Some Characterizations for the Involute Curves in Dual Space

Süleyman ŞENYURT

(Department of Mathematics, Faculty of Arts and Sciences, Ordu University, 52100, Ordu/Turkey)

Mustafa BİLİCİ

(Department of Mathematics, Faculty of Education, Ondokuz Mayis University, Turkey)

Mustafa ÇALIŞKAN

(Department of Mathematics, Faculty of Science, Gazi University, Turkey)

E-mail: senyurtsuleyman@hotmail.com, mbilici@omu.edu.tr, mustafacaliskan@gazi.edu.tr

Abstract: In this paper, we investigate some characterizations of involute – evolute curves in dual space. Then the relationships between dual Frenet frame and Darboux vectors of these curves are found.

Key Words: Dual curve, involute, evolute, dual space.

AMS(2010): 53A04, 45F10

§1. Introduction

Involute-evolute curve couple was originally defined by Christian Huygens in 1668. In the theory of curves in Euclidean space, one of the important and interesting problems is the characterizations of a regular curve. In particular, the involute of a given curve is a well known concept in the classical differential geometry (for the details see [7]). For classical and basic treatments of Involute-evolute curve couple, we refer to [1], [5], [7-9] and [13].

The relationships between the Frenet frames of the involute-evolute curve couple have been found as depend on the angle between binormal vector B and Darboux vector W of evolute curve, [1]. In the light of the existing literature, similar studies have been constructed on Lorentz and Dual Lorentz space,[2-4, 10-12].

In this paper, The relationships between dual Frenet frame and Darboux vectors of these curves have been found. Additionally, some important results concerning these curves are given.

§2. Preliminaries

Dual numbers were introduced by W.K. Clifford (1849-79) as a tool for his geometrical investi-

---

gations. The set $ID = \left\{ A = a + \varepsilon a^* \mid a, a^* \epsilon IR, \ \varepsilon^2 = 0 \right\}$ is called dual numbers set. On this set product and addition operations are described as

$$
\begin{aligned}
(a + \varepsilon a^*) + (b + \varepsilon b^*) &= (a + b) + \varepsilon (a^* + b^*), \\
(a + \varepsilon a^*) \cdot (b + \varepsilon b^*) &= ab + \varepsilon (ab^* + a^* b),
\end{aligned}
$$

respectively. The elements of the set $ID^3 = \left\{ \overrightarrow{A} \mid \overrightarrow{A} = \overrightarrow{a} + \varepsilon \overrightarrow{a^*}, \ \overrightarrow{a}, \overrightarrow{a^*} \epsilon IR^3 \right\}$ are called dual vectors. On this set, addition and scalar product operations are described as

$$
\begin{aligned}
\oplus \ &: \ ID^3 \times ID^3 \to ID^3 \ , \ \overrightarrow{A} \oplus \overrightarrow{B} = \left(\overrightarrow{a} + \overrightarrow{b} \right) + \epsilon \left(\overrightarrow{a^*} + \overrightarrow{b^*} \right), \\
\odot \ &: \ ID \times ID^3 \to ID^3 \ , \quad \widetilde{\lambda} \odot \overrightarrow{A} = \lambda \overrightarrow{a} + \epsilon \left(\lambda \overrightarrow{a^*} + \lambda^* \overrightarrow{a} \right),
\end{aligned}
$$

respectively. Algebraic construction $\left(ID^3, \oplus, ID, +, ., \odot \right)$ is a modul. This modul is called *ID-Modul*.

The inner product and vectorel product of dual vectors $\overrightarrow{A}, \overrightarrow{B} \epsilon ID^3$ are defined by respectively,

$$
\begin{aligned}
\langle \, , \rangle \ &: \ ID^3 \times ID^3 \to ID \ , \ \left\langle \overrightarrow{A}, \overrightarrow{B} \right\rangle = \left\langle \overrightarrow{a}, \overrightarrow{b} \right\rangle + \varepsilon \left(\left\langle \overrightarrow{a}, \overrightarrow{b^*} \right\rangle + \left\langle \overrightarrow{a^*}, \overrightarrow{b} \right\rangle \right) \\
\wedge \ &: \ ID^3 \times ID^3 \to ID^3 \ , \ \overrightarrow{A} \wedge \overrightarrow{B} = \left(\overrightarrow{a} \wedge \overrightarrow{b} \right) + \varepsilon \left(\overrightarrow{a} \wedge \overrightarrow{b^*} + \overrightarrow{a^*} \wedge \overrightarrow{b} \right).
\end{aligned}
$$

For $\overrightarrow{A} \neq 0$, the norm $\left\| \overrightarrow{A} \right\|$ of $\overrightarrow{A} = \overrightarrow{a} + \varepsilon \overrightarrow{a^*}$ is defined by

$$
\left\| \overrightarrow{A} \right\| = \sqrt{\left\langle \overrightarrow{A}, \overrightarrow{A} \right\rangle} = \left\| \overrightarrow{a} \right\| + \varepsilon \frac{\left\langle \overrightarrow{a}, \overrightarrow{a^*} \right\rangle}{\left\| \overrightarrow{a} \right\|} \ , \ \left\| \overrightarrow{a} \right\| \neq 0.
$$

The angle between unit dual vectors $\overrightarrow{A}$ and $\overrightarrow{B}$ $\Phi = \varphi + \varepsilon \varphi^*$ is called dual angle and this angle is denoted by ([6])

$$
\left\langle \overrightarrow{A}, \overrightarrow{B} \right\rangle = \cos \Phi = \cos \varphi - \varepsilon \varphi^* \sin \varphi
$$

Let

$$
\begin{aligned}
\widetilde{\alpha} \ &: \ I \subset IR \to ID^3 \\
s \ &\to \ \widetilde{\alpha}(s) = \alpha(s) + \varepsilon \alpha^*(s)
\end{aligned}
$$

be differential unit speed dual curve in dual space ID^3. Denote by $\{T, N, B\}$ the moving dual Frenet frame along the dual space curve $\widetilde{\alpha}(s)$ in the dual space ID^3. Then T, N and B are the dual tangent, the dual principal normal and the dual binormal vector fields, respectively. The function $\kappa(s) = k_1 + \varepsilon k_1^*$ and $\tau(s) = k_2 + \varepsilon k_2^*$ are called dual curvature and dual torsion of $\widetilde{\alpha}$,

respectively. Then for the dual curve $\widetilde{\alpha}$ the Frenet formulae are given by,

$$
\begin{aligned}
T^{'}(s) &= \kappa(s)N(s) \\
N^{'}(s) &= -\kappa(s)T(s) + \tau(s)B(s) \\
B^{'}(s) &= -\tau(s)N(s)
\end{aligned}
\tag{2.1}
$$

The formulae (2.1) are called the Frenet formulae of dual curve. In this palace curvature and torsion are calculated by,

$$
\kappa(s) = \sqrt{\langle T^{'}, T^{'}\rangle} \ , \ \tau(s) = \frac{\det\left(T, T^{'}, T^{''}\right)}{\langle T^{'}, T^{'}\rangle}
\tag{2.2}
$$

If α is not unit speed curve, then curvature and torsion are calculated by

$$
\kappa(s) = \frac{\left\|\alpha^{'}(s) \wedge \alpha^{''}(s)\right\|}{\left\|\alpha^{'}(s)\right\|^3} \ , \ \tau(s) = \frac{\det\left(\alpha^{'}(s), \alpha^{''}(s), \alpha^{'''}(s)\right)}{\left\|\alpha^{'}(s) \wedge \alpha^{''}(s)\right\|^2}
\tag{2.3}
$$

By separating formulas (2.1) into real and dual part, we obtain

$$
\begin{aligned}
t^{'}(s) &= k_1 n \\
n^{'}(s) &= -k_1 t + k_2 b \\
b^{'}(s) &= -k_2 n
\end{aligned}
\tag{2.4}
$$

$$
\begin{aligned}
t^{*'}(s) &= k_1 n^* + k_1^* n \\
n^{*'}(s) &= -k_1 t^* - k_1^* t + k_2 b^* + k_2^* b \\
b^{*'}(s) &= -k_2 n^* - k_2^* n
\end{aligned}
\tag{2.5}
$$

§3. Some Characterizations Involute of Dual Curves

Definition 3.1 *Let $\widetilde{\alpha} : I \to ID^3$ and $\widetilde{\beta} : I \to ID^3$ be dual unit speed curves. If the tangent lines of the dual curve $\widetilde{\alpha}$ is orthogonal to the tangent lines of the dual curve $\widetilde{\beta}$, the dual curve $\widetilde{\beta}$ is called involute of the dual curve $\widetilde{\alpha}$ or the dual curve $\widetilde{\alpha}$ is called evolute of the dual curve $\widetilde{\beta}$ (see Fig.1). According to this definition, if the tangent of the dual curve $\widetilde{\alpha}$ is denoted by T and the tangent of the dual curve $\widetilde{\beta}$ is denoted by $\bar{T}$, we can write*

$$
\left\langle T, \bar{T}\right\rangle = 0
\tag{3.1}
$$

Theorem 3.1 *Let $\widetilde{\alpha}$ and $\widetilde{\beta}$ be dual curves. If the dual curve $\widetilde{\beta}$ involute of the dual curve $\widetilde{\alpha}$, we can write*

$$
\widetilde{\beta}(s) = \widetilde{\alpha}(s) + [(c_1 - s) + \varepsilon c_2] T(s) \ , \ c_1, c_2 \epsilon IR.
$$

Proof Then by the definition we can assume that

$$\widetilde{\beta}(s) = \widetilde{\alpha}(s) + \lambda T(s) \quad , \quad \lambda(s) = \mu(s) + \varepsilon \mu^*(s) \tag{3.2}$$

for some function $\lambda(s)$. By taking derivative of the equation (3.2) with respect to s and applying the Frenet formulae (2.1) we have

$$\frac{d\widetilde{\beta}}{ds} = \left(1 + \frac{d\lambda}{ds}\right) T + \lambda \kappa N$$

where s and s^* are arc parameter of the dual curves $\widetilde{\alpha}$ and $\widetilde{\beta}$, respectively. It follows that

$$\overline{T}\frac{ds^*}{ds} = \left(1 + \frac{d\lambda}{ds}\right) T + \lambda \kappa N \tag{3.3}$$

the inner product of (3.3) with T is

$$\frac{ds^*}{ds}\left\langle T, \overline{T}\right\rangle = \left(1 + \frac{d\lambda}{ds}\right)\langle T, T\rangle + \lambda\langle T, N\rangle \tag{3.4}$$

From the definition of the involute-evolute curve couple, we can write

$$\left\langle T, \overline{T}\right\rangle = 0$$

By substituting the last equation in (3.4) we get

$$1 + \frac{d\lambda}{ds} = 0 \quad \text{and} \quad \frac{d}{ds}(\mu(s) + \varepsilon\mu^*(s)) = -1 \tag{3.5}$$

Straightforward computation gives

$$\mu'(s) = -1 \quad \text{and} \quad \mu^{*'}(s) = 0$$

integrating last equation, we get

$$\mu(s) = c_1 - s \quad \text{and} \quad \mu^*(s) = c_2 \tag{3.6}$$

By substituting (3.6) in (3.2), we get

$$\widetilde{\beta}(s) - \widetilde{\alpha}(s) = [(c_1 - s) + \varepsilon c_2]\, T(s). \tag{3.7}$$

This completes the proof. □

Corollary 3.1 *The distance between the dual curves $\widetilde{\beta}$ and $\widetilde{\alpha}$ is $|c_1 - s| \mp \varepsilon c_2$.*

Proof By taking the norm of the equation (3.7) we get

$$d\left(\widetilde{\alpha}(s), \widetilde{\beta}(s)\right) = |c_1 - s| \mp \varepsilon c_2 \tag{3.8}$$

This completes the proof. □

Fig.1

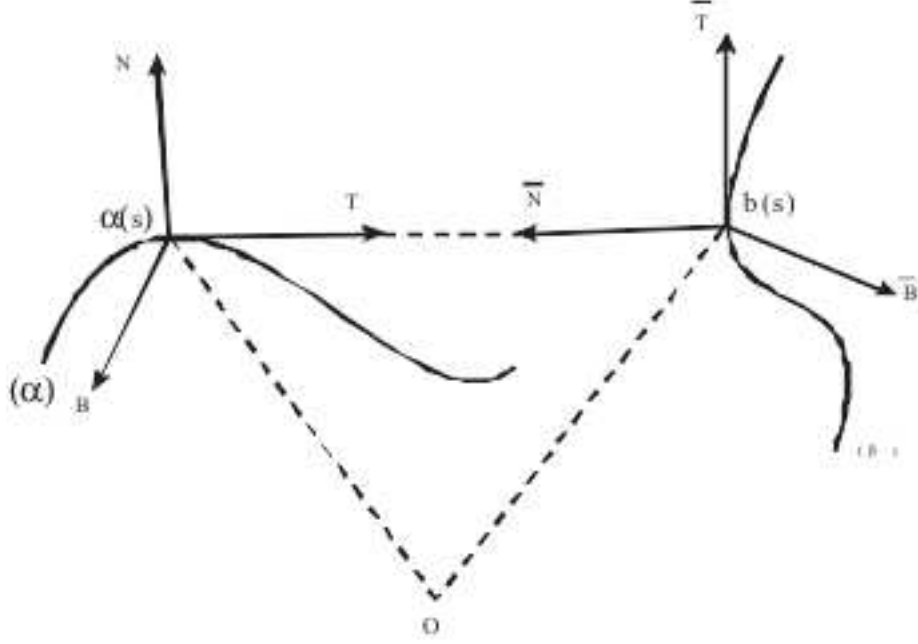**Theorem** 3.2 *Let* $\widetilde{\alpha}, \widetilde{\beta}$ *be dual curves. If the dual curve* $\widetilde{\beta}$ *involute of the dual curve* $\widetilde{\alpha}$, *then the relationships between the dual Frenet vectors of the dual curves* $\widetilde{\alpha}$ *and* $\widetilde{\beta}$

$$
\begin{aligned}
\overline{T} &= N \\
\overline{N} &= -\cos\Phi T + \sin\Phi B \\
\overline{B} &= \sin\Phi T + \cos\Phi B
\end{aligned}
$$

Proof By differentiating the equation (3.2) with respect to s we obtain

$$\widetilde{\beta}'(s) = \lambda\kappa(s)N(s) \ , \ \lambda = (c_1 - s) + \varepsilon c_2 \tag{3.9}$$

and

$$\left\|\widetilde{\beta}'(s)\right\| = \lambda\kappa(s)$$

Thus, the tangent vector of $\widetilde{\beta}$ is found

$$\overline{T} = \frac{\widetilde{\beta}'(s)}{\left\|\widetilde{\beta}'(s)\right\|} = \frac{\lambda\kappa(s)N(s)}{\lambda\kappa(s)}$$

If we arrange the last equation we obtain

$$\overline{T} = N(s) \tag{3.10}$$

By differentiating the equation (3.9) with respect to s we obtain

$$\widetilde{\beta}'' = -\lambda\kappa^2 T + \left(\lambda\kappa' - \kappa\right) N + \lambda\kappa\tau B$$

If the cross product $\widetilde{\beta}' \wedge \widetilde{\beta}''$ is calculated we have

$$\widetilde{\beta}' \wedge \widetilde{\beta}'' = \lambda^2\kappa^2\tau T + \lambda^2\kappa^3 B \qquad (3.11)$$

The norm of vector $\widetilde{\beta}' \wedge \widetilde{\beta}''$ is found

$$\left\|\widetilde{\beta}' \wedge \widetilde{\beta}''\right\| = \lambda^2\kappa^2\sqrt{\kappa^2 + \tau^2} \qquad (3.12)$$

For the dual binormal vector of the dual curve $\widetilde{\beta}$ we can write

$$\bar{B} = \frac{\widetilde{\beta}' \wedge \widetilde{\beta}''}{\left\|\widetilde{\beta}' \wedge \widetilde{\beta}''\right\|}$$

By substituting (3.11) and (3.12) in the last equation we get

$$\bar{B} = \frac{\tau}{\sqrt{\kappa^2 + \tau^2}} T + \frac{\kappa}{\sqrt{\kappa^2 + \tau^2}} B \qquad (3.13)$$

For the dual principal normal vector of the dual curve $\widetilde{\beta}$ we can write

$$\bar{N} = \bar{B} \wedge \bar{T}$$

and

$$\bar{N} = -\frac{\kappa}{\sqrt{\kappa^2 + \tau^2}} T + \frac{\tau}{\sqrt{\kappa^2 + \tau^2}} B \qquad (3.14)$$

Let Φ $\left(\Phi = \varphi + \varepsilon\varphi^* , \varepsilon^2 = 0\right)$ be dual angle between the dual Darboux vector W of $\widetilde{\alpha}$ and dual unit binormal vector B in this situation we can write

$$\sin\Phi = \frac{\tau}{\kappa^2 + \tau^2} , \quad \cos\Phi = \frac{\kappa}{\kappa^2 + \tau^2}. \qquad (3.15)$$

By substituting (3.15) in (3.12) and (3.13) the proof is completed. $\square$

The real and dual parts of $\bar{T}, \bar{N}, \bar{B}$ are

$$\begin{aligned}
\bar{T} &= N \\
\bar{N} &= -\cos\Phi T + \sin\Phi B \\
\bar{B} &= \sin\Phi T + \cos\Phi B
\end{aligned}$$

is separated into the real and dual part, we can obtain

$$\overline{t} = n,$$
$$\overline{n} = -\cos\varphi t + \sin\varphi b,$$
$$\overline{b} = \sin\varphi t + \cos\varphi b$$

$$\overline{t}^* = n^*$$
$$\overline{n}^* = -\cos\varphi t^* + \sin\varphi b^* + \varphi^* (\sin\varphi t + \cos\varphi b)$$
$$\overline{b}^* = \sin\varphi t^* + \cos\varphi b^* + \varphi^* (\cos\varphi t - \sin\varphi b)$$

On the way

$$\begin{cases} \sin\Phi = \sin(\varphi + \varepsilon\varphi^*) = \sin\varphi + \varepsilon\varphi^*\cos\varphi \\ \cos\Phi = \cos(\varphi + \varepsilon\varphi^*) = \cos\varphi - \varepsilon\varphi^*\sin\varphi \end{cases}$$

If the equation

$$\sin\Phi = \frac{\tau}{\kappa^2 + \tau^2}$$

is separated into the real and dual part, we can obtain

$$\sin\varphi = \frac{k_2}{k_1^2 + k_2^2}$$
$$\cos\varphi = \frac{k_1^2 + k_2^* - 2k_1 k_2 k_1^* - 2k_2^2 k_2^*}{\varphi (k_1^2 + k_2^2)^2}$$

If the equation

$$\cos\Phi = \frac{\kappa}{\kappa^2 + \tau^2}$$

is separated into the real and dual part, we can obtain

$$\cos\varphi = \frac{k_1}{k_1^2 + k_2^2}$$
$$\sin\varphi = \frac{2k_1^2 + k_1^* + 2k_1 k_2 k_2^* - k_1^2 k_1^* - k_2^2 k_1^*}{\varphi (k_1^2 + k_2^2)^2}$$

Theorem 3.3   *Let* $\widetilde{\alpha}, \widetilde{\beta}$ *be dual curves. If the dual curve* $\widetilde{\beta}$ *involute of the dual curve* $\widetilde{\alpha}$ *, curvature and torsion of the dual curve* $\widetilde{\beta}$ *are*

$$\overline{\kappa}^2(s) = \frac{\kappa^2(s) + \tau^2(s)}{\lambda^2(s)\kappa^2(s)} \;,\; \overline{\tau}(s) = \frac{\kappa(s)\tau^{'}(s) - \kappa^{'}(s)\tau(s)}{\lambda(s)\kappa(s)(\kappa^2(s) + \tau^2(s))} \tag{3.16}$$

Proof By the definition of involute we can write

$$\widetilde{\beta}(s) = \widetilde{\alpha}(s) + |\lambda| T(s) \tag{3.17}$$

By differentiating the equation (3.17) with respect to s we obtain

$$\frac{d\widetilde{\beta}}{ds^*}\frac{ds^*}{ds} = T(s) + |\lambda|'\, T(s) + |\lambda|\,\kappa(s)\,N(s),$$

$$\frac{d\widetilde{\beta}}{ds^*}\frac{ds^*}{ds} = T(s) - T(s) + |\lambda|\,\kappa(s)\,N(s),$$

$$\overline{T}(s)\frac{ds^*}{ds} = |\lambda|\,\kappa(s)\,N(s). \qquad (3.18)$$

Since the direction of $\overline{T}(s)$ is coincident with $N(s)$ we have

$$\overline{T}(s) = N(s). \qquad (3.19)$$

Taking the inner product of (3.18) with T and necessary operation are made we get

$$\frac{ds^*}{ds} = |\lambda(s)|\,\kappa(s). \qquad (3.20)$$

By taking derivative of (3.19) and applying the Frenet formulae (2.1) we have

$$\overline{T}(s) = N(s) \Rightarrow \overline{T}'(s)\frac{ds^*}{ds} = -\kappa T + \tau B. \qquad (3.21)$$

From (3.20) and (3.21), we have

$$\overline{T}'(s) = \frac{-\kappa T + \tau B}{|\lambda(s)|\,\kappa(s)}.$$

From the last equation we can write

$$\overline{\kappa}(s)\,\overline{N}(s) = \frac{-\kappa T + \tau B}{|\lambda(s)|\,\kappa(s)}.$$

Taking the inner product the last equation with each other we have

$$\left\langle \overline{\kappa}(s)\,\overline{N}(s), \overline{\kappa}(s)\,\overline{N}(s) \right\rangle = \left\langle \frac{-\kappa T + \tau B}{|\lambda(s)|\,\kappa(s)}, \frac{-\kappa T + \tau B}{|\lambda(s)|\,\kappa(s)} \right\rangle.$$

Thus, we find

$$\overline{\kappa}^2(s) = \frac{\kappa^2(s) + \tau^2(s)}{\lambda^2(s)\,\kappa^2(s)}.$$

We know that

$$\widetilde{\beta}' \wedge \widetilde{\beta}'' = \lambda^2\kappa^2\tau T + \lambda^2\kappa^3 B.$$

Taking the norm the last equation, we get

$$\left\| \widetilde{\beta}' \wedge \widetilde{\beta}'' \right\| = \kappa^4\lambda^4\left(\kappa^2 + \tau^2\right).$$

By substituting these equations in (2.3), we get

$$\bar{\tau} = \frac{\begin{vmatrix} 0 & \kappa\lambda & 0 \\ -\kappa^2\lambda & (\kappa\lambda)^{'} & \kappa\tau\lambda \\ (-\kappa^2\lambda)^{'} - \kappa(\kappa\lambda)^{'} & -\kappa^3\lambda + (\kappa\lambda)^{''} - \kappa\tau^2\lambda & (\kappa\lambda)^{'}\tau + (\kappa\tau\lambda)^{'} \end{vmatrix}}{\left\| \widetilde{\beta'} \wedge \widetilde{\beta''} \right\|^2},$$

$$\bar{\tau} = \frac{\kappa\tau^{'} - \kappa^{'}\tau}{\kappa\left|\lambda\right|(\kappa^2 + \tau^2)}.$$

This completes the proof. □

If the equation (3.16) is separated into the real and dual part, we can obtain

$$\bar{k}_1 = \frac{\sqrt{k_1^2 + k_2^2}}{\mu k_1},$$

$$\bar{k}_1^* = \frac{\left(\mu^2 k_1^2\right)\left(2k_1 k_1^* + 2k_2 k_2^*\right) - \left(2k_1 k_1^* \mu^2\right)\left(k_1^2 + k_2^2\right)}{2\mu^3 k_1^3 \sqrt{k_1^2 + k_2^2}},$$

$$\bar{k}_2 = \frac{k_1 k_2^{'} - k_2 k_1^{'}}{\mu k_1\left(k_1^2 + k_2^2\right)},$$

$$\bar{k}_2^* = \frac{\left(k_1 k_2^{'*} + k_2^{'} k_1^* - k_1^{'} k_2^* - k_2 k_1^{'*}\right)}{\left(\mu k_1^3 + k_1 k_2^2 \mu\right)}$$

$$- \frac{\left[2\left(k_1 k_1^* + k_2 k_2^*\right)k_1\mu + \left(k_1^2 + k_2^2\right)\left(k_1^*\mu + k_1\mu^*\right)\right]\left(k_1 k_2^{'} - k_2 k_1^{'}\right)}{\left(\mu k_1^3 + k_1 k_2^2 \mu\right)^2}.$$

Theorem 3.4 *Let $\widetilde{\alpha}, \widetilde{\beta}$ be dual curves and the dual curve $\widetilde{\beta}$ involute of the dual curve $\widetilde{\alpha}$. If W and $\bar{W}$ are Darboux vectors of the dual curves $\widetilde{\alpha}$ and $\widetilde{\beta}$ we can write*

$$\bar{W} = \frac{1}{\lambda\kappa}\left(W + \Phi^{'}N\right) \tag{3.22}$$

Proof Since $\bar{W}$ is Darboux vector of $\widetilde{\beta}(s)$ we can write

$$\bar{W}(s) = \bar{\tau}(s)\bar{T}(s) + \bar{\kappa}(s)\bar{B}(s) \tag{3.23}$$

By substituting $\bar{\tau}, \bar{T}, \bar{\kappa}, \bar{B}$ in the last equation, we get

$$\bar{W}(s) = \frac{\kappa\tau^{'} - \kappa^{'}\tau}{\kappa\left|\lambda\right|(\kappa^2 + \tau^2)}N(s) + \frac{\sqrt{\kappa^2 + \tau^2}}{\kappa\left|\lambda\right|}\left(\sin\Phi T + \cos\Phi B\right). \tag{3.24}$$

By substituting (3.15) in (3.24), we get

$$\bar{W}\left(s\right) = \frac{\kappa\tau^{'} - \kappa^{'}\tau}{\kappa\left|\lambda\right|\left(\kappa^2 + \tau^2\right)}N\left(s\right) + \frac{\sqrt{\kappa^2 + \tau^2}}{\kappa\left|\lambda\right|}\left(\frac{\tau T + \kappa B}{\sqrt{\kappa^2 + \tau^2}}\right).$$

The necessary operation are made, we get

$$\bar{W}\left(s\right) = \frac{\tau T + \kappa B}{\kappa\left|\lambda\right|} + \frac{\kappa\tau^{'} - \kappa^{'}\tau}{\kappa\left|\lambda\right|\left(\kappa^2 + \tau^2\right)}N\left(s\right),$$

$$\bar{W}\left(s\right) = \frac{1}{\kappa\left|\lambda\right|}\left(\tau T + \kappa B + \frac{\kappa\tau^{'} - \kappa^{'}\tau}{\kappa^2 + \tau^2}N\right)$$

and

$$\bar{W}\left(s\right) = \frac{1}{\kappa\left|\lambda\right|}\left(W + \frac{\left(\frac{\tau}{\kappa}\right)^{'}\kappa^2}{\kappa^2 + \tau^2}N\right).$$

Furthermore, Since

$$\frac{\sin\Phi}{\cos\Phi} = \frac{\tau\diagup\sqrt{\kappa^2 + \tau^2}}{\kappa\diagup\sqrt{\kappa^2 + \tau^2}},$$

$$\frac{\tau}{\kappa} = \tan\Phi.$$

By taking derivative of the last equation, we have

$$\Phi^{'}\sec^2\Phi = \left(\frac{\tau}{\kappa}\right)^{'}.$$

By a straightforward calculation, we get

$$\Phi^{'} = \left(\frac{\tau}{\kappa}\right)^{'}\frac{\kappa}{\kappa^2 + \tau^2},$$

$$\bar{W}\left(s\right) = \frac{1}{\kappa\left|\lambda\right|}\left(W + \Phi^{'}N\right),$$

which completes the proof. □

If the equation (3.22) is separated into the real and dual part, we can obtain

$$\bar{w} = \frac{w + \varphi^{'}n}{\mu k_1},$$

$$\bar{w}^{*} = \frac{\mu k_1\left(w^{*} + \varphi^{'}n + \varphi^{'*}n\right) - \left(\mu k_1^{*} + \mu^{*}k_1\right)\left(w + \varphi^{'}n\right)}{\mu^2 k_1^2}.$$

If the equation (3.24) is separated into the real and dual part, we can obtain

$$\bar{w} = \frac{\sqrt{k_1^2 + k_2^2}}{\mu k_1}\left(\sin\varphi t + \cos\varphi b\right),$$

$$\overline{w}^* = \frac{\sqrt{k_1^2 + k_2^2}}{\mu k_1} \left(\sin\varphi t^* + \cos\varphi b^* + \varphi^* \left(\cos\varphi t - \sin\varphi b \right) \right)$$

$$+ \frac{\mu k_1 \left(k_1 k_1^* + k_2 k_2^* \right) - \left(k_1^2 + k_2^2 \right) \left(\mu k_1^* + \mu^* k_1 \right)}{\sqrt{k_1^2 + k_2^2} \mu^2 k_1^2} \left(\sin\varphi t + \cos\varphi b \right) \cdot$$

Theorem 3.5 *Let* $\widetilde{\alpha}, \widetilde{\beta}$ *be dual curves and the dual curve* $\widetilde{\beta}$ *involute of the dual curve* $\widetilde{\alpha}$ *. If* C *and* $\overline{C}$ *are unit vectors of the direction of* W *and* $\overline{W}$, *respectively*

$$\overline{C} = \frac{\Phi'}{\sqrt{\Phi'^2 + \kappa^2 + \tau^2}} N + \frac{\sqrt{\kappa^2 + \tau^2}}{\sqrt{\Phi'^2 + \kappa^2 + \tau^2}} C. \tag{3.25}$$

Proof Since $\widetilde{\beta}$ the dual angle between $\overline{W}$ and $\overline{B}$ we can write

$$\overline{C}(s) = \sin\widetilde{\beta} \overline{T}(s) + \cos\widetilde{\beta} \overline{B}(s).$$

In here, we want to find the statements $\sin\widetilde{\beta}$ and $\cos\widetilde{\beta}$, we know that

$$\sin\widetilde{\beta} = \frac{\overline{\tau}}{\left\| \overline{W} \right\|} = \frac{\overline{\tau}}{\sqrt{\overline{\kappa}^2 + \overline{\tau}^2}} \cdot$$

By substituting $\overline{\tau}$ and $\overline{\kappa}$ in the last equation and necessary operations are made, we get

$$\sin\widetilde{\beta} = \frac{\Phi'}{\sqrt{\Phi'^2 + \kappa^2 + \tau^2}} \cdot \tag{3.26}$$

Similarly,

$$\cos\widetilde{\beta} = \frac{\sqrt{\kappa^2 + \tau^2}}{\sqrt{\Phi'^2 + \kappa^2 + \tau^2}} \cdot \tag{3.27}$$

Thus we find

$$\overline{C} = \frac{\Phi'}{\sqrt{\Phi'^2 + \kappa^2 + \tau^2}} \overline{T} + \frac{\sqrt{\kappa^2 + \tau^2}}{\sqrt{\Phi'^2 + \kappa^2 + \tau^2}} C,$$

which completes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

If the equation (3.25) is separated into the real and dual part, we can obtain

$$\overline{c} = \frac{\varphi' n + \sqrt{k_1^2 + k_2^2} c}{\sqrt{\varphi' + k_1^2 + k_2^2}},$$

$$\overline{c}^* = \frac{\varphi' n^* + \varphi'^* n + \sqrt{k_1^2 + k_2^2} c^* + \frac{k_1 k_1^* + k_2 k_2^*}{\sqrt{k_1^2 + k_2^2}} c - \frac{\varphi' n \left(\sqrt{k_1^2 + k_2^2} \right) c \left(\varphi' \varphi'^* + k_1 k_1^* + k_2 k_2^* \right)}{\sqrt{\varphi' + k_1^2 + k_2^2}}}{\sqrt{\varphi' + k_1^2 + k_2^2}} \cdot$$

If the equation (3.26) and (3.27) are separated into the real and dual part, we can obtain

$$\sin\bar{\varphi} = \frac{\varphi^{'}}{\sqrt{\varphi^{'} + k_1^2 + k_2^2}},$$

$$\cos\bar{\varphi} = \frac{\left(\Phi^{'2} + \kappa^2 + \tau^2\right)\Phi^{'*} - \varphi^{'}\varphi^{*} + k_1 k_1^* + k_2 k_2^* \varphi^{'}}{\bar{\varphi}^{*}\left(\Phi^{'2} + \kappa^2 + \tau^2\right)^{\frac{3}{2}}},$$

$$\cos\bar{\varphi} = \sqrt{\frac{k_1^2 + k_2^2}{\varphi^{'2} + k_1^2 + k_2^2}},$$

$$\sin\bar{\varphi} = \frac{\left(\varphi^{'}\varphi^{*} + k_1 k_1^* + k_2 k_2^*\right)\sqrt{k_1^2 + k_2^2} - \left(\varphi^{'2} + k_1^2 + k_2^2\right)\left(k_1 k_1^* + k_2 k_2^*\right)}{\bar{\varphi}^{*}\left(\Phi^{'2} + \kappa^2 + \tau^2\right)^{\frac{3}{2}}\sqrt{k_1^2 + k_2^2}}.$$

Corollary 3.2 *Let $\widetilde{\alpha}, \widetilde{\beta}$ be dual curves and the dual curve $\widetilde{\beta}$ involute of the dual curve $\widetilde{\alpha}$. If evolute curve $\widetilde{\alpha}$ is helix,*

(1) *The vectors $\bar{W}$ and $\bar{B}$ of the involute curve $\widetilde{\beta}$ are linearly dependent;*

(2) $C = \bar{C}$;

(3) $\widetilde{\beta}$ *is planar.*

Proof (1) If the evolute curve $\widetilde{\alpha}$ is helix, then we have

$$\frac{\tau}{\kappa} = \tan\Phi = cons \text{ or } \Phi^{'} = 0$$

and then we have

$$\sin\bar{\Phi} = 0,$$
$$\cos\bar{\Phi} = 1.$$

Thus, we get

$$\bar{\Phi} = 0. \tag{3.28}$$

(2) Substituting by the equation (3.28) into the equation (3.25), we have

$$C = \bar{C}.$$

(3) For being is a helix , then we have

$$\frac{\tau}{\kappa} = cons, \quad \left(\frac{\tau}{\kappa}\right)^{'} = 0. \tag{3.29}$$

On the other hand, from the equation (3.16), we can write

$$\frac{\overline{\tau}}{\overline{\kappa}} = \frac{\frac{\kappa \tau' - \kappa' \tau}{\lambda \kappa (\kappa^2 + \tau^2)}}{\frac{\left(\kappa^2 + \tau^2\right)^{\frac{1}{2}}}{\lambda \kappa}} = \frac{\left(\frac{\tau}{\kappa}\right)' \kappa^2}{\left(\kappa^2 + \tau^2\right)^{\frac{3}{2}}}. \tag{3.30}$$

Substituting by the equation (3.29) into the equation (3.30), then we find

$$\overline{\tau} = 0,$$

which completes the proof. □

References

[1] Bilici M. and Çalışkan M., Some characterizations for the pair of involute-evolute curves in Euclidean space, *Bulletin of Pure and Applied Sciences*, Vol.21E, No.2, 289-294, 2002.

[2] Bilici M. and Çalışkan M., On the involutes of the spacelike curve with a timelike binormal in Minkowski 3-space, *International Mathematical Forum*, Vol. 4, No.31, 1497-1509, 2009.

[3] Bilici M. and Çalışkan M., Some new notes on the involutes of the timelike curves in Minkowski3-space, *Int.J.Contemp.Math. Sciences*, Vol.6, No.41, 2019-2030, 2011.

[4] Bükçü B and Karacan M.K., On the involute and evolute curves of the spacelike curve with a spacelike binormal in Minkowski 3 space, *Int. J. Contemp. Math. Sciences*, Vol. 2, No. 5, 221 - 232, 2007.

[5] Fenchel W., On the differential geometry of closed space curves, *Bull. Amer. Math. Soc.*, Vol.57, No.1, 44-54, 1951.

[6] Hacısalihoğlu H. H., Acceleration Axes in Spatial Kinematics I, *Communications, Série A: Mathématiques, Physique et Astronomie*, Tome 20 A, pp. 1-15, Année 1971.

[7] Hacısalihoğlu H.H., *Differential Geometry*, (Turkish) Ankara University of Faculty of Science, 2000.

[8] Millman R.S. and Parker G.D., *Elements of Differential Geometry*, Prentice-Hall Inc., Englewood Cliffs, New Jersey, 1977.

[9] Sabuncuoğlu, A., *Differential Geometry* (Turkish), Nobel Publishing, 2006.

[10] Şenyurt S. and Gür S., On the dual spacelike-spacelike involute-evolute curve couple on dual Lorentzian space, *International Journal of Mathematical Engineering and Science*, ıssn:2277-6982, vol.1, Issue :5, 14-29, 2012.

[11] Şenyurt S. and Gür S., Timelike - spacelike involute - evolute curve couple on dual Lorentzian space, *J. Math. Comput. Sci.*, Vol.2, No. 6, 1808-1823, 2012.

[12] Şenyurt S. and Gür S., Spacelike - timelike involute- evolute curve couple on dual Lorentzian space, *J. Math. Comput. Sci.*, Vol.3, No.4,1054-1075, 2013.

[13] Yüce S. and Bektaş Ö., Special involute-evolute partner D-curves in E^3 , *European Journal of Pure and Applied Mathematics*, Vol. 6, No. 1, 20-29, 2013.