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Equitable Total Coloring of Some Graphs

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Abstract: In this paper we determine the equitable total chromatic number χ_{et} for the double star graph $K_{1,n,n}$ and the fan graph $F_{m,n}$.

Key Words: Equitable total coloring, double star graph, fan graph.

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§1. Introduction

In this paper, we consider only finite simple graphs without loops or multiple edges. Let $G(V, E)$ be a graph with the set of vertices V and the edge set E. Total coloring $\chi_t(G)$ was introduced by Vizing [7] and Behzad [1]. They both conjectured that for any graph G the following inequality holds: $\Delta(G) + 1 \leq \chi_t(G) \leq \Delta(G) + 2$. It is obvious that $\Delta(G) + 1$ is the best possible lower bound. This conjecture is proved so far for some specific classes of graphs. In general the equitable total coloring problem is more difficult than the total coloring problem. In 1994, Fu [4] gave the concepts of an equitable total coloring and the equitable total chromatic number of a graph. For a simple graph $G(V, E)$, let f be a proper $k-$total coloring of G

$$||T_i| - |T_j|| \leq 1, \; i, j = 1, 2, \ldots, k.$$

The partition $\{T_i\} = \{V_i \cup E_i : 1 \leq i \leq k\}$ is called a $k-$equitable total coloring ($k-$ETC of G in brief), and

$$\chi_{et}(G) = \min\{k | k - \text{ETC of } G\}$$

is called the equitable total chromatic number [2-6, 10] of G, where $\forall x \in T_i = V_i \cup E_i$, $f(x) = i, \; i = 1, 2, \cdots, k$. It is obvious that $\chi_{et}(G) \geq \Delta + 1$. Furthermore Fu presented a conjecture concerning the equitable total chromatic number (simply denoted by ETCC)

Conjecture 1.1([4]) *For any simple graph $G(V, E)$,*

$$\chi_{et}(G) \leq \Delta(G) + 2.$$

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These Researchers in [2, 3, 5, 6, 8-10] have concentrated in providing the equitable total chromatic number for specific families of graphs.

Lemma 1.2([10]) *For any simple graph $G\,(V, E)$,*

$$\chi_{et}\,(G) \geq \chi_t(G) \geq \Delta\,(G) + 1.$$

Lemma 1.3([4]) *For complete graph K_p with order p,*

$$\chi_{et}\,(K_p) = \begin{cases} p, & p \equiv 1 \text{mod } 2 \\ p+1, & p \equiv 0 \text{mod } 2. \end{cases}$$

Lemma 1.4([3]) *For $n \geq 13$ the total equitable chromatic number of Hypo-Mycielski Graph, $\chi_{et}(HM(W_n)) = n + 2$.*

In [9], equitable total chromatic numbers of some join graphs were given. Gong Kun et.al [2] proved some results on the equitable total chromatic number of $W_m \vee K_n$, $F_m \vee K_n$ and $S_m \vee K_n$. In 2012, Ma Gang and Ma Ming [6] proved some results concerning the equtiable total chromatic number of $P_m \vee S_n$, $P_m \vee F_n$ and $P_m \vee W_n$.

In the present paper, we find the equitable total chromatic number χ_{et} for the double star graph $K_{1,n,n}$ and the fan graph $F_{m,n}$.

§2. Preliminaries

Definition 2.1 *A double star $K_{1,n,n}$ is a tree obtained from the star $K_{1,n}$ by adding a new pendant edge of the existing n pendant vertices. It has 2n+1 vertices and 2n edges. Let $V\,(K_{1,n,n}) = \{v\} \cup \{v_1, v_2, \ldots, v_n\} \cup \{u_1, u_2, \ldots, u_n\}$ and $E\,(K_{1,n,n}) = \{e_1, e_2, \ldots, e_n\} \cup \{s_1, s_2, \ldots, s_n\}$.*

Definition 2.2 *A Fan graph $\overline{K_m} + P_n$ where P_n is path on n vertices. All the vertices of the fan corresponding to the path P_n are labeled from m to n consecutively. The vertices in the fan corresponding $\overline{K_m}$ is labeled $m + n$.*

§3. Equitable Total Coloring of Double Star Graphs

Theorem 3.1 *For any positive integer n,*

$$\chi_{et}\,(K_{1,n,n}) = n + 1.$$

Proof Let $V\,(K_{1,n,n}) = \{u_0\} \cup \{u_i : 1 \leq i \leq n\} \cup \{v_i : 1 \leq i \leq n\}$ and $E\,(K_{1,n,n}) = \{e_i : 1 \leq i \leq n\} \cup \{s_i : 1 \leq i \leq n\}$, where $e_i\,(1 \leq i \leq n)$ is the edge $u_0u_i\,(1 \leq i \leq n)$ and $s_i(1 \leq i \leq n)$ is the edge $u_iv_i\,(1 \leq i \leq n)$. Now we partition the edge and vertex sets in $K_{1,n,n}$ as follows.

$$\begin{aligned}
T_1 &= \{e_1, u_n, s_{n-1}, v_{n-2}\} \\
T_2 &= \{e_2, u_1, s_n, v_{n-1}\} \\
T_3 &= \{e_3, u_2, v_n\} \\
T_4 &= \{e_4, u_3, s_1\} \\
T_k &= \{e_k, u_{k-1}, s_{k-3}, v_{k-4}\}\,(5 \le k \le n) \\
T_{n+1} &= \{u_0, s_{n-2}, v_{n-3}\}
\end{aligned}$$

Clearly T_1, T_2, T_3, T_4, T_k and T_{n+1} are independent sets of $K_{1,n,n}$. Also $|T_1| = |T_2| = |T_k| = 4\,(5 \le k \le n)$ and $|T_3| = |T_4| = |T_{n+1}| = 3$, it holds the inequality $||T_i| - |T_j|| \le 1$ for every pair (i, j). This implies $\chi_{et}(K_{1,n,n}) \le n+1$. Since the set of edges $\{e_1, e_2, \ldots e_n\}$ and u_0 receives distinct color, $\chi_{et}(K_{1,n,n}) \ge \chi_t(K_{1,n,n}) \ge n+1$. Hence $\chi_{et}(K_{1,n,n}) \ge n+1$. Therefore $\chi_{et}(K_{1,n,n}) = n+1$. □

§4. Equitable Total Coloring of Fan Graphs

Theorem 4.1 *For any positive integer n, m $(n \ge m)$, then*

$$\chi_{et}(F_{m,n}) = \begin{cases} \Delta + 2 \text{ if } n = m \\ \Delta + 2 \text{ if } n - m = 2 \\ \Delta + 1 \text{ if } n - m = 1 \\ n + 2 \text{ if } n - m \ge 3 \end{cases}$$

Proof Let $V(F_{m,n}) = \{u_i : 1 \le i \le m\} \cup \{v_j : 1 \le j \le n\}$ and $E(F_{m,n}) = \bigcup_{i=1}^{m} \{e_{i,j} : 1 \le j \le n\} \cup \{e_j : 1 \le j \le n-1\}$, where $e_j\,(1 \le j \le n-1)$ is the edge $v_j v_{j+1}\,(1 \le j \le n-1)$ and $e_{i,j}$ is the edge $u_i v_j\,(1 \le i \le m, 1 \le j \le n)$. Now we partition the edge and vertex sets of $F_{m,n}$ in the following cases.

Case 1. $n = m$

$$\begin{aligned}
T_1 &= \{e_{1,1}\} \cup \{e_{4,n}, e_{5,n-1}, e_{6,n-2}, \ldots, e_{m,4}\} \cup \{v_3\} \\
T_2 &= \{e_{1,2}, e_{2,1}\} \cup \{e_{5,n}, e_{6,n-1}, e_{7,n-2}, \ldots, e_{m,5}\} \cup \{v_4\} \\
T_3 &= \{e_{1,3}, e_{2,2}, e_{3,1}\} \cup \{e_{6,n}, e_{7,n-1}, e_{8,n-2}, \ldots, e_{m,6}\} \cup \{v_5\} \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
T_{\Delta-4} &= \{e_{1,n-2}, e_{2,n-3}, \ldots, e_{m-2,1}\} \cup \{v_n\} \\
T_{\Delta-3} &= \{e_{1,n-1}, e_{2,n-2}, \ldots, e_{m-1,1}\}
\end{aligned}$$

$$\begin{aligned}
T_{\Delta-2} &= \{e_{1,n}, e_{2,n-1}, \ldots, e_{m,1}\} \\
T_{\Delta-1} &= \{e_{2,n}, e_{3,n-1}, \ldots, e_{m,2}\} \cup \{v_1\} \\
T_{\Delta} &= \{e_{3,n}, e_{4,n-1}, \ldots, e_{m,3}\} \cup \{v_2\} \\
T_{\Delta+1} &= \left\{e_{2i} : 1 \le i \le \left\lfloor \frac{n-1}{2} \right\rfloor\right\} \cup \left\{u_{2i-1} : 1 \le i \le \left\lceil \frac{n}{2} \right\rceil\right\} \\
T_{\Delta+2} &= \left\{e_{2i-1} : 1 \le i \le \left\lceil \frac{n-1}{2} \right\rceil\right\} \cup \left\{u_{2i} : 1 \le i \le \left\lfloor \frac{n}{2} \right\rfloor\right\}
\end{aligned}$$

Clearly $T_1, T_2, T_3, \ldots T_{\Delta+2}$ are independent sets of $F_{m,n}$. It holds the inequality $||T_i| - |T_j|| \le 1$ for every pair (i, j). This implies $\chi_{et}(F_{m,n}) \le \Delta + 2$. Each edge $v_j v_{j+1}\,(1 \le j \le n-1)$ is adjacent with $2m + 2$ edges and incident with two vertices,it forms a triangle with atleast one vertex of $\{u_i : 1 \le i \le m\}$. Therefore equitable total coloring needs $\Delta + 2$ colors.$\chi_{et}(F_{m,n}) \ge \chi_t(F_{m,n}) \ge \Delta + 2$. Hence $\chi_{et}(F_{m,n}) \ge \Delta + 2$. Therefore $\chi_{et}(F_{m,n}) = \Delta + 2$.

Case 2. $n - m = 2$

$$\begin{aligned}
T_1 &= \{e_{1,1}\} \cup \{e_{2,n}, e_{3,n-1}, e_{4,n-2}, \ldots, e_{m,4}\} \cup \{v_3\} \\
T_2 &= \{e_{1,2}, e_{2,1}\} \cup \{e_{3,n}, e_{4,n-1}, e_{5,n-2}, \ldots, e_{m,5}\} \cup \{v_4\} \\
T_3 &= \{e_{1,3}, e_{2,2}, e_{3,1}\} \cup \{e_{4,n}, e_{5,n-1}, e_{6,n-2}, \ldots, e_{m,6}\} \cup \{v_5\} \\
& \cdots\cdots\cdots\cdots \\
& \cdots\cdots\cdots\cdots \\
T_{\Delta-2} &= \{e_{1,n-2}, e_{2,n-3}, \ldots, e_{m,1}\} \cup \{v_n\} \\
T_{\Delta-1} &= \{e_{1,n-1}, e_{2,n-2}, \ldots, e_{m,2}\} \cup \{v_1\} \\
T_{\Delta} &= \{e_{1,n}, e_{2,n-1}, \ldots, e_{m,3}\} \cup \{v_2\} \\
T_{\Delta+1} &= \left\{u_{2i-1} : 1 \le i \le \left\lfloor \frac{n-1}{2} \right\rfloor\right\} \cup \left\{e_{2i} : 1 \le i \le \left\lfloor \frac{n-1}{2} \right\rfloor\right\} \\
T_{\Delta+2} &= \left\{u_{2i} : 1 \le i \le \left\lfloor \frac{n-2}{2} \right\rfloor\right\} \cup \left\{e_{2i-1} : 1 \le i \le \left\lceil \frac{n-1}{2} \right\rceil\right\}
\end{aligned}$$

Clearly $T_1, T_2, T_3, \ldots T_{\Delta+2}$ are independent sets of $F_{m,n}$. It holds the inequality $||T_i| - |T_j|| \le 1$ for every pair (i, j). This implies $\chi_{et}(F_{m,n}) \le \Delta + 2$. Since each edge $v_j v_{j+1}\,(1 \le j \le n-1)$ is adjacent with $2m+2$ edges and incident with two vertices,it forms a triangle with atleast one vertex of $\{u_i : 1 \le i \le m\}$. Therefore equitable total coloring needs $\Delta + 2$ colors. $\chi_{et}(F_{m,n}) \ge \chi_t(F_{m,n}) \ge \Delta + 2$. Hence $\chi_{et}(F_{m,n}) \ge \Delta + 2$. Therefore $\chi_{et}(F_{m,n}) = \Delta + 2$.

Case 3. $n - m = 1$

$$\begin{aligned}
T_1 &= \{e_{1,1}\} \cup \{e_{2,n}, e_{3,n-1}, e_{4,n-2}, \ldots, e_{m,3}\} \cup \{v_2\} \\
T_2 &= \{e_{1,2}, e_{2,1}\} \cup \{e_{3,n}, e_{4,n-1}, e_{5,n-2}, \ldots, e_{m,4}\} \cup \{v_3\} \\
T_3 &= \{e_{1,3}, e_{2,2}, e_{3,1}\} \cup \{e_{4,n}, e_{5,n-1}, e_{6,n-2}, \ldots, e_{m,5}\} \cup \{v_4\} \\
& \cdots\cdots\cdots\cdots \\
& \cdots\cdots\cdots\cdots
\end{aligned}$$

$$
\begin{aligned}
T_{\Delta-2} &= \{e_{1,n-1}, e_{2,n-2}, \ldots, e_{m,1}\} \cup \{v_n\} \\
T_{\Delta-1} &= \{e_{1,n}, e_{2,n-1}, \ldots, e_{m,2}\} \cup \{v_1\} \\
T_{\Delta} &= \left\{e_{2i} : 1 \le i \le \left\lfloor \frac{n-1}{2} \right\rfloor\right\} \cup \left\{u_{2i-1} : 1 \le i \le \left\lceil \frac{n-1}{2} \right\rceil\right\} \\
T_{\Delta+1} &= \left\{e_{2i-1} : 1 \le i \le \left\lceil \frac{n-1}{2} \right\rceil\right\} \cup \left\{u_{2i} : 1 \le i \le \left\lfloor \frac{n-1}{2} \right\rfloor\right\}
\end{aligned}
$$

Clearly $T_1, T_2, T_3, \ldots T_{\Delta+1}$ are independent sets of $F_{m,n}$. It holds the inequality $||T_i| - |T_j|| \le 1$ for every pair (i,j). This implies $\chi_{et}(F_{m,n}) \le \Delta + 1$. Since at each vertex $v_j\,(2 \le j \le n-1)$ there exist Δ mutually adjacent edges and $v_j\,(2 \le j \le n-1)$ needs one more color. $\chi_{et}(F_{m,n}) \ge \chi_t(F_{m,n}) \ge \Delta + 1$. Hence $\chi_{et}(F_{m,n}) \ge \Delta + 1$. Therefore $\chi_{et}(F_{m,n}) = \Delta + 1$.

Case 4. $n - m \ge 3$

$$
\begin{aligned}
T_1 &= \{e_{1,1}\} \cup \{e_{2,n}, e_{3,n-1}, e_{4,n-2}, \ldots, e_{m,4}\} \cup \{v_3\} \\
T_2 &= \{e_{1,2}, e_{2,1}\} \cup \{e_{3,n}, e_{4,n-1}, e_{5,n-2}, \ldots, e_{m,5}\} \cup \{v_4\} \\
T_3 &= \{e_{1,3}, e_{2,2}, e_{3,1}\} \cup \{e_{4,n}, e_{5,n-1}, e_{6,n-2}, \ldots, e_{m,6}\} \cup \{v_5\} \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
T_{n-2} &= \{e_{1,n-2}, e_{2,n-3}, \ldots, e_{m,1}\} \cup \{v_n\} \\
T_{n-1} &= \{e_{1,n-1}, e_{2,n-2}, \ldots, e_{m,2}\} \cup \{v_1\} \\
T_n &= \{e_{1,n}, e_{2,n-1}, \ldots, e_{m,3}\} \cup \{v_2\} \\
T_{n+1} &= \left\{u_{2i-1} : 1 \le i \le \left\lceil \frac{m}{2} \right\rceil\right\} \cup \left\{e_{2i} : 1 \le i \le \left\lfloor \frac{n-1}{2} \right\rfloor\right\} \\
T_{n+2} &= \left\{u_{2i} : 1 \le i \le \left\lfloor \frac{m}{2} \right\rfloor\right\} \cup \left\{e_{2i-1} : 1 \le i \le \left\lceil \frac{n-1}{2} \right\rceil\right\}
\end{aligned}
$$

Clearly $T_1, T_2, T_3, \ldots T_{n+2}$ are independent sets of $F_{m,n}$. It holds the inequality $||T_i| - |T_j|| \le 1$ for every pair (i,j). This implies $\chi_{et}(F_{m,n}) \le n + 2$. Since at each vertex $u_i\,(1 \le i \le m)$ there exist n mutually adjacent edges and $u_i\,(1 \le i \le m)$ needs one more color. $\chi_{et}(F_{m,n}) \ge \chi_t(F_{m,n}) \ge n + 2$. Hence $\chi_{et}(F_{m,n}) \ge n + 2$. Therefore $\chi_{et}(F_{m,n}) = n + 2$. □

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