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One Modulo N Gracefulness of Some Arbitrary Supersubdivision and Removal Graphs

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Abstract: A graph G is said to be one modulo N graceful (where N is a positive integer) if there is a function ϕ from the vertex set of G to $\{0, 1, N, (N+1), 2N, (2N+1), \cdots, N(q-1), N(q-1)+1\}$in such a way that (i) ϕ is $1-1$ (ii) ϕ induces a bijection ϕ^* from the edge set of G to $\{1, N+1, 2N+1, \cdots, N(q-1)+1\}$where $\phi^*(uv)$=$|\phi(u)-\phi(v)|$. In this paper we prove that arbitrary supersubdivision of disconnected path and cycle $P_n \cup C_r$ is one modulo N graceful for all positive integer N. Also we prove that the graph $P_n^+ - v_k^{(1)}$ is one modulo N graceful for every positive integer N.

Key Words: Graceful, modulo N graceful, disconnected graphs, arbitrary supersubdivision graphs, $P_n \bigcup C_n$ and $P_n^+ - v_k^{(1)}$.

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§1. Introduction

S. W. Golomb [3] introduced graceful labelling. Odd gracefulness was introduced by R. B. Gnanajothi [4]. C. Sekar [11] introduced one modulo three graceful labelling. In [8,9], we introduced the concept of one modulo N graceful where N is any positive integer. In the case $N=2$, the labelling is odd graceful and in the case $N=1$ the labelling is graceful. Joseph A. Gallian [2] surveyed numerous graph labelling methods. Recently G. Sethuraman and P. Selvaraju [5] have introduced a new method of construction called supersubdivision of a graph. Let G be a graph with n vertices and t edges. A graph H is said to be a supersubdivision of G if H is obtained by replacing every edge e_i of G by the complete bipartite graph $K_{2,m}$ for some positive integer m in such a way that the ends of e_i are merged with the two vertices part of $K_{2,m}$ after removing the edge e_i from G. A supersubdivision H of a graph G is said to be an arbitrary supersubdivision of the graph G if every edge of G is replaced by an arbitrary $K_{2,m}$ (m may vary for each edge arbitrarily). A graph G is said to be connected if any two vertices of G are joined by a path. Otherwise it is called disconnected graph.

G. Sethuraman and P. Selvaraju [6] proved that every connected graph has some supersub-

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division that is graceful. They pose the question as to whether some supersubdivision is valid for disconnected graphs. [10] We proved that an arbitrary supersubdivision of disconnected paths are graceful. Barrientos and Barrientos [1] proved that any disconnected graph has a supersubdivision that admits an α-labeling. They also proved that every supersubdivision of a connected graph admits an α-labeling.

In this paper we prove that arbitrary supersubdivision of disconnected path and cycle $P_n \cup C_r$ is one modulo N graceful for all positive integer N. When $N = 1$ we get an affirmative answer for their question. Also we prove that the graph $P_n^+ - v_k^{(1)}$ is one modulo N graceful for every positive integer N.

§2. Main Results

Definition 2.1 *A graph G with q edges is said to be one modulo N graceful (where N is a positive integer) if there is a function ϕ from the vertex set of G to $\{0, 1, N, (N+1), 2N, (2N+1), \ldots, N(q-1), N(q-1)+1\}$ in such a way that (i) ϕ is $1-1$ (ii) ϕ induces a bijection ϕ^* from the edge set of G to $\{1, N+1, 2N+1, \ldots, N(q-1)+1\}$where $\phi^*(uv)$=$|\phi(u) - \phi(v)|$.*

Definition 2.2 *In the complete bipartite graph $K_{2,m}$ we call the part consisting of two vertices, the 2-vertices part of $K_{2,m}$ and the part consisting of m vertices the m-vertices part of $K_{2,m}$.Let G be a graph with p vertices and q edges. A graph H is said to be a supersubdivision of G if H is obtained by replacing every edge e of G by the complete bipartite graph $K_{2,m}$ for some positive integer m in such a way that the ends of e are merged with the two vertices part of $K_{2,m}$ after removing the edge e from G. H is denoted by $SS(G)$.*

Definition 2.3 *A supersubdivision H of a graph G is said to be an arbitrary supersubdivision of the graph G if every edge of G is replaced by an arbitrary $K_{2,m}$ (m may vary for each edge arbitrarily). H is denoted by $ASS(G)$.*

Definition 2.4 *Let $v_1, v_2, \ldots, v_n$ be the vertices of a path of length n and $v_1^{(1)}, v_2^{(1)}, \ldots, v_n^{(1)}$ be the pendant vertices attached with $v_1, v_2, \ldots, v_n$ respectively. The removal of a pendant vertex $v_k^{(1)}$ where $1 \leq\ k \leq n$ from P_n^+yields the graph $P_n^+ - v_k^{(1)}$.*

Theorem 2.5 *Arbitrary supersubdivision of disconnected path and cycle $P_n \cup C_r$ is one modulo N graceful provided the arbitrary supersubdivision is obtained by replacing each edge of G by $K_{2,m}$ with $m \geqslant 2$.*

Proof Let P_n be a path with successive vertices $v_1, v_2, \cdots, v_n$ and let e_i $(1 \leq i \leq n-1)$ denote the edge $v_i v_{i+1}$ of P_n. Let C_r be a cycle with successive vertices $v_{n+1}, v_{n+2}, \cdots, v_{n+r}$ and let $e_i (n+1 \leq i \leq n+r)$ denote the edge $v_i v_{i+1}$.

Let H be an arbitrary supersubdivision of the disconnected graph $P_n \cup C_r$ where each edge e_i of $P_n \cup C_r$ is replaced by a complete bipartite graph K_{2,m_i} with $m_i \geqslant 2$ for $1 \leq i \leq n-1$ and $n+1 \leq i \leq n+r$. Here the edge $v_{n+r}v_{n+1}$ is replaced by $k_{2,r-1}$. We observe that H has $M = 2(m_1 + m_2 + \cdots + m_{n-1} + m_{n+1} + \cdots + m_{n+r})$ edges.

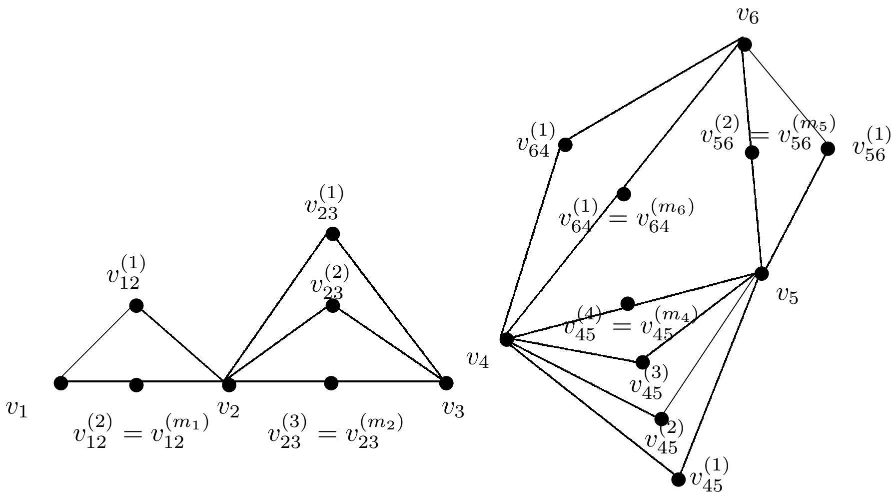

Figure 1 Supersubdivision of $P_3 \cup C_3$

Define

$\phi(v_i) = N(i-1),\ i = 1, 2, 3, \cdots, n,$

$\phi(v_i) = N(i),\ i = n+1, n+2, n+3, \cdots, n+r$, and for $k = 1, 2, 3, \ldots, m_i$,

$$\phi(v_{i,i+1}^{(k)}) = \begin{cases} N(M-2k+1)+1 & \text{if } i = 1 \\ N(M-2+i)+1-2N(m_1+m_2+\cdots+m_{i-1}+k-1) & \text{if } i = 2, 3, \ldots n-1 \\ N(M-1+i)+1-2N(m_1+m_2+\cdots+m_{n-1}+k-1) & \text{if } i = n+1 \\ N(M-1+i)+1-2N[(m_1+m_2+\cdots+m_{n-1})+ & \\ (m_{n+1}+\cdots+m_{i-1})+k-1] & \text{if } i = n+2, n+3, \ldots n+r-1 \end{cases}$$

and for $k = 1, 2, 3, \cdots, m_{n+r}$, $\phi(v_{n+r,n+1}^{(k)}) = N(n+r-k+m_{n+r})+1$

From the definition of ϕ it is clear that

$$\{\phi(v_i), i = 1, 2, \cdots, n+r\} \bigcup \{\phi(v_{i,i+1}^{(k)}), i = 1, 2, \cdots, n+r-1 \text{ and } k = 1, 2, 3, \cdots, m_i\} \bigcup \{\phi(v_{n+r,n+1}^{(k)}), k = 1, 2, 3, \cdots, m_i\}$$

$$= \{0, N, 2N, \cdots, N(n-1)\} \bigcup \{N(n+1), N(n+2), \cdots, N(n+r)\}$$

$$\bigcup \{N[M-2k+1]+1, N[M-2m_1]+1, N[M-2m_1-2]+1 \cdots,$$

$$N[M-2(m_1+m_2)+2]+1, N[M-2(m_1+m_2)+1]+1,$$

$$N[M-2(m_1+m_2)-1]+1, \ldots, N[M-2(m_1+m_2+m_3)+3]+1,$$

$$\ldots, N[M-3+n-2(m_1+m_2+\cdots+m_{n-2})]+1,$$

$$N[M-5+n-2(m_1+m_2+\cdots+m_{n-2})]+1, \ldots,$$

$$N[M-1+n-2(m_1+m_2+\cdots+m_{n-1})]+1,$$

$$N[M+n-2(m_1+m_2+\cdots+m_{n-1})]+1,$$

$$N[M+n-2(m_1+m_2+\cdots+m_{n-1}+1)]+1, \ldots,$$

$$N[M+n-2(m_1+m_2+\cdots+m_{n-1}+m_{n+1}-1)]+1,$$

$$N[M+1+n-2(m_1+m_2+\cdots+m_{n-1}+m_{n+1})]+1,$$

$N[M-1+n-2(m_1+m_2+\cdots+m_{n-1}+m_{n+1})]+1,\cdots,$
$N[M+3+n-2(m_1+m_2+\cdots+m_{n-1}+m_{n+1}+m_{n+2})]+1,$
$N[M+2+n-2(m_1+m_2+\cdots+m_{n-1}+m_{n+1}+m_{n+2})]+1,$
$N[M+n-2(m_1+m_2+\cdots+m_{n-1}+m_{n+1}+m_{n+2})]+1,\ldots,$
$N[M+4+n-2(m_1+m_2+\cdots+m_{n-1}+m_{n+1}+m_{n+2}+m_{n+3})]+1,$
$N[M-2+n+r-2(m_1+m_2+\cdots+m_{n-1}+m_{n+1}+m_{n+2})]+1,$
$N[M-2+n+r-2[(m_1+m_2+\cdots+m_{n-1})+(m_{n+1}+m_{n+2}+\cdots+m_{n+r-2})]]+1,$
$N[M-4+n+r-2[(m_1+m_2+\cdots+m_{n-1})+(m_{n+1}+m_{n+2}+\cdots+m_{n+r-2})]]+1,$
$\cdots,N[M+n+r-2[(m_1+m_2+\cdots+m_{n-1})+(m_{n+1}+m_{n+2}+\cdots+m_{n+r-1})]]+1\}$
$\bigcup\{N(n+r-1+m_{n+r})+1,N(n+r-2+m_{n+r})+1,\cdots,N(n+r)+1\}$

Thus it is clear that the vertices have distinct labels. Therefore ϕ is $1-1$. We compute the edge labels as follows:

For $k=1,2,\cdots,m_1$, $\phi^*(v_{1,2}^{(k)}v_1)=\mid\phi(v_{1,2}^{(k)})-\phi(v_1)\mid=N(M-2k+1)+1$, $\phi^*(v_{1,2}^{(k)}v_2)=\mid\phi(v_{1,2}^{(k)})-\phi(v_2)\mid=N(M-2k)+1$.

For $k=1,2,\cdots,m_i$ and $i=2,3,\cdots,n-1$, $\phi^*(v_{i,i+1}^{(k)}v_i)=\mid\phi(v_{i,i+1}^{(k)})-\phi(v_i)\mid=N(M-2k+1)-2N(m_1+m_2+\cdots+m_{i-1})+1$, $\phi^*(v_{i,i+1}^{(k)}v_{i+1})=\mid\phi(v_{i,i+1}^{(k)})-\phi(v_{i+1})\mid=N(M-2k)-2N(m_1+m_2+\cdots+m_{i-1})+1$.

For $k=1,2,\cdots,m_{n+1}$, $\phi^*(v_{n+1,n+2}^{(k)}v_{n+1})=\mid\phi(v_{n+1,n+2}^{(k)})-\phi(v_{n+1})\mid=N(M-2k+1)-2N(m_1+m_2+\cdots+m_{n-1})+1$, $\phi^*(v_{n+1,n+2}^{(k)}v_{n+2})=\mid\phi(v_{n+1,n+2}^{(k)})-\phi(v_{n+2})\mid=N(M-2k)-2N(m_1+m_2+\cdots+m_{n-1})+1$.

For $k=1,2,\ldots,m_i$ and $j=n+2,n+3,\cdots,n+r$, $\phi^*(v_{i,i+1}^{(k)}v_i)=\mid\phi(v_{i,i+1}^{(k)})-\phi(v_i)\mid=N(M-2k+1)-2N\{(m_1+m_2+\cdots+m_{n-1})+(m_{n+1}+m_{n+2}+\cdots+m_{i-1})\}+1$, $\phi^*(v_{i,i+1}^{(k)}v_{i+1})=\mid\phi(v_{i,i+1}^{(k)})-\phi(v_{i+1})\mid=N(M-2k)-2N\{(m_1+m_2+\cdots+m_{n-1})+(m_{n+1}+m_{n+2}+\cdots+m_{i-1})\}+1$.

For $k=1,2,\cdots,m_{n+r}$, $\phi^*(v_{n+r,n+1}^{(k)}v_{n+r})=\mid\phi(v_{n+r,n+1}^{(k)})-\phi(v_{n+r})\mid=N(m_{n+r}-k)+1$, $\phi^*(v_{n+r,n+1}^{(k)}v_{n+1})=\mid\phi(v_{n+r,n+1}^{(k)})-\phi(v_{n+1})\mid=N(m_{n+r}+r-k-1)+1$.

It is clear from the above labelling that the m_i+2 vertices of K_{2,m_i} have distinct labels and the $2m_i$ edges of K_{2,m_i} also have distinct labels for $1\le i\le n-1$ and $n+1\le i\le n+r-1$. Therefore the vertices of each $K_{2,m_i}, 1\le i\le n-1$ and $n+1\le i\le n+r-1$ in the arbitrary supersubdivision H of $P_n\cup C_r$ have distinct labels and also the edges of each $K_{2,m_i}, 1\le i\le n-1$ and $n+1\le i\le n+r-1$ in the arbitrary supersubdivision graph H of $P_n\cup C_r$ have distinct labels. Clearly H is one modulo N graceful. Hence arbitrary supersubdivisions of disconnected path and cycle $P_n\cup C_r$ is one modulo N graceful, for every positive integer N.

Consequently, every disconnected graph has some supersubdivision that is one modulo N graceful. □

Example 2.6 A odd graceful labelling of $ASS(P_3\cup C_4)$ is shown in Figure 2.

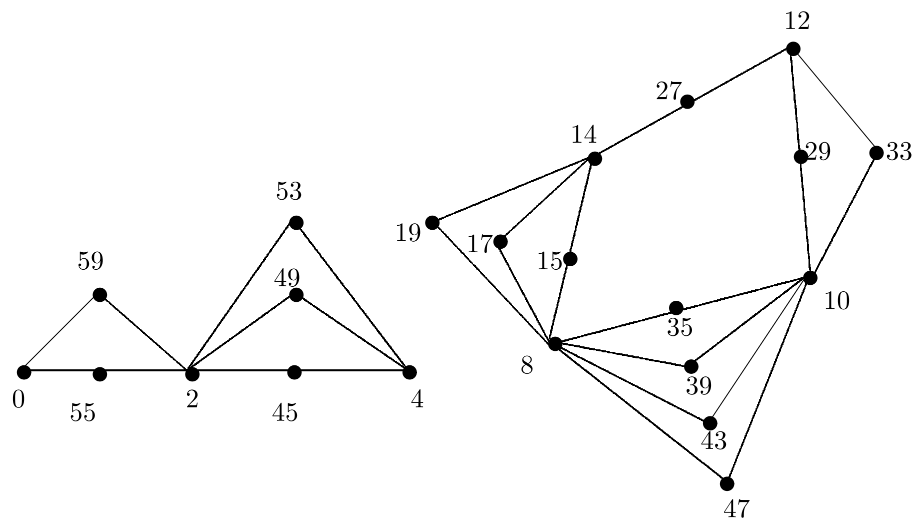

Figure 2

Example 2.7 A graceful labelling of $ASS(P_3 \cup C_3)$ is shown in Figure 3.

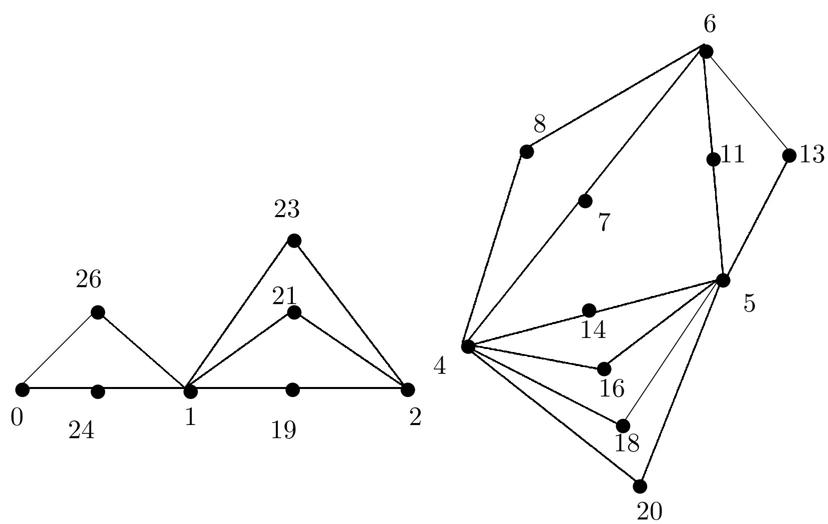

Figure 3

Theorem 2.8 *For any pendant vertex $v_k^{(1)} \in V(P_n^+)$, the graph $P_n^+ - v_k^{(1)}$ is one modulo N graceful for every positive integer N.*

Proof Let $v_1, v_2, \cdots, v_n$ be the vertices of a path of length n and $v_1^{(1)}, v_2^{(1)}, \cdots, v_n^{(1)}$ the pendant vertices attached with $v_1, v_2, \cdots, v_n$ respectively. Consider the graph $P_n^+ - v_k^{(1)}$, where $1 \leq k \leq n$. It has $2n-1$ vertices and $2n-2$ edges.

Case 1. n is even and k is even

Define

$$\phi(v_{2i-1}) = \begin{cases} N(2n-3)+1-2N(i-1) & \text{for i} = 1, 2, \cdots, \frac{\text{k}}{2} \\ N(2n-3)+1-2N(\frac{k}{2}-1)-N-2N(i-(\frac{k}{2}+1)) & \text{for i} = \frac{k}{2}+1, \cdots, \frac{\text{n}}{2} \end{cases},$$

$\phi(v_{2i}) = N(2i-1)$ for $i = 1, 2, \cdots, \frac{n}{2}$,

$$\phi(v_{2i}^{(1)}) = \begin{cases} 2N(n-2)+1-2N(i-1) & \text{for i} = 1, 2, \cdots, \frac{\text{k}}{2}-1 \\ 2N(n-2)+1-2N(\frac{k}{2}-2)-3N-2N(i-(\frac{k}{2}+1)) & \text{fori} = \frac{\text{k}}{2}+1, \frac{\text{k}}{2}+2, \cdots, \frac{\text{n}}{2} \end{cases},$$

$\phi(v_{2i-1}^{(1)}) = 2N(i-1)$ for $i = 1, 2, \cdots, \frac{n}{2}$.

From the definition of ϕ it is clear that

$$\begin{aligned} &\{\phi(v_{2i-1}),\ i = 1, 2, \cdots, \frac{n}{2}\} \bigcup \{\phi(v_{2i}),\ i = 1, 2, \cdots, \frac{n}{2}\} \\ &\bigcup \{\phi(v_{2i}^{(1)}), i = 1, 2, \cdots, \frac{k}{2}-1, \frac{k}{2}+1, \frac{k}{2}+2, \cdots, \frac{n}{2}\} \\ &\bigcup \{\phi(v_{2i-1}^{(1)}),\ i = 1, 2, \cdots, \frac{n}{2}\} \\ &= \{N(2n-3)+1, N(2n-5)+1, \cdots, N(2n-k-1)+1, N(2n-k-2)+1, \\ &\quad N(2n-k-4)+1, \ldots, Nn+1\} \bigcup \{N, 3N, \cdots, N(n-1)\} \\ &\quad \bigcup \{2N(n-2)+1, 2N(n-3)+1, \cdots, N(2n-k)+1, N(2n-k-3)+1, \\ &\quad N(2n-k-5)+1, \cdots, N(n-1)+1\} \bigcup \{0, 2N, \ldots, N(n-2)\} \end{aligned}$$

Thus it is clear that the vertices have distinct labels. Therefore ϕ is $1-1$. We compute the edge labels as follows.

For $i = 1, 2, \cdots, \frac{k}{2}$, $\phi^*(v_{2i-1}v_{2i}) = \mid \phi(v_{2i-1}) - \phi(v_{2i}) \mid = N(2n-4i)+1$, $\phi^*(v_{2i-1}v_{2i-1}^{(1)}) = \mid \phi(v_{2i-1}) - \phi(v_{2i-1}^{(1)}) \mid = N(2n-4i+1)+1$.

For $i = 1, 2, \cdots, \frac{k}{2}-1$, $\phi^*(v_{2i+1}v_{2i}) = \mid \phi(v_{2i+1}) - \phi(v_{2i}) \mid = N(2n-4i-2)+1$, $\phi^*(v_{2i}^{(1)}v_{2i}) = \mid \phi(v_{2i}^{(1)}) - \phi(v_{2i}) \mid = N(2n-4i-1)+1$.

For $i = \frac{k}{2}+1, \frac{k}{2}+2, \cdots, \frac{n}{2}$, $\phi^*(v_{2i-1}v_{2i}) = \mid \phi(v_{2i-1}) - \phi(v_{2i}) \mid = N(2n-4i+1)+1$, $\phi^*(v_{2i-1}v_{2i-1}^{(1)}) = \mid \phi(v_{2i-1}) - \phi(v_{2i-1}^{(1)}) \mid = N(2n-4i+2)+1$, $\phi^*(v_{2i}^{(1)}v_{2i}) = \mid \phi(v_{2i}^{(1)}) - \phi(v_{2i}) \mid = N(2n-4i)+1$.

For $i = \frac{k}{2}+1, \frac{k}{2}+2, \cdots, \frac{n}{2}-1$, $\phi^*(v_{2i+1}v_{2i}) = \mid \phi(v_{2i+1}) - \phi(v_{2i}) \mid = N(2n-4i-1)+1$.

This show that the edges have the distinct labels $\{1, N+1, 2N+1, \cdots, N(q-1)+1\}$, where $q = 2n-2$. Hence for every positive integer N, $P_n^+ - v_k^{(1)}$ is one modulo N graceful if n is even and k is even.

Example 2.9 A one modulo 10 graceful labelling of $P_{10}^{+} - v_6^{(1)}$ is shown in Figure 4.

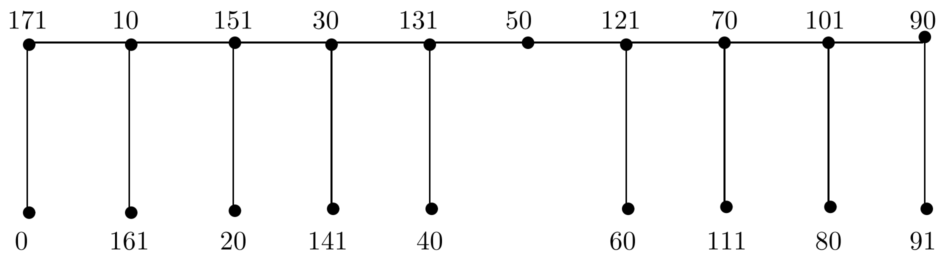

Figure 4

Case 2. n is even and k is odd

Define

$$\phi(v_{2i}) = \begin{cases} N(2i-1) & \text{for i} = 1, 2, \cdots, \frac{\text{k}-1}{2} \\ N(k-2) + N + 2N(i - (\frac{(k+1)}{2})) & for\ i = \frac{k+1}{2}, \frac{k+3}{2}, \cdots, \frac{n}{2} \end{cases},$$

$\phi(v_{2i-1}) = N(2n-3) + 1 - 2N(i-1)$ for $i = 1, 2, \cdots, \frac{n}{2}$,

$$\phi(v_{2i-1}^{(1)}) = \begin{cases} 2N(i-1) & \text{for i} = 1, 2, \cdots, \frac{\text{k}-1}{2} \\ 2N(\frac{k-1}{2} - 1) + 3N + 2N(i - (\frac{k+3}{2})) & \text{fori} = \frac{\text{k}+3}{2}, \frac{\text{k}+5}{2}, \cdots, \frac{\text{n}}{2} \end{cases},$$

$\phi(v_{2i}^{(1)}) = 2N(n-2) + 1 - 2N(i-1)$ for $i = 1, 2, \cdots, \frac{n}{2}$.

The proof is similar to that of Case 1. Hence for every positive integer N, $P_n^{+} - v_k^{(1)}$ is one modulo N graceful if n is even and k is odd.

Example 2.10 A one modulo 4 graceful labelling of $P_{12}^{+} - v_9^{(1)}$ is shown in Figure 5.

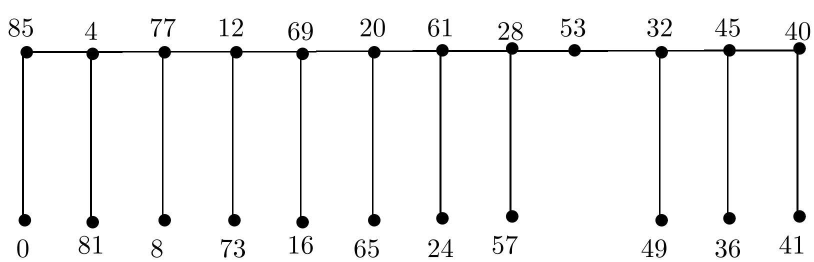

Figure 5

Case 3. n is odd and k is even

Define

$$\phi(v_{2i-1})=\begin{cases} N(2n-3)+1-2N(i-1) & \text{for i} = 1,2,\cdots,\frac{k}{2} \\ N(2n-3)+1-2N(\frac{k}{2}-1)-N-2N(i-(\frac{k}{2}+1)) & \text{fori} = \frac{k}{2}+1,\cdots,\frac{n-1}{2} \end{cases},$$

$\phi(v_{2i})=N(2i-1)$ for $i=1,2,\ldots,\frac{n-1}{2}$,

$$\phi(v_{2i}^{(1)})=\begin{cases} 2N(n-2)+1-2N(i-1) & \text{for i} = 1,2,\ldots,\frac{k}{2}-1 \\ 2N(n-2)+1-2N(\frac{k}{2}-2)-3N-2N(i-(\frac{k}{2}+1)) & \text{fori} = \frac{k}{2}+1,\cdots,\frac{n-1}{2} \end{cases},$$

$\phi(v_{2i-1}^{(1)})=2N(i-1)$ for $i=1,2,\ldots,\frac{n-1}{2}$.

From the definition of ϕ it is clear that

$$\{\phi(v_{2i-1}),\ i=1,2,\cdots,\frac{n-1}{2}\}\bigcup\{\phi(v_{2i}),\ i=1,2,\cdots,\frac{n-1}{2}\}$$
$$\bigcup\{\phi(v_{2i}^{(1)}),\ i=1,2,\cdots,\frac{k}{2}-1,\frac{k}{2}+1,\frac{k}{2}+2,\cdots,\frac{n-1}{2}\}\bigcup\{\phi(v_{2i-1}^{(1)}),\ i=1,2,\cdots,\frac{n-1}{2}\}$$
$$=\{N(2n-3)+1,N(2n-5)+1,\ldots,N(2n-k-1)+1,N(2n-k-2)+1,$$
$$N(2n-k-4)+1,\ldots,N(n-1)+1\}\bigcup\{N,3N,\ldots,N(n-2)\}$$
$$\bigcup\{2N(n-2)+1,2N(n-3)+1,\ldots,N(2n-k)+1,N(2n-k-3)+1,$$
$$N(2n-k-5)+1,\ldots,Nn+1\}\bigcup\{0,2N,\ldots,N(n-1)\}$$

Thus it is clear that the vertices have distinct labels. Therefore ϕ is $1-1$. We compute the edge labels as follows:

For $i=1,2,\cdots,\frac{k}{2}$, $\phi^*(v_{2i-1}v_{2i})=\mid\phi(v_{2i-1})-\phi(v_{2i})\mid=N(2n-4i)+1$, $\phi^*(v_{2i-1}v_{2i-1}^{(1)})=\mid\phi(v_{2i-1})-\phi(v_{2i-1}^{(1)})\mid=N(2n-4i+1)+1$.

For $i=1,2,\cdots,\frac{k}{2}-1$, $\phi^*(v_{2i+1}v_{2i})=\mid\phi(v_{2i+1})-\phi(v_{2i})\mid=N(2n-4i-2)+1$, $\phi^*(v_{2i}^{(1)}v_{2i})=\mid\phi(v_{2i}^{(1)})-\phi(v_{2i})\mid=N(2n-4i-1)+1$.

For $i=\frac{k}{2}+1,\frac{k}{2}+2,\cdots,\frac{n-1}{2}$, $\phi^*(v_{2i-1}v_{2i})=\mid\phi(v_{2i-1})-\phi(v_{2i})\mid=N(2n-4i+1)+1$, $\phi^*(v_{2i}^{(1)}v_{2i})=\mid\phi(v_{2i}^{(1)})-\phi(v_{2i})\mid=N(2n-4i)+1$.

For $i=\frac{k}{2}+1,\frac{k}{2}+2,\cdots,\frac{n-1}{2}$, $\phi^*(v_{2i+1}v_{2i})=\mid\phi(v_{2i+1})-\phi(v_{2i})\mid=N(2n-4i-1)+1$.

For $i=\frac{k}{2}+1,\frac{k}{2}+2,\cdots,\frac{n+1}{2}$, $\phi^*(v_{2i-1}v_{2i-1}^{(1)})=\mid\phi(v_{2i-1})-\phi(v_{2i-1}^{(1)})\mid=N(2n-4i+2)+1$.

This show that the edges have the distinct labels $\{1,N+1,2N+1,\cdots,N(q-1)+1\}$, where $q=2n-2$. Hence for every positive integer N, $P_n^+-v_k^{(1)}$ is one modulo N graceful if n is odd and k is even.

Example 2.11 A one modulo 3 graceful labelling of $P_{13}^+-v_2^{(1)}$ is shown in Figure 6.

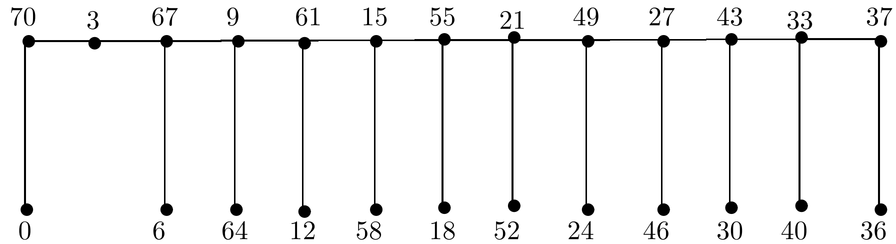

Figure 6

Case 4. n is odd and k is odd

Define

$$\phi(v_{2i}) = \begin{cases} N(2i-1) & \text{for i} = 1,2,\cdots,\frac{\text{k}-1}{2} \\ N(k-2)+N+2N(i-(\frac{(k+1)}{2})) & \text{fori} = \frac{\text{k}+1}{2},\frac{\text{k}+3}{2},\cdots,\frac{\text{n}-1}{2} \end{cases},$$

$\phi(v_{2i-1}) = N(2n-3)+1-2N(i-1)$ for $i = 1,2,\ldots,\frac{n-1}{2}$,

$$\phi(v_{2i-1}^{(1)}) = \begin{cases} 2N(i-1) & \text{for i} = 1,2,\cdots,\frac{\text{k}-1}{2} \\ 2N(\frac{k-1}{2}-1)+3N+2N(i-(\frac{k+3}{2})) & \text{fori} = \frac{\text{k}+3}{2},\frac{\text{k}+5}{2},\cdots,\frac{\text{n}-1}{2} \end{cases},$$

$\phi(v_{2i}^{(1)}) = 2N(n-2)+1-2N(i-1)$ for $i = 1,2,\ldots,\frac{n-1}{2}$.

The proof is similar to that of Case 3. Hence for every positive integer N, $P_n^+ - v_k^{(1)}$ is one modulo N graceful if n is odd and k is odd. □

Example 2.12 A one modulo 5 graceful labelling of $P_{11}^+ - v_5^{(1)}$ is shown in Figure 7.

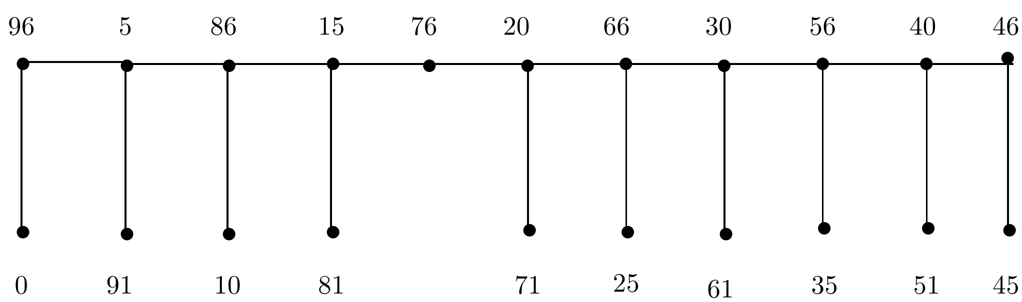

Figure 7

§3. Conclusion

Subdivision or supersubdivision or arbitrary supersubdivision of certain graphs which are not graceful may be graceful. The method adopted in making a graph one modulo N graceful will provide a new approach to have graceful labelling of graphs and it will be helpful to attack standard conjectures and unsolved open problems.

References

[1] C. Barrientos and S. Barrientos, On graceful supersubdivisions of graphs, *Bull. Inst. Combin. Appl.*, 70 (2014) 77-85.

[2] Joseph A. Gallian, A Dynamic Survey of Graph Labeling, *The Electronic Journal of Combinatorics*, **18** (2011), #DS6.

[3] S.W.Golomb, How to number a graph in Graph theory and computing R.C. Read, ed., Academic press, New York (1972)23-27.

[4] R. B. Gnanajothi, *Topics in Graph Theory*, Ph.D. Thesis, Madurai Kamaraj University, 1991.

[5] G. Sethuraman and P. Selvaraju, Gracefulness of arbitrary supersubdivisions of graphs, *Indian J. Pure Appl. Math.*, 32 (2001) 1059-1064.

[6] G. Sethuraman and P. Selvaraju, Super-subdivisions of connected graphs are graceful, preprint.

[7] Z. Liang, On the gracefulness of the graph $C_m \cup P_n$, *Ars Combin.*, 62(2002), 273-280.

[8] V. Ramachandran, C. Sekar, One modulo N gracefullness of arbitrary supersubdivisions of graphs, *International J. Math. Combin.*, Vol.2 (2014) 36-46.

[9] V. Ramachandran, C. Sekar, One modulo N gracefulness of supersubdivision of ladder, *Journal of Discrete Mathematical Sciences and Cryptography* (Accepted).

[10] C. Sekar and V. Ramachandren, Graceful labelling of arbitrary supersubdivision of disconnected graph, *Ultra Scientist*, 25(2)A (2013) 315-318.

[11] C. Sekar, *Studies in Graph Theory*, Ph.D. Thesis, Madurai Kamaraj University, 2002.