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Upper Singed Domination Number of Graphs

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Abstract: A function $f : V(G) \to \{-1, 1\}$ defined on the vertices of a graph G is a signed dominating function (SDF) if $f(N[v]) \geq 1$, $\forall\ v \in V$, where $N[v]$ is the closed neighborhood of v. A SDF f is minimal if there does not exists signed dominating function g, $g \neq f$ such that $g(v) \leq f(v)$ for each $v \in V$. The signed domination number of a graph G is the minimum weight of a minimal SDF on G and upper signed domination number of G is the maximum weight of a minimal SDF on G. In this paper, we obtain the upper signed domination number of path, cycle and complete bipartite graph.

Key Words: Signed (minus) dominating function, signed (minus) dominating function.

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§1. Introduction

For all terminology and notation in graph theory we refer the reader to [2]. However, unless mentioned otherwise, we shall consider here only connected simple graphs.

Let $G = (V, E)$ be a simple graph, the *open neighborhood* of a vertex v is $N(v) = \{u : uv \in E(G)\}$ and *closed neighborhood* of v is $N[v] = N(v) \cup \{v\}$. For any real valued function $f : V \to \mathbb{R}$ and $S \subseteq V$, let $f(S) = \sum\limits_{\forall u \in S} f(u)$ and then the *weight* of f is defined as $wt(f) := f(V)$.

A function $f : V \to \{-1, 0, 1\}$, is said to be a *minus dominating function* (MDF) if $f(N[v]) \geq 1$, $\forall\ v \in V$ and the function $f : V \to \{-1, 1\}$ is called a *signed dominating function* (SDF) of G if $f(N[v]) \geq 1$, $\forall\ v \in V$. A SDF (MDF) f on a graph G is minimal if there does not exist an SDF (MDF) g ($g \neq f$) for which $g(v) \leq f(v)$ for every $v \in V$.

The minus domination number for a graph G, denoted by $\gamma^-(G)$ and defined as $\gamma^-(G) = min\{wt(f) :\ f$ is a minus dominating function on $G\}$. Likewise, the upper minus domination

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number for a graph G, denoted by $\Gamma^{-}(G)$ and defined as

$$\Gamma^{-}(G) = max\{wt(f) : \ f \text{ is a minimal minus dominating functionon } G\}.$$

The sign domination number for a graph G, denoted by $\gamma_s(G)$ and defined as $\gamma_s(G) = min\{wt(f) : \ f$ is a sign dominating function on $G\}$. Likewise, the upper sign domination number for a graph G, denoted by $\Gamma_s(G)$ and defined as

$$\Gamma_s(G) = max\{wt(f) : \ f \text{ is a minimal minus dominating function on} G\}.$$

In [4], Dunbar et.al. characterized the minimal signed dominating function which is as follows:

Proposition 1.1(Dunbar et.al.[4]) *A SDF g on a graph G is minimal if and only if for every vertex $v \in V$ with $g(v) = 1$, there exist a vertex $u \in N[v]$ with $g(N[u]) \in \{1, 2\}$.*

In [5], Henning and Slater posed an open problem to find the good bound for upper signed domination number. Towards solving this problem Favaron [6] found the following sharp bound for the regular graphs.

Theorem 1.2(Favaron [6]) *If G is a k-regular graph, $k \geq 1$ of order n, then*

$$\Gamma_s(G) \leq \begin{cases} \dfrac{n(k+1)}{k+3} & if\ k\, is\ even; \\ \dfrac{n(k+1)^2}{k^2+4k-1} & if\ k\ is\ odd. \end{cases}$$

In 2001, Wang and Mao [1] gave upper bound for nearly regular graphs.

Theorem 1.3(Wang and Mao [1]) *If G is a nealy $(k+1)$-regular graph of order n, then*

$$\Gamma_s(G) \leq \begin{cases} \dfrac{n(k+2)^2}{k^2+6k+4} & if\ k\ is\ even; \\ \dfrac{n(k^2+3k+4)}{k^2+5k+2} & if\ k\ is\ odd \end{cases}$$

and this bound is sharp.

The next result which was stated in [3] provides the best possible bound for a graph in terms of minimum degree δ and maximum degree Δ of the graph.

Theorem 1.4(Tang and Chen [3]) *If G is a graph of order n, then $\Gamma_s(G) \leq \dfrac{(\delta\Delta+4\Delta-\delta)n}{\delta\Delta+4\Delta+\delta}$ for δ even and $\Gamma_s(G) \leq \dfrac{(\delta\Delta+3\Delta-\delta+1)n}{\delta\Delta+3\Delta+\delta-1}$ for δ odd. Furthermore, if G is an Eulerian graph then $\Gamma_s(G) \leq \dfrac{(\delta\Delta+2\Delta-\delta)n}{\delta\Delta+2\Delta+\delta}$.*

It is easy to observe that if a graph has a pendent vertex then by Theorems 1.3 and 1.4, $\Gamma_s \leq n$ which is not a good bound, however, from a survey of literature and to the best of our knowledge, the upper signed domination number of basic graphs like path, cycle, caterpillar and bipartite graphs are not known. Thus, in this paper we have find the upper signed domination number of path, cycle and complete bipartite graph.

§2. Upper Singed Domination Number of Path and Cycle

In this section we give the upper signed domination number of path and cycle.

Theorem 2.1 *For every path P_n of order n,* $\Gamma_s(P_n) = n - 2\left\lfloor \frac{n}{5} \right\rfloor$.

Proof If f is a minimal SDF of P_n with weight Γ_s, then

$$\Gamma_s = |P_f| - |M_f|,$$

where $P_f = \{u \in V(P_n) : f(u) = +1\}$ and $M_f = \{u \in V(P_n) : f(u) = -1\}$. Therefore,

$$\Gamma_s = n - 2|M_f|.$$

In order to prove the result it is suffices to show that $|M_f| = \left\lfloor \frac{n}{5} \right\rfloor$. Let $n = 5k + l$ for some non negative integers k and l. Let $g : V(P_n) \to \{-1, 1\}$ be a function such that,

$$M_g = \begin{cases} \{v_{5i}\} \cup \{v_{n-2}\} & \forall, 1 \le i \le k-1 \quad if\ k = 0, 1; \\ \{v_{5i}\} & \forall, 1 \le i \le k \quad if\ k = 2, 3, 4, \end{cases}$$

and $P_f = V(P_n) \backslash M_g$. One can check that given function G is a minimal SDF with $|M_g| = \left\lfloor \frac{n}{5} \right\rfloor$. Therefore,

$$|M_f| \le |M_g| = \left\lfloor \frac{n}{5} \right\rfloor. \tag{1}$$

If v_i and v_j are two vertices in M_f such that there is no other vertices between v_i and v_j in M_f. Now, suppose that the distance between v_i and v_j is less than or equal to two *i.e.*, $d(v_i, v_j) \le 2$. Then there exists a vertex v_x adjacent to v_i and v_j. Since $f(v_i) = f(v_j) = -1$,

$$f(N[v_x]) = f(v_i) + f(v_x) + f(v_j) = -1 + 1 - 1 < 0,$$

which is a contradiction to the assumption that f is an SDF. Therefore, $d(v_i, v_j) \ge 3$.

On the other hand, if the distance between v_i and v_j is greater than or equal to six *i.e.*, $d(v_i, v_j) \ge 6$ then there exist a sub path $P_i = \{v_i, v_{i+1}, v_{i+2}, v_{i+3}, v_{i+4}, v_{i+5}, \cdots, v_{i+t}, v_j\}$ for every $t \ge 5$, such that all the vertices $\{v_{i+1}, v_{i+2}, v_{i+3}, v_{i+4}, v_{i+5}, \cdots, v_{i+t}\}$ are positive and $f(v_i) = f(v_j) = -1$. By Proposition 1.1 a SDF g on a graph G is minimal if and only if for every vertex $v \in V$ with $g(v) = 1$, there exist a vertex $u \in N[v]$ with $g(N[u]) \in \{1, 2\}$, but $f(v_{i+3}) = 1$ and $f(N[v_{i+2}]) = f(N[v_{i+3}]) = f(N[v_{i+4}]) = 3$ (see Figure 1) therefore Proposition 1.1 implies that f can not be minimal SDF, this contradicts the assumption that f is a minimal SDF. Therefore $d(v_i, v_j) \le 5$.

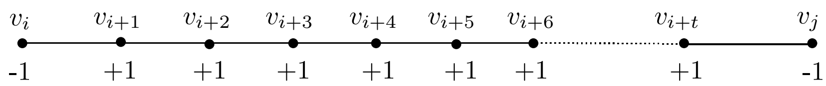

Figure 1

Hence

$$3 \leq d(v_i, v_j) \leq 5,$$

from this one can conclude that

$$|M_f| \geq \left\lfloor \frac{n}{5} \right\rfloor. \tag{2}$$

From (1) and (2)

$$|M_f| = \left\lfloor \frac{n}{5} \right\rfloor.$$

Hence,

$$\Gamma_s(P_n) = n - 2|M_f| = n - 2\left\lfloor \frac{n}{5} \right\rfloor. \qquad \square$$

Corollary 2.2 *For every cycle C_n of order n, $\Gamma_s(C_n) = n - 2\lceil \frac{n}{5} \rceil$.*

Proof The proof of this corollary can be given by the arguments analogues to those used in the above Theorem 2.1. $\square$

§3. Upper Signed Domination Number of Complete Bipartite Graphs

Theorem 3.1 *If $K_{m,n}$, the complete bipartite graph with $m \leq n$, then $\Gamma_s = (m+n) - 2\left\lfloor \frac{m}{2} \right\rfloor$.*

Proof Consider $K_{m,n} = (U, W)$ the complete bipartite graph with partite sets U and W having $|U| = m \leq n = |W|$ $(m, n \geq 2)$ and f be a minimal SDF with weight $\Gamma_s(K_{m,n})$, then

$$\Gamma_s(K_{m,n}) = |P_f| - |M_f| = (m+n) - 2|M_f|.$$

Where P_f and M_f are as defined in Theorem 2.1. In order to establish the desired result, it is sufficient to show that $|M_f| = \left\lfloor \frac{m}{2} \right\rfloor$.

Let $|U \cap M_f| = m^-$ and $|W \cap M_f| = n^-$. Since $K_{m,n}$ is a complete bipartite graph with $m \leq n$, then $m^- \leq \left\lfloor \frac{m}{2} \right\rfloor$ and $n^- \leq \lfloor \frac{n}{2} \rfloor$. This gives,

$$|M_f| \leq \left\lfloor \frac{m}{2} \right\rfloor.$$

Suppose $|M_f| < \left\lfloor \frac{m}{2} \right\rfloor$, then there exists a positive integer k such that

$$|M_f| = \left\lfloor \frac{m}{2} \right\rfloor - k$$

$$m^- + n^- = \left\lfloor \frac{m}{2} \right\rfloor - k. \tag{3}$$

Consider,

$$\begin{aligned} f(N[w_i]) &= \sum_{u_i \in U} f(u_i) + f(w_i) \\ &= \sum_{u_i \notin M_f} f(u_i) + \sum_{u_i \in M_f} f(u_i) + f(w_i) \\ &= m - m^- - m^- + f(w_i) \\ &= m - 2m^- + f(w_i) \\ &= m - 2\left\lfloor \frac{m}{2} \right\rfloor + 2k + 2n^- + f(w_i) && \text{by equation (3),} \\ &\geq 3 && \forall, w_i \in W. \end{aligned}$$

Following the above procedure, we calculate the value of $f(N[u_i])$

$$\begin{aligned} f(N[u_i]) &= \sum_{w_i \in W} f(w_i) + f(u_i) \\ &= \sum_{w_i \notin M_f} f(w_i) + \sum_{w_i \in M_f} f(w_i) + f(u_i) \\ &= n - n^- - n^- + f(u_i) \\ &= n - 2n^- + f(u_i) \\ &= n - 2\left\lfloor \frac{m}{2} \right\rfloor + 2k + 2m^- + f(u_i) && \text{by equation (3),} \\ &\geq 3 && \forall, u_i \in U. \end{aligned}$$

This implies that, if $|M_f| < \left\lfloor \frac{m}{2} \right\rfloor$ then $f(N[v]) \geq 3$ for all $v \in V(K_{m,n})$ and by Proposition 1.1 a SDF g on a graph G is minimal if and only if for every vertex $v \in V$ with $g(v) = 1$, there exist a vertex $u \in N[v]$ with $g(N[u]) \in \{1, 2\}$, but $f(N[v]) \geq 3$ for all $v \in V(K_{m,n})$ hence f is not a minimal SDF, which is a contradiction to the assumption that f is an minimal SDF. Therefore,

$$|M_f| = \left\lfloor \frac{m}{2} \right\rfloor.$$

This implies

$$\begin{aligned} \Gamma_s(K_{m,n}) &= (m+n) - 2|M_f| \\ &= (m+n) - 2\left\lfloor \frac{m}{2} \right\rfloor. \end{aligned}$$

Hence the result. □

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