

Some Results in Fuzzy and Anti Fuzzy Group Theory

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Abstract: This paper is to further investigate some properties of an anti fuzzy subgroup of a group in relation to pseudo coset. It also uses isomorphism theorems to establish some results in relation to level subgroups of a fuzzy subgroup μ of a group G .

Key Words: Fuzzy group, level subgroup, Smarandache fuzzy algebra, anti fuzzy group, anti fuzzy subgroup, group homomorphism, group isomorphism.

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§1. Introduction

Major part of this work leans on the work of [5]. There are some new results using isomorphism theorems with some results in [5].

§2. Preliminaries

Definition 2.1 Let X be a non-empty set. A fuzzy subset μ of the set G is a function $\mu : G \rightarrow [0, 1]$.

Definition 2.2 Let G be a group and μ a fuzzy subset of G . Then μ is called a fuzzy subgroup of G if

- (i) $\mu(xy) \geq \min\{\mu(x), \mu(y)\}$;
- (ii) $\mu(x^{-1}) = \mu(x)$;
- (iii) μ is called a fuzzy normal subgroup if $\mu(xy) = \mu(yx)$ for all x and y in G .

Definition 2.3 Let G be a group and μ a fuzzy subset of G . Then μ is called an anti fuzzy subgroup of G if

- (i) $\mu(xy) \leq \max\{\mu(x), \mu(y)\}$;
- (ii) $\mu(x^{-1}) = \mu(x)$.

Definition 2.4 Let μ and λ be any two fuzzy subsets of a set X . Then

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- (i) λ and μ are equal if $\mu(x) = \lambda(x)$ for every x in X ;
- (ii) λ and μ are disjoint if $\mu(x) \neq \lambda(x)$ for every x in X ;
- (iii) $\lambda \subseteq \mu$ if $\mu(x) \geq \lambda(x)$.

Definition 2.5 Let μ be a fuzzy subset (subgroup) of X . Then, for some t in $[0, 1]$, the set $\mu_t = \{x \in X : \mu(x) \geq t\}$ is called a level subset (subgroup) of the fuzzy subset (subgroup) μ .

Remark 2.5.1 The set μ_t if it is group can be represented as G_μ^t .

Definition 2.6 Let μ be a fuzzy subgroup of a group G . The set $H = \{x \in G : \mu(x) = \mu(e)\}$ is such that $o(\mu) = o(H)$.

Definition 2.7 Let μ be a fuzzy subgroup of a group G . μ is said to be normal if $\sup \mu(x) = 1$ for all x in G . It is said to be normalized if there is an x in G such that $\mu(x) = 1$.

Definition 2.8 Let G be a group and μ a fuzzy subset of G . Then μ is called an anti fuzzy subgroup of G if and only if $\mu(xy^{-1}) \leq \max\{\mu(x), \mu(y)\}$, and μ is called an anti fuzzy normal subgroup if $\mu(xy) = \mu(yx)$ for all x and y .

Definition 2.9 Let μ be a fuzzy subset of X . Then, for $t \in [1, 0]$, the set $\mu_t = \{x \in X : \mu(x) \leq t\}$ is called a lower level subset of the fuzzy subset μ .

Definition 2.10 Let μ be an anti fuzzy subgroup of X . Then, for $t \in [1, 0]$, the set $\mu_t = \{x \in X : \mu(x) \leq t\}$ is called a lower level subgroup of μ .

Definition 2.11 Let μ be an anti fuzzy subgroup of a group G of finite order. Then, the image of μ is $Im(\mu) = \{t_i \in I : \mu(x) = t_i \text{ for some } x \text{ in } G\}$, where $I = [0, 1]$.

Definition 2.12 Let μ be an anti fuzzy subgroup of a group G . For a in G , the anti fuzzy coset $a\mu$ of G determined by a and μ is defined by $(a\mu)(x) = \mu(a^{-1}x)$ for all x in G .

Definition 2.13 Let μ be an anti fuzzy subgroup of a group G . For a and b in G , the anti fuzzy middle coset $a\mu b$ of G is defined by $(a\mu b)(x) = \mu(a^{-1}xb^{-1})$ for all x in G .

Definition 2.14 Let μ be an anti fuzzy subgroup of G and an element a in G . Then pseudo anti fuzzy coset $(a\mu)^p$ is defined by $(a\mu)^p(x) = p(a)\mu(x)$ for all x in G and p in P .

Definition 2.15 The Cartesian product $\lambda \times \mu : X \times Y \rightarrow [0, 1]$ of two anti fuzzy subgroups is defined by $(\lambda \times \mu)(x, y) = \max\{\lambda(x), \mu(y)\}$ for all (x, y) in $X \times Y$ and R_λ is a binary anti fuzzy relation defined by $R_\lambda(x, y) = \max\{\lambda(x), \lambda(y)\}$. The anti fuzzy relation R_λ is said to be a similarity relation if

- (i) $R_\lambda(x, x) = 1$;
- (ii) $R_\lambda(x, y) = R_\lambda(y, x)$;
- (iii) $\max\{R_\lambda(x, y), R_\lambda(y, z)\} \leq R_\lambda(x, z)$.

Definition 2.16 Let G be a finite group of order n and μ a fuzzy subgroup of G . Then for t_1, t_2 in $[0, 1]$ such that $t_1 \leq t_2$, $\mu_{t_2} \subseteq \mu_{t_1}$.

Definition 2.17 Let G be a finite group of order n and μ an anti fuzzy subgroup of G . Then for $t_1, t_2 \in [0, 1]$ such that $t_1 \leq t_2, \mu_{t_1} \subseteq \mu_{t_2}$.

Definition 2.18 Let f be a group homomorphism from a group G to H . Then there is an isomorphism $\phi : f(G) \rightarrow G/\text{Ker}f$, where ϕ is the canonical isomorphism associated with f .

Definition 2.19 Let G be a group and H, K normal subgroups of G such that $H \leq K$. Then there is a natural isomorphism $G/K \cong (G/H)/(K/H)$.

Proposition 2.20 Let G be a group and μ a fuzzy subset of G . Then μ is a fuzzy subgroup of G if and only if μ_t^t is a level subgroup of G for every t in $[0, \mu(e)]$, where e is the identity of G .

Proposition 2.21 H as described in 2.6 can be realized as a level subgroup.

Theorem 2.22 G is a Dedekind or Hamiltonian group if and only if every fuzzy subgroup of G is fuzzy normal subgroup. (A Dedekind and Hamiltonian groups have all the subgroups to be normal).

§3. Briefly on Properties of Anti Fuzzy Subgroup

Proposition 3.1 Any two pseudo cosets of an anti fuzzy subgroup of a group G are either identical or disjoint.

Proof Assume that $(a\mu)^p$ and $(b\mu)^p$ are any two identical pseudo anti fuzzy cosets of μ for any a and b in G . Then, $(a\mu)^p(x) = (b\mu)^p(x)$ for all x in G . Assume also on the contrary that they are disjoint. Then, there is no y in G such that $(a\mu)^p(y) = (b\mu)^p(y)$ which implies that $p(a)\mu(y) \neq p(b)\mu(y)$. The consequence is that $p(a) \neq p(b)$. This makes the assumption $(a\mu)^p(x) = (b\mu)^p(x)$ false.

Conversely, assume that $(a\mu)^p$ and $(b\mu)^p$ are disjoint, then $p(a)\mu(y) \neq p(b)\mu(y)$ for every y in G . But if it is assumed that this is also identical, then $p(a)\mu(y) = p(b)\mu(y)$ and that means $p(a) = p(b)$ so that $p(a)\mu(y) \neq p(b)\mu(y)$ cannot be true. $\square$

Proposition 3.2 Let μ be an anti fuzzy subgroup of any group G . Let $\{\mu_i\}$ be a partition of μ . Then

- (i) each μ_i is normal if μ is normalized;
- (ii) each μ_i is normal if μ is normal.

Proof Note that for each $i, \mu_i \subseteq \mu$ which implies that $\mu_i(x) \leq \mu(x)$ for all x in G .

(i) Since μ is normalized, there is an x_0 in G such that $\mu_i(x) \leq \mu(x) \leq \mu(x_0) = 1$ for each i . Whence, $\mu_i(x) \leq 1$. Then $\sup \mu_i(x) = 1$.

(ii) Since μ is normal, $\sup \mu(x) = 1$, then $\mu(x) \leq 1$. Note that $\mu_i(x) \leq \mu(x) \leq 1$. Then $\mu_i(x) \leq 1$ and $\sup \mu_i(x) = 1$. $\square$

Proposition 3.3 Let μ be an anti fuzzy subgroup of any group G . Then $\mu(e) \leq 1$ even if μ is

normalized.

Proof Note that for all x in G , $0 \leq \mu(x) \leq 1$.

$$\mu(e) = \mu(xx^{-1}) \leq \max\{\mu(x), \mu(x^{-1})\} = \mu(x) \text{ since } \mu(x) = \mu(x^{-1}) \text{ for all } x \text{ in } G.$$

But since μ is normal, there is an x_0 in G such that $\mu(e) \leq \mu(x) \leq \mu(x_0) = 1$. Hence $\mu(e) \leq 1$. $\square$

Proposition 3.4 *Let μ be an anti fuzzy subgroup of any group G and $R_\mu : G \times G \rightarrow [0, 1]$ be given by $R_\mu(x, y) = \mu(xy^{-1})$. R_μ is not a similarity relation.*

Proof The reference [4] has shown that this is a similarity relation when μ is a fuzzy subgroup of G . But

$$R_\mu(x, x) = \mu(xx^{-1}) = \mu(e) \leq 1.$$

R_μ is not symmetric, hence not a similarity relation. $\square$

§4. Application of Isomorphism Theorems of Groups to Fuzzy Subgroups

Proposition 4.1 *Let f be a group homomorphism between G and H . Let μ be a fuzzy subgroup of H . Then G is isomorphic to a level subgroup of H .*

Proof Since f is a homomorphism, it is defined on G .

$$\text{Ker } f = \{x \in G : f(x) = e_H\} \Leftrightarrow \{x \in G : \mu f(x) = \mu(e_H) \leq 1\}.$$

Hence, $\mu f(x) \leq 1$ for all x in G since μ is a fuzzy subgroup of H and $f(x)$ is in H .

$$\text{Ker } f = G \text{ so that } \mu f(G) \leq 1.$$

Also, note that

$$f(G) = \{y = f(x) \in H : \mu f(x) = \mu(y) = \mu(e_H)\}.$$

By 2.21 and 2.6, $f(G)$ is a level subgroup, say H_μ^t of H .

$$G/G = G \cong H_\mu^t$$

by Definition 2.18. $\square$

Remark 4.2 It can be said then that every group G is isomorphic to a level subgroup of a group H if there is a group homomorphism between G and H and μ a fuzzy subgroup of H exists.

Proposition 4.3 *Let G be a Dedekind or an Hamiltonian group and μ a fuzzy subgroup of G . For $t_1, t_2 \in [0, 1]$ such that $t_1 < t_2$ and $G/G_{1\mu}^t \cong (G/G_{2\mu}^t)/(G_{1\mu}^t/G_{2\mu}^t)$.*

Proof By Proposition 2.20, $G_{1\mu}^t$ and $G_{2\mu}^t$ are subgroups of G and by Theorem 2.22, they are normal subgroups. Also by Definition 2.16,

$$G_{2\mu}^t \leq G_{1\mu}^t.$$

Then, $f : G/G_{2\mu}^t \rightarrow G/G_{1\mu}^t$ is a group homomorphism and

$$Im(f) = G/G_{1\mu}^t \text{ if } f(gG_{2\mu}^t) = gG_{1\mu}^t.$$

Also, it can be shown that $Ker f = G_{1\mu}^t/G_{2\mu}^t$.

Then apply Definition 2.19 so that $G/G_{1\mu}^t \cong (G/G_{2\mu}^t)/(G_{1\mu}^t/G_{2\mu}^t)$. $\square$

Remarks 4.4 It is equally of note that if μ is an anti fuzzy subgroup of a group G ,

$$\text{for } t_1 < t_2, G_{1\mu}^t \leq G_{2\mu}^t$$

by Definition 2.17.

Following the same argument as in Proposition 4.3,

$$G/G_{2\mu}^t \cong (G/G_{1\mu}^t)/(G_{2\mu}^t/G_{1\mu}^t). \quad \square$$

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References

- [1] A.O.Kuku, *Abstract Algebra*, Ibadan University Press, Nigeria, 1992.
- [2] M. Artin, *Algebra* (Second Edition), PHI Learning Private Limited, New Delhi-110001, 2012.
- [3] R. Muthuraj et. al., A Study on Anti Fuzzy Sub-Bigroup, *IJCA* (0975-8887), Volume 2, No.1(2010), 31-34.
- [4] Shobha Shukla, Pseudo Fuzzy Coset, *IJSRP* (2205-3153), Volume 3, Issue 1(2013), 1-2.
- [5] W.B. Vasantha Kandasamy, *Smarandache Fuzzy Algebra*, American Research Press, Rehoboth (2003).