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Fixed Point Theorems of Two-Step Iterations for Generalized Z-Type Condition in CAT(0) Spaces

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Abstract: In this paper, we establish some strong convergence theorems of modified two-step iterations for generalized Z-type condition in the setting of CAT(0) spaces. Our results extend and improve the corresponding results of [3, 6, 28] and many others from the current existing literature.

Key Words: Strong convergence, modified two-step iteration scheme, fixed point, CAT(0) space.

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§1. Introduction

A metric space X is a CAT(0) space if it is geodesically connected and if every geodesic triangle in X is at least as 'thin' as its comparison triangle in the Euclidean plane. It is well known that any complete, simply connected Riemannian manifold having non-positive sectional curvature is a CAT(0) space. Fixed point theory in a CAT(0) space was first studied by Kirk (see [19, 20]). He showed that every nonexpansive (single-valued) mapping defined on a bounded closed convex subset of a complete CAT(0) space always has a fixed point. Since, then the fixed point theory for single-valued and multi-valued mappings in CAT(0) spaces has been rapidly developed, and many papers have appeared (see, e.g., [2], [9], [11]-[13], [17]-[18], [21]-[22], [24]-[26] and references therein). It is worth mentioning that the results in CAT(0) spaces can be applied to any CAT(k) space with $k \leq 0$ since any CAT(k) space is a CAT(m) space for every $m \geq k$ (see [7].

Let (X, d) be a metric space. A *geodesic path* joining $x \in X$ to $y \in X$ (or, more briefly, a geodesic from x to y) is a map c from a closed interval $[0, l] \subset \mathbb{R}$ to X such that $c(0) = x$, $c(l) = y$ and $d(c(t), c(t')) = |t - t'|$ for all $t, t' \in [0, l]$. In particular, c is an isometry, and $d(x, y) = l$. The image α of c is called a geodesic (or metric) *segment* joining x and y. We say X is (i) a *geodesic space* if any two points of X are joined by a geodesic and (ii) a *uniquely geodesic* if there is exactly one geodesic joining x and y for each $x, y \in X$, which we will denoted by $[x, y]$, called the segment joining x to y.

A *geodesic triangle* $\triangle(x_1, x_2, x_3)$ in a geodesic metric space (X, d) consists of three points

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in X (the vertices of $\triangle$) and a geodesic segment between each pair of vertices (the *edges* of $\triangle$). A *comparison triangle* for geodesic triangle $\triangle(x_1, x_2, x_3)$ in (X, d) is a triangle $\overline{\triangle}(x_1, x_2, x_3) := \triangle(\overline{x_1}, \overline{x_2}, \overline{x_3})$ in $\mathbb{R}^2$ such that $d_{\mathbb{R}^2}(\overline{x_i}, \overline{x_j}) = d(x_i, x_j)$ for $i, j \in \{1, 2, 3\}$. Such a triangle always exists (see [7]).

1.1 CAT(0) Space

A geodesic metric space is said to be a $CAT(0)$ space if all geodesic triangles of appropriate size satisfy the following $CAT(0)$ comparison axiom.

Let $\triangle$ be a geodesic triangle in X, and let $\overline{\triangle} \subset \mathbb{R}^2$ be a comparison triangle for $\triangle$. Then $\triangle$ is said to satisfy the $CAT(0)$ inequality if for all $x, y \in \triangle$ and all comparison points $\overline{x}, \overline{y} \in \overline{\triangle}$,

$$d(x, y) \leq d_{\mathbb{R}^2}(\overline{x}, \overline{y}). \tag{1.1}$$

Complete $CAT(0)$ spaces are often called *Hadamard spaces* (see [16]). If x, y_1, y_2 are points of a $CAT(0)$ space and y_0 is the mid point of the segment $[y_1, y_2]$ which we will denote by $(y_1 \oplus y_2)/2$, then the $CAT(0)$ inequality implies

$$d^2\Big(x, \frac{y_1 \oplus y_2}{2}\Big) \leq \frac{1}{2}\ d^2(x, y_1) + \frac{1}{2}\ d^2(x, y_2) - \frac{1}{4}\ d^2(y_1, y_2). \tag{1.2}$$

The inequality (1.2) is the (CN) inequality of Bruhat and Tits [8]. The above inequality was extended in [12] as

$$\begin{aligned} d^2(z, \alpha x \oplus (1-\alpha)y) \quad &\leq \quad \alpha d^2(z, x) + (1-\alpha) d^2(z, y) \\ &\quad\quad -\alpha(1-\alpha) d^2(x, y) \end{aligned} \tag{1.3}$$

for any $\alpha \in [0, 1]$ and $x, y, z \in X$.

Let us recall that a geodesic metric space is a $CAT(0)$ space if and only if it satisfies the (CN) inequality (see [7, page 163]). Moreover, if X is a $CAT(0)$ metric space and $x, y \in X$, then for any $\alpha \in [0, 1]$, there exists a unique point $\alpha x \oplus (1-\alpha)y \in [x, y]$ such that

$$d(z, \alpha x \oplus (1-\alpha)y) \leq \alpha d(z, x) + (1-\alpha) d(z, y), \tag{1.4}$$

for any $z \in X$ and $[x, y] = \big\{\alpha x \oplus (1-\alpha)y : \alpha \in [0, 1]\big\}$.

A subset C of a $CAT(0)$ space X is convex if for any $x, y \in C$, we have $[x, y] \subset C$.

We recall the following definitions in a metric space (X, d). A mapping $T\colon X \to X$ is called an a-contraction if

$$d(Tx, Ty) \leq a\, d(x, y) \text{ for all } x,\ y\ \in X, \tag{1.5}$$

where $a \in (0, 1)$.

The mapping T is called Kannan mapping [15] if there exists $b \in (0, \frac{1}{2})$ such that

$$d(Tx, Ty) \leq b\,[d(x, Tx) + d(y, Ty)] \tag{1.6}$$

for all $x, y \in X$.

The mapping T is called Chatterjea mapping [10] if there exists $c \in (0, \frac{1}{2})$ such that

$$d(Tx, Ty) \leq c\,[d(x, Ty) + d(y, Tx)] \tag{1.7}$$

for all $x, y \in X$.

In 1972, Zamfirescu [29] proved the following important result.

Theorem Z *Let (X, d) be a complete metric space and $T\colon X \to X$ a mapping for which there exists the real number a, b and c satisfying $a \in (0, 1)$, $b, c \in (0, \frac{1}{2})$ such that for any pair $x, y \in X$, at least one of the following conditions holds:*

(z_1) $d(Tx, Ty) \leq a\, d(x, y)$;

(z_2) $d(Tx, Ty) \leq b\,[d(x, Tx) + d(y, Ty)]$;

(z_3) $d(Tx, Ty) \leq c\,[d(x, Ty) + d(y, Tx)]$.

Then T has a unique fixed point p and the Picard iteration $\{x_n\}_{n=0}^{\infty}$ defined by

$$x_{n+1} = Tx_n,\ n = 0, 1, 2, \ldots$$

converges to p for any arbitrary but fixed $x_0 \in X$.

An operator T which satisfies at least one of the contractive conditions (z_1), (z_2) and (z_3) is called a *Zamfirescu operator* or a Z-operator.

In 2004, Berinde [5] proved the strong convergence of Ishikawa iterative process defined by: for $x_0 \in C$, the sequence $\{x_n\}_{n=0}^{\infty}$ given by

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T y_n,$$

$$y_n = (1 - \beta_n)x_n + \beta_n T x_n, \quad n \geq 0, \tag{1.8}$$

to approximate fixed points of Zamfirescu operator in an arbitrary Banach space E. While proving the theorem, he made use of the condition,

$$\|Tx - Ty\| \leq \delta\ \|x - y\| + 2\delta\ \|x - Tx\| \tag{1.9}$$

which holds for any x, $y \in E$ where $0 \leq \delta < 1$.

In 1953, W.R. Mann defined the Mann iteration [23] as

$$u_{n+1} = (1 - a_n)u_n + a_n T u_n, \tag{1.10}$$

where $\{a_n\}$ is a sequence of positive numbers in [0,1].

In 1974, S.Ishikawa defined the Ishikawa iteration [14] as

$$s_{n+1} = (1 - a_n)s_n + a_n T t_n,$$

$$t_n = (1 - b_n)s_n + b_n T s_n, \tag{1.11}$$

where $\{a_n\}$ and $\{b_n\}$ are sequences of positive numbers in [0,1].

In 2008, S.Thianwan defined the new two step iteration [27] as

$$\nu_{n+1} = (1-a_n)w_n + a_n T w_n,$$
$$w_n = (1-b_n)\nu_n + b_n T \nu_n, \tag{1.12}$$

where $\{a_n\}$ and $\{b_n\}$ are sequences of positive numbers in [0,1].

Recently, Agarwal et al. [1] introduced the S-iteration process defined as

$$x_{n+1} = (1-a_n)Tx_n + a_n T y_n,$$
$$y_n = (1-b_n)x_n + b_n T x_n, \tag{1.13}$$

where $\{a_n\}$ and $\{b_n\}$ are sequences of positive numbers in (0,1).

In this paper, inspired and motivated [5, 29], we employ a condition introduced in [6] which is more general than condition (1.9) and establish fixed point theorems of S- iteration scheme in the framework of CAT(0) spaces. The condition is defined as follows:

Let C be a nonempty, closed, convex subset of a CAT(0) space X and $T\colon C \to C$ a self map of C. There exists a constant $L \geq 0$ such that for all $x,\, y \in C$, we have

$$d(Tx,Ty) \leq e^{L\, d(x,Tx)}\Big[\delta\, d(x,y) + 2\delta\, d(x,Tx)\Big], \tag{1.14}$$

where $0 \leq \delta < 1$ and e^x denotes the exponential function of $x \in C$. Throughout this paper, we call this condition as generalized Z-type condition.

Remark 1.1 If $L = 0$, in the above condition, we obtain

$$d(Tx,Ty) \leq \delta\, d(x,y) + 2\delta\, d(x,Tx),$$

which is the Zamfirescu condition used by Berinde [5] where

$$\delta = \max\Big\{a,\, \frac{b}{1-b},\, \frac{c}{1-c}\Big\},\ 0 \leq \delta < 1,$$

while constants a, b and c are as defined in Theorem Z.

Example 1.2 Let X be the real line with the usual norm $\|.\|$ and suppose $C = [0,1]$. Define $T\colon C \to C$ by Tx $= \frac{\text{x+1}}{2}$ for all $x, y \in C$. Obviously T is self-mapping with a unique fixed point 1. Now we check that condition (1.14) is true. If $x, y \in [0,1]$, then $\|Tx - Ty\| \leq e^{L\,\|x-Tx\|}[\delta\ \|x-y\| + 2\delta\ \|x - Tx\|]$ where $0 \leq \delta < 1$. In fact

$$\|Tx - Ty\| \quad = \quad \left\|\frac{x-y}{2}\right\|$$

and

$$e^{L\,\|x-Tx\|}\Big[\delta\,\|x-y\| + 2\delta\,\|x-Tx\|\Big] = e^{L\left\|\frac{x-1}{2}\right\|}\Big[\delta\,\|x-y\| + \delta\,\|x-1\|\Big].$$

Clearly, if we chose $x = 0$ and $y = 1$, then contractive condition (**??**) is satisfied since

$$\|Tx - Ty\| = \left\|\frac{x-y}{2}\right\| = \frac{1}{2},$$

and for $L \geq 0$, we chose $L = 0$, then

$$e^{L\,\|x-Tx\|}\Big[\delta\,\|x-y\| + 2\delta\,\|x-Tx\|\Big] = e^{L\left\|\frac{x-1}{2}\right\|}\Big[\delta\,\|x-y\| + \delta\,\|x-1\|\Big]$$

$$= e^{0(1/2)}(2\delta) = 2\delta, \quad \text{where} \;\; 0 < \delta < 1.$$

Therefore

$$\|Tx - Ty\| \leq e^{L\,\|x-Tx\|}\Big[\delta\,\|x-y\| + 2\delta\,\|x-Tx\|\Big].$$

Hence T is a self mapping with unique fixed point satisfying the contractive condition (1.14).

Example 1.3 Let X be the real line with the usual norm $\|.\|$ and suppose $K = \{0, 1, 2, 3\}$. Define $T\colon K \to K$ by

$$\begin{cases} Tx = 2, & \text{if} \quad x = 0 \\ \quad\; = 3, & \text{otherwise.} \end{cases}$$

Let us take $x = 0$, $y = 1$ and $L = 0$. Then from condition (1.14), we have

$$\begin{aligned} 1 &\leq e^{0(2)}[\delta(1) + 2\delta(2)] \\ &\leq 1(5\delta) = 5\delta \end{aligned}$$

which implies $\delta \geq \frac{1}{5}$. Now if we take $0 < \delta < 1$, then condition (1.14) is satisfied and 3 is of course a unique fixed point of T.

1.2 Modified Two-Step Iteration Schemes in CAT(0) Space

Let C be a nonempty closed convex subset of a complete CAT(0) space X. Let $T\colon C \to C$ be a contractive operator. Then for a given $x_1 = x_0 \in C$, compute the sequence $\{x_n\}$ by the iterative scheme as follows:

$$\begin{aligned} x_{n+1} &= (1-a_n)Tx_n \oplus a_n Ty_n, \\ y_n &= (1-b_n)x_n \oplus b_n Tx_n, \end{aligned} \tag{1.15}$$

where $\{a_n\}$ and $\{b_n\}$ are sequences of positive numbers in (0,1). Iteration scheme (1.15) is called modified S-iteration scheme in CAT(0) space.

$$\begin{aligned} \nu_{n+1} &= (1-a_n)w_n \oplus a_n Tw_n, \\ w_n &= (1-b_n)\nu_n \oplus b_n T\nu_n, \end{aligned} \tag{1.16}$$

where $\{a_n\}$ and $\{b_n\}$ are sequences of positive numbers in [0,1]. Iteration scheme (1.16) is called modified S.Thianwan iteration scheme in CAT(0) space.

$$s_{n+1} = (1 - a_n)s_n \oplus a_n T t_n,$$

$$t_n = (1 - b_n)s_n \oplus b_n T s_n, \tag{1.17}$$

where $\{a_n\}$ and $\{b_n\}$ are sequences of positive numbers in [0,1]. Iteration scheme (1.17) is called modified Ishikawa iteration scheme in CAT(0) space.

We need the following useful lemmas to prove our main results in this paper.

Lemma 1.4([24]) *Let X be a $CAT(0)$ space.*

(i) For $x,\, y \in X$ and $t \in [0,1]$, there exists a unique point $z \in [x,\, y]$ such that

$$d(x,\, z) = t\, d(x,\, y) \text{ and } d(y,\, z) = (1-t)\, d(x,\, y). \tag{A}$$

We use the notation $(1-t)x \oplus ty$ for the unique point z satisfying (A).

(ii) For $x,\, y \in X$ and $t \in [0,1]$, we have

$$d((1-t)x \oplus ty, z) \leq (1-t)d(x,z) + td(y,z).$$

Lemma 1.5([4]) *Let $\{p_n\}_{n=0}^{\infty}$, $\{q_n\}_{n=0}^{\infty}$, $\{r_n\}_{n=0}^{\infty}$ be sequences of nonnegative numbers satisfying the following condition:*

$$p_{n+1} \leq (1 - s_n)p_n + q_n + r_n, \quad \forall\, n \geq 0,$$

where $\{s_n\}_{n=0}^{\infty} \subset [0,1]$. If $\sum_{n=0}^{\infty} s_n = \infty$, $\lim_{n\to\infty} q_n = O(s_n)$ and $\sum_{n=0}^{\infty} r_n < \infty$, then $\lim_{n\to\infty} p_n = 0$.

§2. Strong Convergence Theorems in CAT(0) Space

In this section, we establish some strong convergence theorems of modified two-step iterations to converge to a fixed point of generalized Z-type condition in the framework of CAT(0) spaces.

Theorem 2.1 *Let C be a nonempty closed convex subset of a complete $CAT(0)$ space X and let $T\colon C \to C$ be a self mapping satisfying generalized Z-type condition given by (1.14) with $F(T) \neq \emptyset$. For any $x_0 \in C$, let $\{x_n\}_{n=0}^{\infty}$ be the sequence defined by (1.15). If $\sum_{n=0}^{\infty} a_n = \infty$ and $\sum_{n=0}^{\infty} a_n b_n = \infty$, then $\{x_n\}_{n=0}^{\infty}$ converges strongly to the unique fixed point of T.*

Proof From the assumption $F(T) \neq \emptyset$, it follows that T has a fixed point in C, say u. Since T satisfies generalized Z-type condition given by (1.14), then from (1.14), taking $x = u$

and $y = x_n$, we have

$$\begin{aligned} d(Tu, Tx_n) &\leq e^{L\, d(u,Tu)}\Big(\delta\, d(u, x_n) + 2\delta\, d(u, Tu)\Big) \\ &= e^{L\, d(u,u)}\Big(\delta\, d(u, x_n) + 2\delta\, d(u, u)\Big) \\ &= e^{L\,(0)}\Big(\delta\, d(u, x_n) + 2\delta\,(0)\Big), \end{aligned}$$

which implies that

$$d(Tx_n, u) \leq \delta\, d(x_n, u). \tag{2.1}$$

Similarly by taking $x = u$ and $y = y_n$ in (1.14), we have

$$d(Ty_n, u) \leq \delta\, d(y_n, u), \tag{2.2}$$

Now using (1.15), (2.2) and Lemma 1.4(ii), we have

$$\begin{aligned} d(y_n, u) &= d((1-b_n)x_n \oplus b_n Tx_n, u) \\ &\leq (1-b_n)d(x_n, u) + b_n\, d(Tx_n, u) \\ &\leq (1-b_n)d(x_n, u) + b_n\delta\, d(x_n, u) \\ &= (1 - b_n + b_n\,\delta)d(x_n, u). \end{aligned} \tag{2.3}$$

Now using (1.15), (2.1), (2.3) and Lemma 1.4(ii), we have

$$\begin{aligned} d(x_{n+1}, u) &= d((1-a_n)Tx_n \oplus a_n Ty_n, u) \\ &\leq (1-a_n)d(Tx_n, u) + a_n d(Ty_n, u) \\ &\leq (1-a_n)\delta\, d(x_n, u) + a_n\delta\, d(y_n, u) \\ &\leq (1 - a_n + a_n\,\delta)d(x_n, u) + a_n\delta(1 - b_n + b_n\,\delta)d(x_n, u) \\ &= [1 - (1-\delta)a_n]d(x_n, u) + a_n\delta[1 - (1-\delta)b_n]d(x_n, u) \\ &= [1 - (1-\delta)a_n + a_n\delta(1 - (1-\delta)b_n)]d(x_n, u) \\ &= [1 - \{(1-\delta)a_n + \delta(1-\delta)a_n b_n\}]d(x_n, u) = (1-\mu_n)d(x_n, u) \end{aligned} \tag{2.4}$$

where $\mu_n = (1-\delta)a_n + \delta(1-\delta)a_n b_n$. Since $0 \leq \delta < 1$; a_n, $b_n \in (0,1)$; $\sum_{n=0}^{\infty} a_n = \infty$ and $\sum_{n=0}^{\infty} a_n b_n = \infty$, it follows that $\sum_{n=0}^{\infty} \mu_n = \infty$. Setting $p_n = d(x_n, u)$, $s_n = \mu_n$ and by applying Lemma 1.5, it follows that $\lim_{n\to\infty} d(x_n, u) = 0$. Thus $\{x_n\}_{n=0}^{\infty}$ converges strongly to a fixed point of T.

To show uniqueness of the fixed point u, assume that u_1, $u_2 \in F(T)$ and $u_1 \neq u_2$. Applying generalized Z-type condition given by (1.14) and using the fact that $0 \leq \delta < 1$, we obtain

$$\begin{aligned} d(u_1, u_2) &= d(Tu_1, Tu_2) \\ &\leq e^{L\, d(u_1,Tu_1)}\Big\{\delta\, d(u_1, u_2) + 2\delta\, d(u_1, Tu_1)\Big\} \\ &= e^{L\, d(u_1,u_1)}\Big\{\delta\, d(u_1, u_2) + 2\delta\, d(u_1, u_1)\Big\} \end{aligned}$$

$$= e^{L(0)}\Big\{\delta\, d(u_1, u_2) + 2\delta(0)\Big\}$$
$$= \delta\, d(u_1, u_2) < d(u_1, u_2),$$

which is a contradiction. Therefore $u_1 = u_2$. Thus $\{x_n\}_{n=0}^{\infty}$ converges strongly to the unique fixed point of T. □

Theorem 2.2 *Let C be a nonempty closed convex subset of a complete CAT(0) space X and let $T\colon C \to C$ be a self mapping satisfying generalized Z-type condition given by (1.14) with $F(T) \neq \emptyset$. For any $x_0 \in C$, let $\{x_n\}_{n=0}^{\infty}$ be the sequence defined by (1.16). If $\sum_{n=0}^{\infty} a_n = \infty$, then $\{x_n\}_{n=0}^{\infty}$ converges strongly to the unique fixed point of T.*

Proof The proof of Theorem 2.2 is similar to that of Theorem 2.1. □

Theorem 2.3 *Let C be a nonempty closed convex subset of a complete CAT(0) space X and let $T\colon C \to C$ be a self mapping satisfying generalized Z-type condition given by (1.14) with $F(T) \neq \emptyset$. For any $x_0 \in C$, let $\{x_n\}_{n=0}^{\infty}$ be the sequence defined by (1.17). If $\sum_{n=0}^{\infty} a_n = \infty$ and $\sum_{n=0}^{\infty} a_n b_n = \infty$, then $\{x_n\}_{n=0}^{\infty}$ converges strongly to the unique fixed point of T.*

Proof The proof of Theorem 2.3 is also similar to that of Theorem 2.1. □

If we take $L = 0$ in condition (1.14), then we obtain the following result as corollary which extends the corresponding result of Berinde [5] to the case of modified S-iteration scheme and from arbitrary Banach space to the setting of CAT(0) spaces.

Corollary 2.4 *Let C be a nonempty closed convex subset of a complete CAT(0) space X and let $T\colon C \to C$ a Zamfirescu operator. For any $x_0 \in C$, let $\{x_n\}_{n=0}^{\infty}$ be the sequence defined by (1.15). If $\sum_{n=0}^{\infty} a_n = \infty$ and $\sum_{n=0}^{\infty} a_n b_n = \infty$, then $\{x_n\}$ converges strongly to the unique fixed point of T.*

Remark 2.5 Our results extend and improve upon, among others, the corresponding results proved by Berinde [3], Yildirim et al. [28] and Bosede [6] to the case of generalized Z-type condition, modified S-iteration scheme and from Banach space or normed linear space to the setting of CAT(0) spaces.

§3. Conclusion

The generalized Z-type condition is more general than Zamfirescu operators. Thus the results obtained in this paper are improvement and generalization of several known results in the existing literature (see, e.g., [3, 6, 28] and some others).

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