset* of S_x. The *forcing x-monophonic number* of S_x, denoted by $f_{m_x}(S_x)$, is the cardinality of a minimum x-forcing subset for S_x. The *forcing x-monophonic number* of G is $f_{m_x}(G) = min\{f_{m_x}(S_x)\}$, where the minimum is taken over all minimum x-monophonic sets S_x in G. We determine bounds for it and find the forcing vertex monophonic number for some special classes of graphs. It is shown that for any three positive integers a, b and c with $2 \leq a \leq b < c$, there exists a connected graph G such that $f_{m_x}(G) = a$, $m_x(G) = b$ and $cm_x(G) = c$ for some vertex x in G, where $cm_x(G)$ is the connected x-monophonic number of G.

Key Words: monophonic path, vertex monophonic number, forcing vertex monophonic number, connected vertex monophonic number, Smarandachely geodetic k-set, Smarandachely hull k-set.

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§1. Introduction

By a *graph* $G = (V, E)$ we mean a finite undirected connected graph without loops or multiple edges. The *order* and *size* of G are denoted by p and q respectively. For basic graph theoretic terminology we refer to Harary [6]. For vertices x and y in a connected graph G, the *distance* $d(x, y)$ is the length of a shortest $x-y$ path in G. An $x-y$ path of length $d(x, y)$ is called an *$x-y$ geodesic*. The *neibourhood* of a vertex v is the set $N(v)$ consisting of all vertices u which are adjacent with v. The *closed neibourhood* of a vertex v is the set $N[v] = N(v) \bigcup \{v\}$. A vertex v is a *simplicial vertex* if the subgraph induced by its neighbors is complete.

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The *closed interval* $I[x,y]$ consists of all vertices lying on some $x-y$ geodesic of G, while for $S \subseteq V$, $I[S] = \bigcup_{x,y \in S} I[x,y]$. A set S of vertices is a *geodetic set* if $I[S] = V$, and the minimum cardinality of a geodetic set is the *geodetic number* $g(G)$. The geodetic number of a graph was introduced in [1,8] and further studied in [2,5]. A geodetic set of cardinality $g(G)$ is called a $g-set$ of G. Generally, for an integer $k \geq 0$, a subset $S \subseteq V$ is called a *Smarandachely geodetic k-set* if $I[S \bigcup S^+] = V$ and a *Smarandachely hull k-set* if $I_h(S \bigcup S^+) = V$ for a subset $S^+ \subset V$ with $|S^+| \leq k$. Let $k = 0$. Then a Smarandachely geodetic 0-set and Smarandachely hull 0-set are nothing else but the geodetic set and hull set, respectively.

The concept of vertex geodomination number was introduced in [9] and further studied in [10]. For any vertex x in a connected graph G, a set S of vertices of G is an *x-geodominating set* of G if each vertex v of G lies on an $x-y$ geodesic in G for some element y in S. The minimum cardinality of an x-geodominating set of G is defined as the *x-geodomination number* of G and is denoted by $g_x(G)$. An x-geodominating set of cardinality $g_x(G)$ is called a $g_x - set$.

A chord of a path P is an edge joining any two non-adjacent vertices of P. A path P is called a *monophonic path* if it is a chordless path. A set S of vertices of a graph G is a *monophonic set* of G if each vertex v of G lies on an $x-y$ monophonic path in G for some $x, y \in S$. The minimum cardinality of a monophonic set of G is the *monophonic number* of G and is denoted by $m(G)$.

The concept of vertex monophonic number was introduced in [11]. For a connected graph G of order $p \geq 2$ and a vertex x of G, a set $S_x \subseteq V(G)$ is an *x-monophonic set* of G if each vertex v of G lies on an $x-y$ monophonic path for some element y in S_x. The minimum cardinality of an x-monophonic set of G is defined as the *x-monophonic number* of G, denoted by $m_x(G)$. An x-monophonic set of cardinality $m_x(G)$ is called a $m_x - set$ of G. The concept of upper vertex monophonic number was introduced in [13]. An x-monophonic set S_x is called a *minimal x-monophonic set* if no proper subset of S_x is an x-monophonic set. The *upper x-monophonic number*, denoted by $m_x^+(G)$, is defined as the minimum cardinality of a minimal x-monophonic set of G. The connected x-monophonic number was introduced and studied in [12]. A *connected x-monophonic set* of G is an x-monophonic set S_x such that the subgraph $G[S_x]$ induced by S_x is connected. The minimum cardinality of a connected x-monophonic set of G is the *connected x-monophonic number* of G and is denoted by $cm_x(G)$. A connected x-monophonic set of cardinality $cm_x(G)$ is called a $cm_x - set$ of G.

The following theorems will be used in the sequel.

Theorem 1.1([11]) *Let x be a vertex of a connected graph G.*

(1) *Every simplicial vertex of G other than the vertex x (whether x is simplicial vertex or not) belongs to every $m_x - set$;*

(2) *No cut vertex of G belongs to any $m_x - set$.*

Theorem 1.2([11]) (1) *For any vertex x in a cycle $C_p (p \geq 4)$, $m_x(C_p) = 1$;*

(2) *For the wheel $W_p = K_1 + C_{p-1} (p \geq 5)$, $m_x(W_p) = p-1$ or 1 according as x is K_1 or x is in C_{p-1}.*

Theorem 1.3([11]) *For $n \geq 2$, $m_x(Q_n) = 1$ for every vertex x in Q_n.*

Throughout this paper G denotes a connected graph with at least two vertices.

§2. Vertex Forcing Subsets in Vertex Monophonic Sets of a Graph

Let x be any vertex of a connected graph G. Although G contains a minimum x-monophonic set there are connected graphs which may contain more than one minimum x-monophonic set. For example, the graph G given in Figure 2.1 contains more than one minimum x-monophonic set. For each minimum x-monophonic set S_x in a connected graph G there is always some subset T of S_x that uniquely determines S_x as the minimum x-monophonic set containing T. Such sets are called "vertex forcing subsets" and we discuss these sets in this section. Also, forcing concepts have been studied for such diverse parameters in graphs as the geodetic number [3], the domination number [4] and the graph reconstruction number [7].

Definition 2.1 *Let x be any vertex of a connected graph G and let S_x be a minimum x-monophonic set of G. A subset T of S_x is called an x-forcing subset for S_x if S_x is the unique minimum x-monophonic set containing T. An x-forcing subset for S_x of minimum cardinality is a minimum x-forcing subset of S_x. The forcing x-monophonic number of S_x, denoted by $f_{m_x}(S_x)$, is the cardinality of a minimum x-forcing subset for S_x. The forcing x-monophonic number of G is $f_{m_x}(G) = min\{f_{m_x}(S_x)\}$, where the minimum is taken over all minimum x-monophonic sets S_x in G.*

Example 2.2 For the graph G given in Figure 2.1, the minimum vertex monophonic sets, the vertex monophonic numbers, the minimum forcing vertex monophonic sets and the forcing vertex monophonic numbers are given in Table 2.1.

Vertex x	Minimum x-monophonic sets	$m_x(G)$	Minimum forcing x-monophonic sets	$f_{m_x}(G)$
u	$\{r,y\},\{r,z\},\{r,s\}$	2	$\{y\},\{z\},\{s\}$	1
v	$\{u,r,y\},\{u,r,z\},\{u,r,s\}$	3	$\{y\},\{z\},\{s\}$	1
w	$\{u,r\}$	2	$\emptyset$	0
y	$\{u,r\}$	2	$\emptyset$	0
z	$\{u,r\}$	2	$\emptyset$	0
s	$\{u,r\}$	2	$\emptyset$	0
t	$\{u,r,w\},\{u,r,y\},\{u,r,z\}$	3	$\{w\},\{y\},\{z\}$	1
r	$\{u,w\},\{u,y\},\{u,z\}$	2	$\{w\},\{y\},\{z\}$	1

Table 2.1

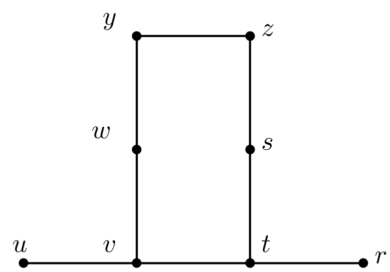

Figure 2.1

Theorem 2.3 *For any vertex x in a connected graph G, $0 \leq f_{m_x}(G) \leq m_x(G)$.*

Proof Let x be any vertex of G. It is clear from the definition of $f_{m_x}(G)$ that $f_{m_x}(G) \geq 0$. Let S_x be a minimum x-monophonic set of G. Since $f_{m_x}(S_x) \leq m_x(G)$ and since $f_{m_x}(G) = min\{f_{m_x}(S_x) : S_x$ is a minimum x-monophonic set in $G\}$, it follows that $f_{m_x}(G) \leq m_x(G)$. Thus $0 \leq f_{m_x}(G) \leq m_x(G)$. □

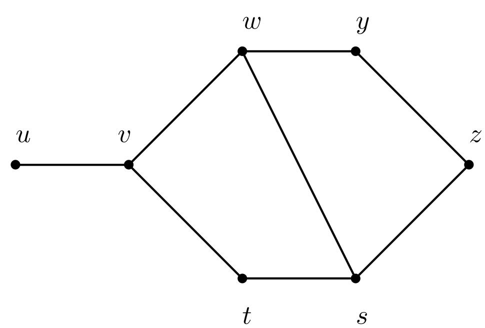

Figure 2.2

Remark 2.4 The bounds in Theorem 2.3 are sharp. For the graph G given in Figure 2.2, $S = \{u, z, t\}$ is the unique minimum w-monophonic set of G and the empty set ϕ is the unique minimum w-forcing subset for S. Hence $f_{m_w}(G) = 0$. Also, for the graph G given in Figure 2.2, $S_1 = \{y\}$ and $S_2 = \{z\}$ are the minimum u-monophonic sets of G and so $m_u(G) = 1$. It is clear that no minimum u-monophonic set is the unique minimum u-monophonic set containing any of its proper subsets. It follows that $f_{m_u}(G) = 1$ and hence $f_{m_u}(G) = m_u(G) = 1$. The inequalities in Theorem 2.3 can be strict. For the graph G given in Figure 2.1, $m_u(G) = 2$ and $f_{m_u}(G) = 1$. Thus $0 < f_{m_u}(G) < m_u(G)$.

In the following theorem we characterize graphs G for which the bounds in Theorem 2.3 are attained and also graphs for which $f_{m_x}(G) = 1$.

Theorem 2.5 *Let x be any vertex of a connected graph G. Then*

(1) $f_{m_x}(G) = 0$ *if and only if* G *has a unique minimum* x*-monophonic set;*

(2) $f_{m_x}(G) = 1$ *if and only if* G *has at least two minimum* x*-monophonic sets, one of which is a unique minimum* x*-monophonic set containing one of its elements, and*

(3) $f_{m_x}(G) = m_x(G)$ *if and only if no minimum* x*-monophonic set of* G *is the unique minimum* x*-monophonic set containing any of its proper subsets.*

Definition 2.6 *A vertex* u *in a connected graph* G *is said to be an* x*-monophonic vertex if* u *belongs to every minimum* x*-monophonic set of* G.

For the graph G in Figure 2.1, $S_1 = \{u, r, y\}$, $S_2 = \{u, r, z\}$ and $S_3 = \{u, r, s\}$ are the minimum v-monophonic sets and so u and r are the v-monophonic vertices of G. In particular, every simplicial vertex of G other than x is an x-monophonic vertex of G.

Next theorem follows immediately from the definitions of an x-monophonic vertex and forcing x-monophonic subset of G.

Theorem 2.7 *Let* x *be any vertex of a connected graph* G *and let* $\mathcal{F}_{m_x}$ *be the set of relative complements of the minimum* x*-forcing subsets in their respective minimum* x*-monophonic sets in* G*. Then* $\bigcap_{F \in \mathcal{F}_{m_x}} F$ *is the set of* x*-monophonic vertices of* G.

Theorem 2.8 *Let* x *be any vertex of a connected graph* G *and let* M_x *be the set of all* x*-monophonic vertices of* G*. Then* $0 \leq f_{m_x}(G) \leq m_x(G) - |M_x|$.

Proof Let S_x be any minimum x-monophonic set of G. Then $m_x(G) = |S_x|$, $M_x \subseteq S_x$ and S_x is the unique minimum x-monophonic set containing $S_x - M_x$ and so $f_{m_x}(G) \leq |S_x - M_x| = m_x(G) - |M_x|$. □

Theorem 2.9 *Let* x *be any vertex of a connected graph* G *and let* S_x *be any minimum* x*-monophonic set of* G*. Then*

(1) *no cut vertex of* G *belongs to any minimum* x*-forcing subset of* S_x*;*

(2) *no* x*-monophonic vertex of* G *belongs to any minimum* x*-forcing subset of* S_x.

Proof (1) Since any minimum x-forcing subset of S_x is a subset of S_x, the result follows from Theorem 1.1(2).

(2) Let v be an x-monophonic vertex of G. Then v belongs to every minimum x-monophonic set of G. Let $T \subseteq S_x$ be any minimum x-forcing subset for any minimum x-monophonic set S_x of G. If $v \in T$, then $T' = T - \{v\}$ is a proper subset of T such that S_x is the unique minimum x-monophonic set containing T' so that T' is an x-forcing subset for S_x with $|T'| < |T|$, which is a contradiction to T a minimum x-forcing subset for S_x. Hence $v \notin T$. □

Corollary 2.10 *Let* x *be any vertex of a connected graph* G*. If* G *contains* k *simplicial vertices, then* $f_{m_x}(G) \leq m_x(G) - k + 1$.

Proof This follows from Theorem 1.1(1) and Theorem 2.9(2). □

Remark 2.11 The bound for $f_{m_x}(G)$ in Corollary 2.10 is sharp. For a non-trivial tree T with

k end-vertices, $f_{m_x}(T) = 0 = m_x(T) - k + 1$ for any end-vertex x in T.

Theorem 2.12 (1) *If T is a non-trivial tree, then $f_{m_x}(T) = 0$ for every vertex x in T;*

(2) *If G is the complete graph, then $f_{m_x}(G) = 0$ for every vertex x in G.*

Proof This follows from Theorem 2.9. □

Theorem 2.13 *For every vertex x in the cycle $C_p (p \geq 3), f_{m_x}(C_p) = \begin{cases} 0 & \text{if } p = 3, 4 \\ 1 & \text{if } p \geq 5 \end{cases}$.*

Proof Let $C_p : u_1, u_2, \cdots, u_p, u_1$ be a cycle of order $p \geq 3$. Let x be any vertex in C_p, say $x = u_1$. If $p = 3$ or 4, then C_p has unique minimum x-monophonic set. Then by Theorem 2.5(1), $f_{m_x}(C_p) = 0$. Now, assume that $p \geq 5$. Let y be a non-adjacent vertex of x in C_p. Then $S_x = \{y\}$ is a minimum x-monophonic set of C_p. Hence C_p has more than one minimum x-monophonic set and it follows from Theorem 2.5(1) that $f_{m_x}(C_p) \neq 0$. Now it follows from Theorems 1.2(1) and 2.3 that $f_{m_x}(G) = m_x(G) = 1$. □

Theorem 2.14 *For any vertex x in a complete bipartite graph $K_{m,n}(m, n \geq 2)$, $f_{m_x}(K_{m,n}) = 0$.*

Proof Let (V_1, V_2) be the bipartition of $K_{m,n}$. If $x \in V_1$, then $S_x = V_1 - \{x\}$ is the unique minimum x-monophonic set of G and so by Theorem 2.5(1), $f_{m_x}(G) = 0$. If $x \in V_2$, then $S_x = V_2 - \{x\}$ is the unique minimum x-monophonic set of G and so by Theorem 2.5(1), $f_{m_x}(G) = 0$. □

Theorem 2.15 (1) *If G is the wheel $W_p = K_1 + C_{p-1}(p = 4, 5)$, then $f_{m_x}(G) = 0$ for any vertex x in W_p;*

(2) *If G is the wheel $W_p = K_1 + C_{p-1}(p \geq 6)$, then $f_{m_x}(G) = 0$ or 1 according as x is K_1 or x is in C_{p-1}.*

Proof Let $C_{p-1} : u_1, u_2, \cdots, u_{p-1}, u_1$ be a cycle of order $p - 1$ and let u be the vertex of K_1.

(1) If $p = 4$ or 5, then G has unique minimum x-monophonic set for any vertex x in G and so by Theorem 2.5(1), $f_{m_x}(G) = 0$.

(2) Let $p \geq 6$. If $x = u$, then $S_x = \{u_1, u_2, \cdots, u_{p-1}\}$ is the unique minimum x-monophonic set and so by Theorem 2.5(1), $f_{m_x}(G) = 0$. If $x \in V(C_{p-1})$, say $x = u_1$, then $S_i = \{u_i\}\,(3 \leq i \leq p-2)$ is a minimum x-monophonic set of G. Since $p \geq 6$, there is more than one minimum x-monophonic set of G. Hence it follows from Theorem 2.5(1) that $f_{m_x}(G) \neq 0$. Now it follows from Theorems 1.2(2) and 2.3 that $f_{m_x}(G) = m_x(G) = 1$. □

Theorem 2.16 *For any vertex x in the n-cube Q_n $(n \geq 2)$, then $f_{m_x}(Q_n) = \begin{cases} 0 & \text{if } n = 2 \\ 1 & \text{if } n \geq 3 \end{cases}$.*

Proof If $n = 2$, then Q_n has unique minimum x-monophonic set for any vertex x in Q_n and so by Theorem 2.5(1), $f_{m_x}(Q_n) = 0$. If $n \geq 3$, then it is easily seen that there is more than one minimum x-monophonic set for any vertex x in Q_n. Hence it follows from Theorem 2.5(1) that $f_{m_x}(Q_n) \neq 0$. Now it follows from Theorems 1.3 and 2.3 that $f_{m_x}(Q_n) = m_x(Q_n) = 1$. □

The following theorem gives a realization result for the parameters $f_{m_x}(G)$, $m_x(G)$ and $m_x^+(G)$.

Theorem 2.17 *For any three positive integers a, b and c with $2 \leq a \leq b \leq c$, there exists a connected graph G with $f_{m_x}(G) = a$, $m_x(G) = b$ and $m_x^+(G) = c$ for some vertex x in G.*

Proof For each integer i with $1 \leq i \leq a-1$, let $F_i : u_{0,i}, u_{1,i}, u_{2,i}, u_{3,i}$ be a path of order 4. Let $C_6 : t, u, v, w, x, y, t$ be a cycle of order 6. Let H be a graph obtained from F_i and C_6 by joining the vertex x of C_6 to the vertices $u_{0,i}$ and $u_{3,i}$ of $F_i (1 \leq i \leq a-1)$. Let G be the graph obtained from H by adding $c-a$ new vertices $y_1, y_2, \cdots, y_{c-b}, v_1, v_2, \cdots, v_{b-a}$ and joining each $y_i (1 \leq i \leq c-b)$ to both u and y, and joining each $v_j (1 \leq j \leq b-a)$ with x. The graph G is shown in Figure 2.3.

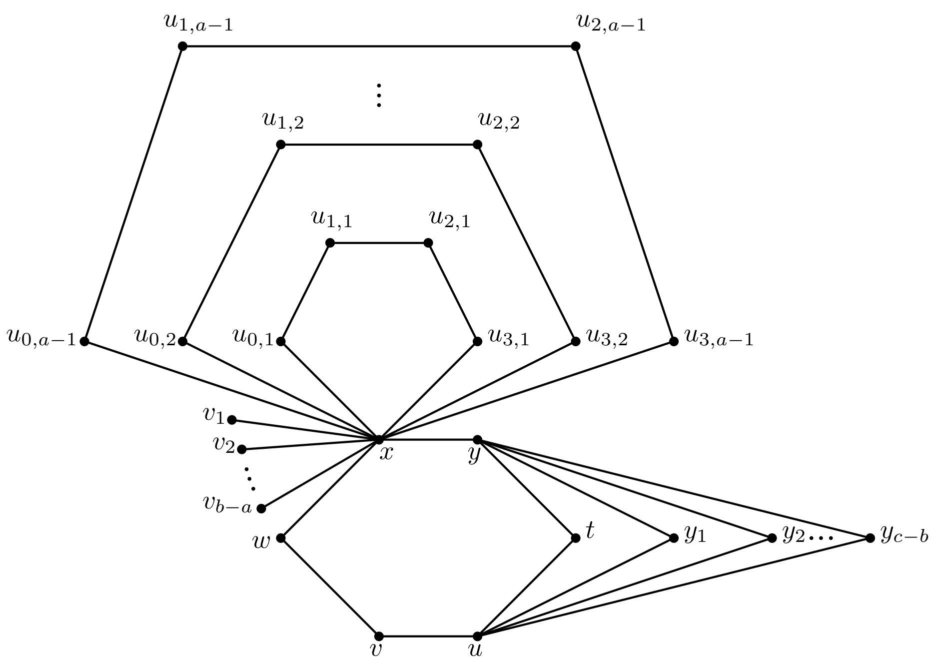

Figure 2.3

Let $S = \{v_1, v_2, \cdots, v_{b-a}\}$ be the set of all simplicial vertices of G. For $1 \leq j \leq a-1$, let $S_j = \{u_{1,j}, u_{2,j}\}$. If $b = c$, then let $S_a = \{u, v, t\}$. Otherwise, let $S_a = \{u, v\}$. Now, we observe that a set S_x of vertices of G is a m_x-set if S_x contains S and exactly one vertex from each set $S_j (1 \leq j \leq a)$ so that $m_x(G) \geq b$. Since $S_x^{'} = S \bigcup \{u, u_{1,1}, u_{1,2}, \cdots, u_{1,a-1}\}$ is an x-monophonic set of G, we have $m_x(G) = b$.

Now, we show that $f_{m_x}(G) = a$. Let $S_x = S \bigcup \{u, u_{1,1}, u_{1,2}, \cdots, u_{1,a-1}\}$ be a m_x-set of G and let T_x be a minimum x-forcing subset of S_x. Since S is the set of all x-monophonic vertices of G and by Theorem 2.8, $f_{m_x}(G) \leq m_x(G) - |S| = a$.

If $|T_x| < a$, then there exists a vertex $y \in S_x$ such that $y \notin T_x$. It is clear that $y \in S_j$ for some $j = 1, 2, \cdots, a$, say $y = u_{1,1}$. Let $S_x' = (S_x - \{u_{1,1}\}) \bigcup \{u_{2,1}\}$. Then $S_x' \neq S_x$ and S_x' is also a minimum x-monophonic set of G such that it contains T_x, which is a contradiction to T_x a minimum x-forcing subset of S_x. Thus $|T_x| = a$ and so $f_{m_x}(G) = a$.

Next, we show that $m_x^+(G) = c$. Let $U_x = S \bigcup \{u_{1,1}, u_{1,2}, \cdots, u_{1,a-1}, t, y_1, y_2, \cdots, y_{c-b}\}$. Clearly U_x is a minimal x-monophonic set of G and so $m_x^+(G) \geq c$. Also, it is clear that every minimal x-monophonic set of G contains at most c elements and hence $m_x^+(G) \leq c$. Therefore, $m_x^+(G) = c$. □

The following theorem gives a realization for the parameters $f_{m_x}(G)$, $m_x(G)$ and $cm_x(G)$.

Theorem 2.18 *For any three positive integers a, b and c with $2 \leq a \leq b < c$, there exists a connected graph G with $f_{m_x}(G) = a$, $m_x(G) = b$ and $cm_x(G) = c$ for some vertex x in G.*

Proof We prove this theorem by considering three cases.

Case 1. $2 \leq a < b < c$.

For each integer i with $1 \leq i \leq a-1$, let $F_i : y_1, u_{1,i}, u_{2,i}, y_3$ be a path of order 4. Let $P_{c-b+2} : y_1, y_2, y_3, \cdots, y_{c-b+2}$ be a path of order $c-b+2$ and let $P : v_1, v_2, v_3$ be a path of order 3. Let H_1 be a graph obtained from $F_i (1 \leq i \leq a-1)$ and P_{c-b+2} by identifying the vertices y_1 and y_3 of all $F_i (1 \leq i \leq a-1)$ and P_{c-b+2}. Let H_2 be the graph obtained from H_1 and P by joining the vertex v_1 of P to the vertex y_2 of H_1 and joining the vertex v_3 of P to the vertex y_3 of H_1. Let G be the graph obtained from H_2 by adding $b-a$ new vertices $z_1, z_2, ..., z_{b-a}$ and joining each $z_i (1 \leq i \leq b-a)$ with the vertex y_{c-b+2}. The graph G is shown in Figure 2.4.

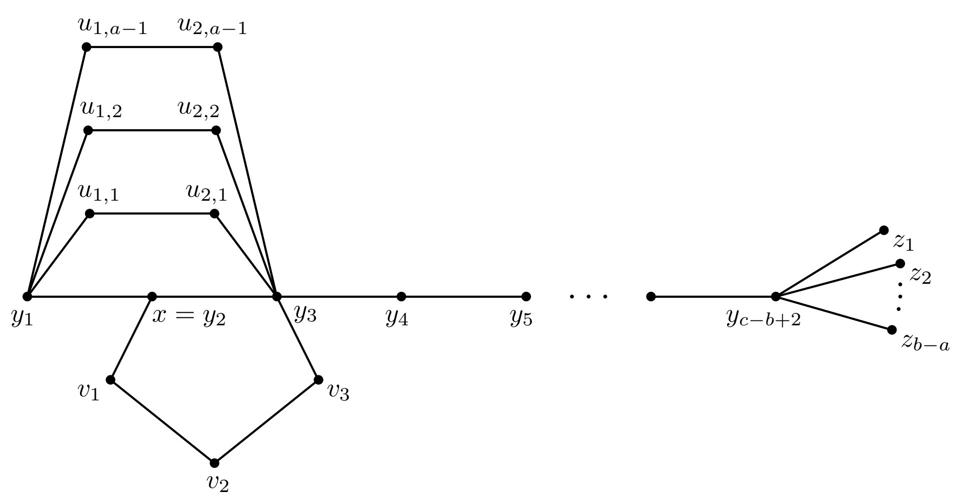

Figure 2.4

Let $x = y_2$ and let $S = \{z_1, z_2, \cdots, z_{b-a}\}$ be the set of all simplicial vertices of G. For $1 \leq j \leq a-1$, let $S_j = \{u_{1,j}, u_{2,j}\}$ and let $S_a = \{v_2, v_3\}$. Now, we observe that a set S_x of vertices of G is a m_x-set if S_x contains S and exactly one vertex from each set $S_j (1 \leq j \leq a)$. Hence $m_x(G) \geq b$. Since $S_x^{'} = S \bigcup \{v_2, u_{1,1}, u_{1,2}, \cdots, u_{1,a-1}\}$ is an x-monophonic set of G with $|S_x^{'}| = b$, it follows that $m_x(G) = b$.

Now, we show that $f_{m_x}(G) = a$. Let $S_x = S \bigcup \{v_2, u_{1,1}, u_{1,2}, ..., u_{1,a-1}\}$ be a m_x-set of G and let T_x be a minimum x-forcing subset of S_x. Since S is the set of all x-monophonic vertices of G and by Theorem 2.8, $f_{m_x}(G) \leq m_x(G) - |S| = a$.

If $|T_x| < a$, then there exists a vertex $y \in S_x$ such that $y \notin T_x$. It is clear that $y \in S_j$ for some $j = 1, 2, \cdots, a$, say $y = u_{1,1}$. Let $S_x^{'} = (S_x - \{u_{1,1}\}) \bigcup \{u_{2,1}\}$. Then $S_x^{'} \neq S_x$ and $S_x^{'}$ is also a minimum x-monophonic set of G such that it contains T_x, which is a contradiction to T_x an x-forcing subset of S_x. Thus $|T_x| = a$ and so $f_{m_x}(G) = a$.

Clearly, $S \bigcup \{v_3, u_{2,1}, u_{2,2}, \cdots, u_{2,a-1}, y_3, y_4, \cdots, y_{c-b+2}\}$ is the unique minimum connected x-monophonic set of G, we have $cm_x(G) = c$.

Case 2. $2 \leq a = b < c$ and $c = b + 1$.

Construct the graph H_2 in Case 1. Then $G = H_2$ has the desired properties (S is the empty set).

Case 3. $2 \leq a = b < c$ and $c \geq b + 2$. For each i with $1 \leq i \leq a - 1$, let $F_i : y_1, u_{i,1}, u_{i,2}, y_3$ be a path of order 4. Let $P_{c-a+1} : y_1, y_2, y_3, \cdots, y_{c-a+1}$ be a path of order $c - a + 1$ and let $C_5 : v_1, v_2, v_3, v_4, v_5, v_1$ be a cycle of order 5. Let H be a graph obtained from F_i and P_{c-a+1} by identifying the vertices y_1 and y_3 of all $F_i (1 \leq i \leq a - 1)$ and P_{c-a+1}. Let G be the graph obtained from H by identifying the vertex y_{c-a+1} of P_{c-a+1} and v_1 of C_5. The graph G is shown in Figure 2.5. Let $x = y_2$.

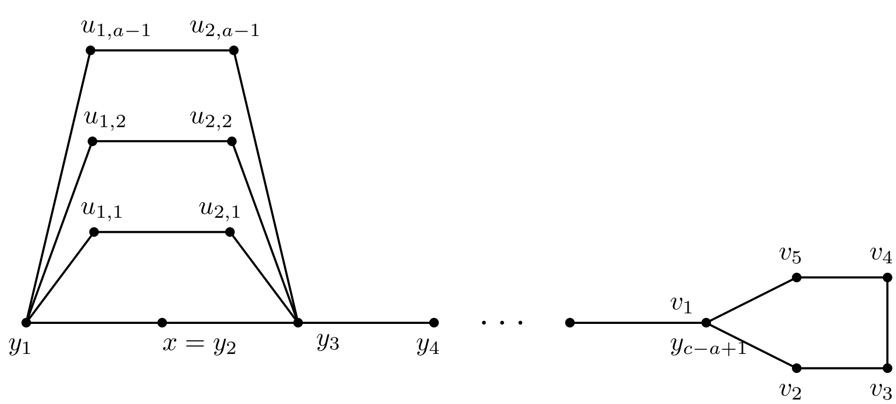

Figure 2.5

For $1 \leq j \leq a - 1$, let $S_j = \{u_{1,j}, u_{2,j}\}$ and let $S_a = \{v_3, v_4\}$. Now, we observe that a set S_x of vertices of G is a m_x-set if S_x contains exactly one vertex from each set $S_j (1 \leq j \leq a)$ so that $m_x(G) \geq a$. Since $S_x^{'} = \{v_3, u_{1,1}, u_{1,2}, \cdots, u_{1,a-1}\}$ is an x-monophonic set of G with $|S_x^{'}| = a$, we have $m_x(G) = a$.

Now, we show that $f_{m_x}(G) = a$. Let $S_x = \{v_3, u_{1,1}, u_{1,2}, \cdots, u_{1,a-1}\}$ be a m_x-set of G and let T_x be a minimum x-forcing subset of S_x. Then by Theorem 2.3, $f_{m_x}(G) \leq m_x(G) = a$.

If $|T_x| < a$, then there exists a vertex $y \in S_x$ such that $y \notin T_x$. It is clear that $y \in S_j$ for some $j = 1, 2, \cdots, a$, say $y = u_{1,1}$. Let $S_x^{'} = (S_x - \{u_{1,1}\}) \bigcup \{u_{1,2}\}$. Then $S_x^{'} \neq S_x$ and $S_x^{'}$ is also a minimum x-monophonic set of G such that it contains T_x, which is a contradiction to T_x an x-forcing subset of S_x. Thus $|T_x| = a$ and so $f_{m_x}(G) = a$.

Let $S = \{v_2, v_3, u_{2,1}, u_{2,2}, \cdots, u_{2,a-1}, y_3, y_4, \cdots, y_{c-a+1}\}$. It is easily verified that S is a minimum connected x-monophonic set of G and so $cm_x(G) = c$. □

Problem 2.19 *For any three positive integers a, b and c with $2 \leq a \leq b = c$, does there exist a connected graph G with $f_{m_x}(G) = a$, $m_x(G) = b$ and $cm_x(G) = c$ for some vertex x in G?*

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