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On Signed Graphs Whose Two Path Signed Graphs are Switching Equivalent to Their Jump Signed Graphs

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Abstract: In this paper, we obtained a characterization of signed graphs whose jump signed graphs are switching equivalent to their two path signed graphs.

Key Words: Smarandachely k-signed graph, t-Path graphs, jump graphs, signed graphs, balance, switching, t-path signed graphs and jump signed graphs.

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§1. Introduction

For standard terminology and notation in graph theory we refer Harary [6] and Zaslavsky [20] for signed graphs. Throughout the text, we consider finite, undirected graph with no loops or multiple edges.

Given a graph Γ and a positive integer t, the *t-path graph* $(\Gamma)_t$ of Γ is formed by taking a copy of the vertex set $V(\Gamma)$ of Γ, joining two vertices u and v in the copy by a single edge $e = uv$ whenever there is a $u - v$ path of length t in Γ. The notion of *t-path graphs* was introduced by Escalante et al. [4]. A graph G for which

$$(\Gamma)_t \cong \Gamma \tag{1}$$

has been termed as *t-path invariant graph* by Esclante et al. in [4], Escalante & Montejano [5] where the explicit solution to (1) has been determined for $t = 2, 3$. The structure of t-path invariant graphs are still remains uninvestigated in literature for all $t \geq 4$.

The *line graph* $L(\Gamma)$ of a graph $\Gamma = (V, E)$ is that graph whose vertices can be put in one-to-one correspondence with the edges of Γ so that two vertices of $L(\Gamma)$ are adjacent if, and only if, the corresponding edges of Γ are adjacent.

The *jump graph* $J(\Gamma)$ of a graph $\Gamma = (V, E)$ is $\overline{L(\Gamma)}$, the complement of the line graph $L(\Gamma)$ of Γ (see [6]).

A *Smarandachely k-signed graph* is an ordered pair $S = (G, \sigma)$ $(S = (G, \mu))$ where $G = (V, E)$ is a graph called *underlying graph of* S and $\sigma : E \to (\overline{e}_1, \overline{e}_2, \cdots, \overline{e}_k)$ $(\mu : V \to$

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$(\overline{e}_1, \overline{e}_2, \cdots, \overline{e}_k))$ is a function, where each $\overline{e}_i \in \{+, -\}$. Particularly, a Smarandachely 2-signed graph is called abbreviated to *signed graph*, where $\Gamma = (V, E)$ is a graph called *underlying graph of* Σ and $\sigma : E \rightarrow \{+, -\}$ is a function. We say that a signed graph is *connected* if its underlying graph is connected. A signed graph $\Sigma = (\Gamma, \sigma)$ is *balanced*, if every cycle in Σ has an even number of negative edges (See [7]). Equivalently, a signed graph is balanced if product of signs of the edges on every cycle of Σ is positive.

Signed graphs Σ_1 and Σ_2 are isomorphic, written $\Sigma_1 \cong \Sigma_2$, if there is an isomorphism between their underlying graphs that preserves the signs of edges.

The theory of balance goes back to Heider [8] who asserted that a social system is balanced if there is no tension and that unbalanced social structures exhibit a tension resulting in a tendency to change in the direction of balance. Since this first work of Heider, the notion of balance has been extensively studied by many mathematicians and psychologists. In 1956, Cartwright and Harary [3] provided a mathematical model for balance through graphs. For more new notions on signed graphs refer the papers (see [12-17, 20]).

A *marking* of Σ is a function $\zeta : V(\Gamma) \rightarrow \{+, -\}$. Given a signed graph Σ one can easily define a marking ζ of Σ as follows: For any vertex $v \in V(\Sigma)$,

$$\zeta(v) = \prod_{uv \in E(\Sigma)} \sigma(uv),$$

the marking ζ of Σ is called *canonical marking* of Σ.

A switching function for Σ is a function $\zeta : V \rightarrow \{+, -\}$. The switched signature is $\sigma^\zeta(e) := \zeta(v)\sigma(e)\zeta(w)$, where e has end points v, w. The switched signed graph is $\Sigma^\zeta := (\Sigma|\sigma^\zeta)$. We say that Σ switched by ζ. Note that $\Sigma^\zeta = \Sigma^{-\zeta}$ (see [1]).

If $X \subseteq V$, switching Σ by X (or simply switching X) means reversing the sign of every edge in the cut set $E(X, X^c)$. The switched signed graph is Σ^X. This is the same as Σ^ζ where $\zeta(v) := -$ if and only if $v \in X$. Switching by ζ or X is the same operation with different notation. Note that $\Sigma^X = \Sigma^{X^c}$.

Signed graphs Σ_1 and Σ_2 are switching equivalent, written $\Sigma_1 \sim \Sigma_2$ if they have the same underlying graph and there exists a switching function ζ such that $\Sigma_1^\zeta \cong \Sigma_2$. The equivalence class of Σ,

$$[\Sigma] := \{\Sigma' : \Sigma' \sim \Sigma\},$$

is called the its switching class.

Similarly, Σ_1 and Σ_2 are switching isomorphic, written $\Sigma_1 \cong \Sigma_2$, if Σ_1 is isomorphic to a switching of Σ_2. The equivalence class of Σ is called its switching isomorphism class.

Two signed graphs $\Sigma_1 = (\Gamma_1, \sigma_1)$ and $\Sigma_2 = (\Gamma_2, \sigma_2)$ are said to be *weakly isomorphic* (see [18]) or *cycle isomorphic* (see [19]) if there exists an isomorphism $\phi : \Gamma_1 \rightarrow \Gamma_2$ such that the sign of every cycle Z in Σ_1 equals to the sign of $\phi(Z)$ in Σ_2. The following result is well known (see [19]).

Theorem 1.(T. Zaslavsky, [19]) *Two signed graphs Σ_1 and Σ_2 with the same underlying graph are switching equivalent if and only if they are cycle isomorphic.*

In [11], the authors introduced the switching and cycle isomorphism for signed digraphs. The notion of t-path graph of a given graph was extended to the class of signed graphs by Mishra [9] as follows:

Given a signed graph Σ and a positive integer t, the t-path signed graph $(\Sigma)_t$ of Σ is formed by taking a copy of the vertex set $V(\Sigma)$ of Σ, joining two vertices u and v in the copy by a single edge $e = uv$ whenever there is a $u - v$ path of length t in S and then by defining its sign to be $-$ whenever in every $u - v$ path of length t in Σ all the edges are negative.

In [13], P. S. K. Reddy introduced a variation of the concept of t-path signed graphs studied above. The motivation stems naturally from one's mathematically inquisitiveness as to ask why not define the sign of an edge $e = uv$ in $(\Sigma)_t$ as the product of the signs of the vertices u and v in Σ. The *t-path signed graph* $(\Sigma)_t = ((\Gamma)_t, \sigma')$ of a signed graph $\Sigma = (\Gamma, \sigma)$ is a signed graph whose underlying graph is $(\Gamma)_t$ called *t-path graph* and sign of any edge $e = uv$ in $(\Sigma)_t$ is $\mu(u)\mu(v)$, where μ is the canonical marking of Σ. Further, a signed graph $\Sigma = (\Gamma, \sigma)$ is called *t-path signed graph*, if $\Sigma \cong (\Sigma')_t$, for some signed graph Σ'. In this paper, we follow the notion of t-path signed graphs defined by P. S. K. Reddy as above.

Theorem 2.(P. S. K. Reddy, [13]) *For any signed graph $\Sigma = (\Gamma, \sigma)$, its t-path signed graph $(\Sigma)_t$ is balanced.*

Corollary 3. *For any signed graph $\Sigma = (\Gamma, \sigma)$, its 2-path signed graph $(\Sigma)_2$ is balanced.*

The *jump signed graph* of a signed graph $S = (G, \sigma)$ is a signed graph $J(S) = (J(G), \sigma')$, where for any edge ee' in $J(S)$, $\sigma'(ee') = \sigma(e)\sigma(e')$. This concept was introduced by M. Acharya and D. Sinha [2] (See also E. Sampathkumar et al. [10]).

Theorem 4.(M. Acharya and D. Sinha, [2]) *For any signed graph $\Sigma = (\Gamma, \sigma)$, its jump signed graph $J(\Sigma)$ is balanced.*

§2. Switching Equivalence of Two Path Signed Graphs and Jump Signed Graphs

The main aim of this paper is to prove the following signed graph equation

$$(\Sigma)_2 \sim J(\Sigma).$$

We first characterize graphs whose two path graphs are isomorphic to their jump graphs.

Theorem 5. *A graph $\Gamma = (V, E)$ satisfies $(\Gamma)_2 \cong J(\Gamma)$ if, and only if, $G = C_4$ or C_5.*

Proof Suppose Γ is a graph such that $(\Gamma)_2 \cong J(\Gamma)$. Hence number of vertices and number of edges of Γ are equal and so Γ must be unicyclic. Let $C = C_m$ be the cycle of length $m \geq 3$ in Γ.

Case 1. $m = 3$.

Let $V(C) = \{u_1, u_2, u_3\}$. Then C is also a cycle in $(\Gamma)_2$, where as in $J(\Gamma)$, the vertices corresponds the edges of C are mutually non-adjacent. Since $(\Gamma)_2 \cong J(\Gamma)$, $J(\Gamma)$ must also contain a C_3 and hence Γ must contain $3K_2$, disjoint union of 3 copies of K_2. Whence Γ must contain either Γ_1, Γ_2 or Γ_3 as shown in Figure 1 as induced subgraph.

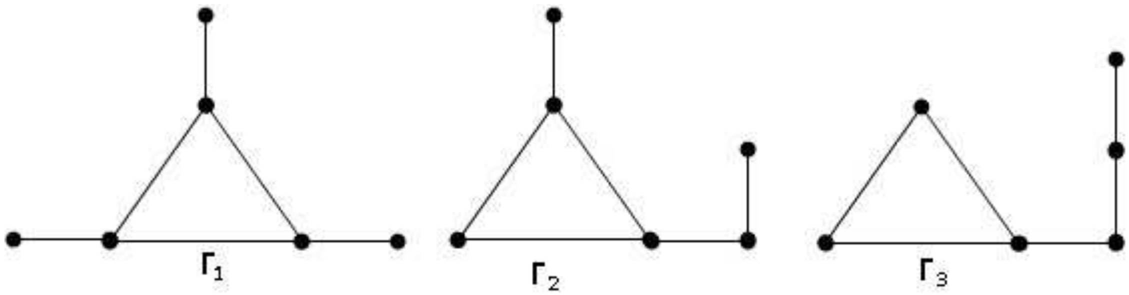

Figure 1.

Subcase 1.1 If Γ contains Γ_2 or Γ_3. Let v be the vertex satisfying $d(v, C) = 2$. Since $d(v, C) = 2$ there exists a vertex u in Γ adjacent to v and a vertex in C say u_1. Now, in $(\Gamma)_2$, the vertex v is not adjacent to u. Since C is also a cycle in $(\Gamma)_2$ and w is adjacent to u_1 in Γ, the vertices w u_2 and u_3 forms a cycle C' in $(\Gamma)_2$. Further, the vertex v is not adjacent to C'. Hence, Γ contains $H = C_3 \cup K_1$ as a induced subgraph. But since $\overline{\Gamma'} = K_{1,3}$ which is a forbidden induced subgraph for line graph $L(\Gamma)$ and $J(\Gamma) = \overline{L(\Gamma)}$, we must have $(\Gamma)_2 \ncong J(\Gamma)$.

Subcase 1.2 If Γ contains Γ_1. Then by subcase(i), $\Gamma = \Gamma_1$. But clearly, $(\Gamma)_2 \ncong J(\Gamma)$.

Case 2. $m \geq 4$.

Suppose that $m \geq 4$ and there exists vertex v in Γ which is not on the cycle C. Let $C = (v_1, v_2, v_3, v_4, ..., v_m, v_1)$. Since Γ is connected v is adjacent to a vertex say, v_i in C. Then the subgraph induced by the vertices v_{i-1}, v, v_{i+1} and v_i in $(\Gamma)_2$ is $K_3 \cup K_1$. Now the graph $\overline{K_3 o K_1}$ is $K_{1,3}$ which is a forbidden induced subgraph for $L(\Gamma) = \overline{J(\Gamma)}$. Hence Γ is not a jump graph. Hence Γ must be a cycle. Clearly $(\Gamma)_2(C_4) = 2K_2 = J(C_4)$ and $J(C_5) = (\Gamma)_2(C_5) = C_5$ it remains to show that for $\Gamma = C_m$ with $m \geq 6$ does not satisfy $J(C_m) = (\Gamma)_2(C_m)$, for $m \geq 6$.

Suppose that $m \geq 6$. Then clearly every vertex in $J(C_m)$ is adjacent to at least $m - 2 \geq 4$ vertices where as in $(\Gamma)_2$, degree of every vertex is 2. This proves the necessary part. The converse part is obvious. □

We now give a characterization of signed graphs whose two path signed graphs are switching equivalent to their jump signed graphs.

Theorem 6. *For any signed graph $\Sigma = (\Gamma, \sigma)$, $(\Sigma)_2 \sim J(\Sigma)$ if, and only if, the underlying graph Γ is either C_5 or C_4.*

Proof Suppose that $(\Sigma)_2 \sim J(\Sigma)$. Then clearly, $(\Gamma)_2 \cong J(\Gamma)$. Hence by Theorem 5, Γ must be either C_4 or C_5.

Conversely, suppose that Σ is a signed graph on C_4 or C_5. Then by Theorem 5, $(\Gamma)_2 \cong J(\Gamma)$.

Since for any signed graph Σ, by Corollary 3 and Theorem 4, $(\Sigma)_2$ and $J(\Sigma)$ are balanced, the result follows by Theorem 1. □

Remark The only possible cases for which $(\Sigma)_2 \cong J(\Sigma)$ are shown in Figure 2.

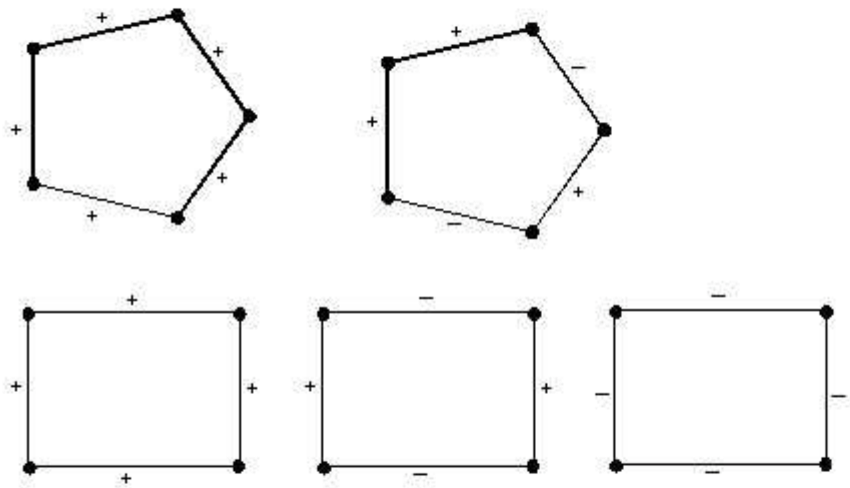

Figure 2.

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