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Regularization and Energy Estimation of Pentahedra (Pyramids) Using Geometric Element Transformation Method

Buddhadev Pal and Arindam Bhattacharyya

(Department of Mathematics, Jadavpur University, Kolkata-700032, India)

E-mail: buddha.pal@rediffmail.com, bhattachar1968@yahoo.co.in

Abstract: By geometric element transformation method (GETMe) always we get a new element. It is based on geometric transformations, which, if applied iteratively, lead to the regularization of a pyramid (under conditions). Energy function is a cost function for pentahedra which is applicable also for hexahedra, octahedra, decahera etc. is defined by a particular process, which we call as base diagonal apex method (BDAMe). Here, we try to investigate the characterization of different cost functions using BDAMe when we transform a pyramid by GETMe.

Key Words: Mesh quality, iterative element regularization, finite element mesh, objective function, cost function.

AMS(2010): 65M50, 65N50

§1. Introduction

In many finite element applications unstructured tessellations of the geometry under consideration play a fundamental role. Therefore, the generation of quality meshes are essential steps of the simulation process, since mesh quality has an impact on solution accuracy and the efficiency of the computational methods involved [1,2].

In [4] the geometric element transformation method (GETMe) has been introduced as new type of geometry-based mesh smoothing for triangular surface meshes. Based on a simple geometric element transformation, which iteratively transforms low quality elements to regular, hence perfect elements, mesh improvement is accomplished by sequentially improving the worst element of the mesh. In [5] this approach has been generalized to a simultaneous approach for triangular or quadrilateral mixed surface meshes in which all mesh elements are transformed simultaneously and node updates are obtained by transformed node averaging. As has been shown in [6,7] such regularizing transformations exist for polygons with an arbitrary number of nodes. Furthermore, the sequential as well as the simultaneous GETMe approach naturally extend to tetrahedral meshes [8].

In finite element simulation the mesh quality is a crucial aspect for good numerical be-

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haviour of the method. In a first stage, some automatic 3-D mesh generators construct meshes with poor quality and in special cases, for example when node movement is required, inverted elements may appear. So, it is necessary to develop a procedure that optimises the pre-existing mesh. This process must be able to smooth and untangle the mesh.

The most usual technique to improve the quality of a valid mesh, that is, one that does not have inverted elements, are based upon local smoothing. In short, these technique consists of finding the new positions that the mesh nodes must hold, in such a way that they optimize an objective function. Usually, objective functions are appropriate to improve the quality of a valid mesh, but they do not work properly when there are inverted elements. To avoid this problem we can proceed as Freitag et al in [9,10,11].

In this paper, we have defined the characterization of energy function of a pyramid using base diagonal apex method (BDAMe). Then we have proved that the energy function always lies between 0 and 1 and discussed regularization properties of a pyramid and tried to regularize by using geometric element transformation method (GETMe). Finally we have studied the characterization of energy function of a particular type of pentahedron using GETMe and BDAMe.

§2. Characterization of Energy Function of a Pyramid

For 3-simplex the cost function which referred to as energy function, which is also discussed in [3]. But we can not estimate energy function for all 3-D shapes. In this paper, we shall try to estimate the energy function of a pentahedron by a particular method, which we have defined as base diagonal apex method (BDAMe).

2.1 Base Diagonal Apex Method (BDAMe)

In this method, we add the two diagonal of the base of the pyramid and then add between the intersection point of the diagonal and the apex of the pyramid. This line (from apex to the intersection point of the diagonals) may be the height of the pyramid or may not be the height of the pyramid, totally depend upon the type of pyramid we choose. If we follow this method, we get four 3-simplex, that is, four tetrahedra. Now each tetrahedron has a cost function or energy function. Therefore, we get four cost functions and then we can easily define the cost function of a pyramid, and to define cost function of 3-D shapes except 3-simplex, we introduce the function $h(v_i, \sigma^n)$, the signed distance from $c(\sigma^n)$ to aff(σ_i^{n-1}) with the convention that $h(v_i, \sigma^n) \geq 0$ when $c(\sigma^n)$ and v_i are on the same side of aff(σ_i^{n-1}). Here σ^n are the n-simplex, $c(\sigma^n)$ the circumcenter of the n-simplex, facet aff(σ_i^{n-1}) and vertex v_i. We work on basically all 3-D figures, so in that case $n = 3$. The magnitude of $h(v_i, \sigma^n)$ can be computed as the distance between $c(\sigma^n)$ and $c(\sigma_i^{n-1})$, and its sign can be computed by testing whether $c(\sigma^n)$ and v_i have the same orientation with respect to aff(σ_i^{n-1}). Now by BDAMe, the pentahedron is the sum of the maximum number of four 3-simplexes. Here we divide the quantity $h(v_i, \sigma^n)$ by the circumradius $R(\sigma^n)$ to get a quantity called cost function or energy function. Note that $-1 < h(v_i, \sigma^n)/R(\sigma^n) < 1$ for finite σ^n, because $R(\sigma^n)^2 = h(v_i, \sigma^n)^2 + R(\sigma_i^{n-1})^2$.

We consider the energy function

$$f_p(\sigma^n) = \frac{1}{4}\Sigma_{j=1}^{4} max_{v\in\sigma^n}|\frac{h(v,\sigma_j^n)}{R(\sigma_j^n)}| \tag{1}$$

Now we prove the following theorem.

Theorem 2.1 *The energy function (using BDAMe)* $f_p(\sigma^n) = \frac{1}{N}\Sigma_{j=1}^{N} max_{v\in\sigma^n}|\frac{h(v,\sigma_j^n)}{R(\sigma_j^n)}|$, *where* N *is maximum total number of tetrahedron, of a 3D-figure (pentahedron, hexahedron, decahedron, octahedron,$\cdots$, etc) always lies between* 0 *and* 1 *that is,* $0 < f_p(\sigma^n) < 1$.

Proof First we break (using BDAMe) the 3D-figure with maximum number of tetrahedra (pairwise disjoint) which can cover the hole 3D-figure. If it is not possible, then we have to break it in maximum number of pentahedra (pairwise disjoint) which can cover the hole 3D-figure and then use BDAMe in pentahedron. Therefore, we can get the cost function (using BDAMe) of any regular 3D-figure. Therefore, overall the total number of tetrahedra gives the value of N. Note that $-1 < h(v_i,\sigma^n)/R(\sigma^n) < 1$ for finite σ^n, because $R(\sigma^n)^2 = h(v_i,\sigma^n)^2 + R(\sigma_i^{n-1})^2$ and when we consider energy function, we take the maximum ratio (h/R) with positive value of each tetrahedron. Therefore, the value $f_p(\sigma^n)$ is always greater than zero and we divide $\Sigma_{j=1}^{N} max_{v\in\sigma^n}|\frac{h(v,\sigma_j^n)}{R(\sigma_j^n)}|$ by N (total number of tetrahedra), hence it is always less than one. Therefore we can write, $0 < f_p(\sigma^n) < 1$. □

For instance, if we consider a hexahedron, which is six pentahedra. After using BDAMe we get 24 tetrahedra. Therefore the energy function

$$f_H(\sigma^n) = \frac{1}{24}\Sigma_{j=1}^{24} max_{v\in\sigma^n}|\frac{h(v,\sigma_j^n)}{R(\sigma_j^n)}|,$$

where H for hexahedron.

§3. Methods of Transformation

Here we use several methods of transformation to regularize the 3-D figure, like pentahedra. By this, regularizing means that if the transformation is applied iteratively to a single element, it becomes regular. Consequently, this section focuses on the properties of the transformations applied to a single pentahedron.

3.1 Transformation of a pentahedron using GETMe

Let $P := (p_1, p_2, p_3, p_4, p_5)^t$ denote a pentahedron with five pairwise disjoint nodes $p_i \in R^3$, $i \in \{1, \cdots, 5\}$, which are positively oriented. Let

$$n_1 := (p_5 - p_2) \times (p_3 - p_2),$$

$$n_2 := (p_5 - p_3) \times (p_4 - p_3),$$
$$n_3 := (p_4 - p_5) \times (p_1 - p_5),$$
$$n_4 := (p_1 - p_5) \times (p_2 - p_5),$$
$$n_5 := (p_4 - p_3) \times (p_2 - p_3)$$

denote the inside oriented face normal of P. A new pentahedron P' with nodes p_i' is derived from P by constructing on each node p_i the opposing face normal n_i scaled by $\sigma/\sqrt{|n_i|}$, where $\sigma \in R_0^+$. That is

$$P' = \begin{pmatrix} p_1' \\ p_2' \\ p_3' \\ p_4' \\ p_5' \end{pmatrix} := \begin{pmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \\ p_5 \end{pmatrix} + \sigma \begin{pmatrix} \frac{1}{\sqrt{|n_1|}} n_1 \\ \frac{1}{\sqrt{|n_2|}} n_2 \\ \frac{1}{\sqrt{|n_3|}} n_3 \\ \frac{1}{\sqrt{|n_4|}} n_4 \\ \frac{1}{\sqrt{|n_5|}} n_5 \end{pmatrix} \tag{2}$$

It is clear that if $\sigma = 0$ then P' and P are same.

3.2 Apex transformation of a pentahedron using GETMe

Apex transformation means, we transform the apex (top vertex) of the pentahedron (pyramid) using geometric element transformation method (GETMe) as discussed in the article (3.1). So, we transform p_5 (apex) to p_5' using only the inside oriented face normal n_5, $n_5 := (p_4 - p_3) \times (p_2 - p_3)$ of P. In that case, a new pentahedron P' with nodes p_i' is derived from P by constructing the node p_5 the opposing face normal n_5 scaled by $\sigma/\sqrt{|n_5|}$, where $\sigma \in R_0^+$. That is

$$P' = \begin{pmatrix} p_1' \\ p_2' \\ p_3' \\ p_4' \\ p_5' \end{pmatrix} := \begin{pmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \\ p_5 \end{pmatrix} + \sigma \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \frac{1}{\sqrt{|n_5|}} n_5 \end{pmatrix} \tag{3}$$

It is also clear that if $\sigma = 0$ then P' and P are same.

3.3 New pentahedron derived from centroid transformation of a pentahedron using GETMe

Let P denote a pentahedron with nodes p_k and $\sigma \in R^+$ an arbitrary scaling factor. The nodes p_k' of the transformed pentahedron P' are given by

$$p_k' := c_k + \frac{\sigma}{\sqrt{|n_k|}} n_k, k \in \{1, ..., 5\}. \tag{4}$$

That is p_k' is obtained by adding the centroid c_k of the kth pentahedron face with the associated normal n_k scaled by $\sigma/\sqrt{|n_k|}$.

3.4 Apex transformation (one step) of the pentahedron using centroid transformation

In this case, we only transform the apex (top vertex) of the pentahedron using method (4). So, we only transform p_5 to p_5' and the transformed pentahedron is given by

$$P' = \begin{pmatrix} p_1' \\ p_2' \\ p_3' \\ p_4' \\ p_5' \end{pmatrix} := \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \end{pmatrix} + \sigma \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \frac{1}{\sqrt{|n_5|}} n_5 \end{pmatrix} \tag{5}$$

Here c_k is the centroid of kth pentahedron face where the associated normal n_k scaled by $\sigma/\sqrt{|n_k|}$, $k \in \{1, ..., 5\}$ and $n_5 := (p_4 - p_3) \times (p_2 - p_3)$ of P.

§4. Procedure of Transformations of a Pentahedron

Now it is very important to note the procedure of transformations of a pentahedron. When we consider the 3-D figure like pentahedron, the three points of each four faces must be coplanar. In this paper the construction of pentahedron is in such a way that the four vertices of the pentahedron lie on a plane which means they are coplanar and this plane forms the base of the pentahedron. That means in Fig 1, Fig 2, Fig 3 and Fig 4, one must verify that the points p_1, p_2, p_3, p_4 are coplanar or not. From [15], we can check whether those points are coplanar or not.

Next we discuss about the procedure of first method given by (2). In [8] when the authors transformed a tetrahedron, they took the face normals n_1, n_2, n_3, n_4 opposite to the points p_1, p_2, p_3, p_4 respectively but our case is not exactly the same as in [8]. So, in this case, if we consider Fig 1, let p_1 be any vertices of the pentahedron. Here we find that there are two opposite faces namely face$\{p_2, p_3, p_5\}$ and face$\{p_3, p_4, p_5\}$. In this method we consider the face which is first when we start from p_1 in the anti-clockwise sense. In Fig 1, face$\{p_2, p_3, p_5\}$ is the opposite face of the point p_1. But when we transform p_5 (apex of the pyramid), for this case, the opposite face always forms the base of the pyramid. In Fig.1, the base is face$\{p_1, p_2, p_3, p_4\}$ of the pyramid $P := (p_1, p_2, p_3, p_4, p_5)^t$ which is the opposite face of the p_5. We follow the same procedure in method (3). Now it is important to note that when we use method (3) then it will be necessary to check that after the transformation, the base points of the pentahedron are coplanar or not.

Now for method (4), the procedure is not the same as in [12]. For this case, we consider the face which is first when we start from p_1 in the anti clockwise sense and take the centroid of the face. In Fig.3 let p_1 be any vertices of the pentahedron, here we see that there are two opposite faces namely face$\{p_2, p_3, p_5\}$ and face$\{p_3, p_4, p_5\}$, but in this case we take face$\{p_2, p_3, p_5\}$ and then consider the centroid of the face$\{p_2, p_3, p_5\}$. But when we transform p_5, for this case the opposite face is always base of the pyramid. particularly, for Fig.3 $c_1 = \{p_2 + p_3 + p_5\}/3$, $c_2 = \{p_3 + p_4 + p_5\}/3$, $c_3 = \{p_4 + p_1 + p_5\}/3$, $c_4 = \{p_2 + p_1 + p_5\}/3$, $c_5 = \{p_1 + p_2 + p_3 + p_4\}/4$. So, in this way we get new element (pyramid) using method (4). But here, in that case, the new elements are not linearly formed that means, when new elements form after transformation, the base of this new element is opposite the original element. It happens according to the iteration. For this procedure it will be necessary to check that after the transformation, the base points of the pentahedron are coplanar or not.

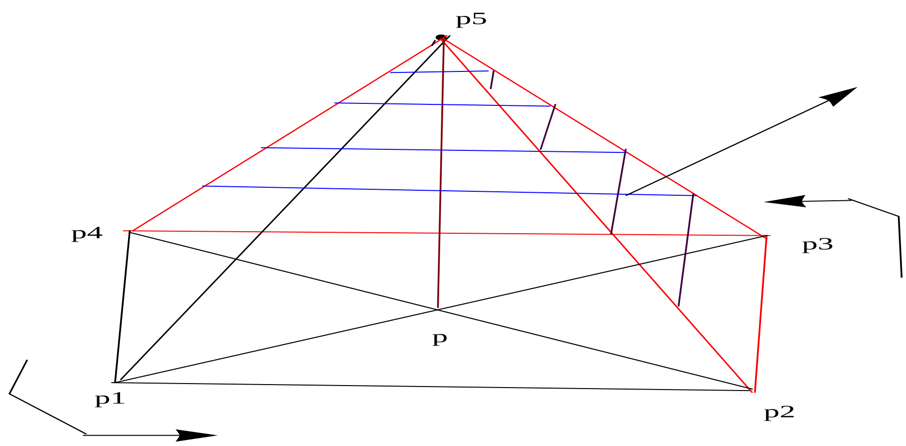

Fig.1 Transformation of a pentahedron using method (2)

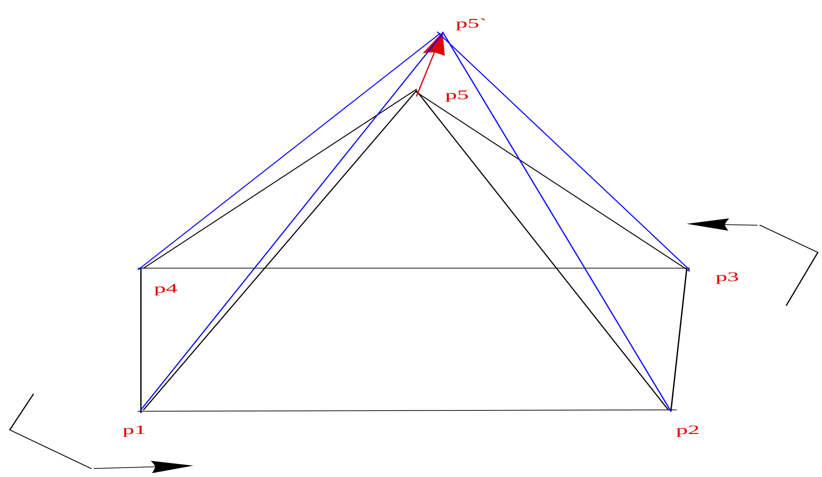

Fig.2 Transformation of a pentahedron using method (3)

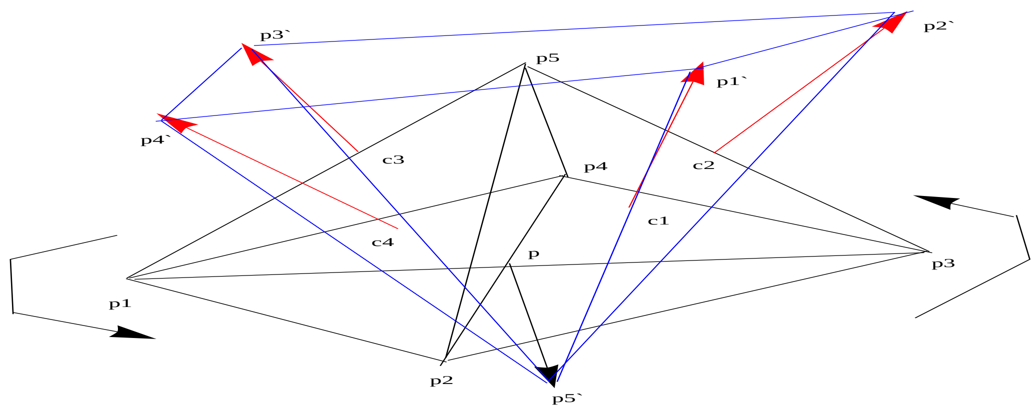

Fig.3 Transformation of a pentahedron using method (4)

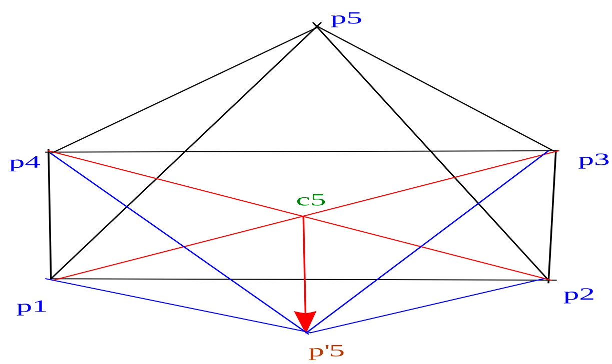

Fig.4 Transformation of a pentahedron using method (5)

§5. Properties of the Transformations

In this section, we discuss the basic properties of the above three transformations.

5.1 The transformations are scale invariant

The transformations given by (2), (3), (4) and (5) are scale invariant that means for $s > 0$,

$(sP)' = sP'$. Since the normals n_i are scaled by $1/\sqrt{|n_i|}$, therefore the transformations are scale invariant. To check this property we shall consider some examples where we choose $\sigma = 0.1$.

Example 1 In this example, we use the transformation (2) and then investigate the property of $(sP)' = sP'$. Let $P := (p_1, p_2, p_3, p_4, p_5)^t$ denote a pentahedron with $p_1 \equiv (0.8, 0, 0.6)$, $p_2 \equiv (0, -0.8, 0.6)$, $p_3 \equiv (-0.8, 0, 0.6)$, $p_4 \equiv (0, 0.8, 0.6)$ and $p_5 \equiv (0, 0, 1)$. Let $s = 0.5$ and applying the transformation (1) both on $(sP)'$ and sP' after that we get the following table.

	Vertex Coordinates of $(sP)'$		
	x	y	z
$(sp_1)'$	0.38	-0.02	0.34
$(sp_2)'$	-0.02	-0.38	0.34
$(sp_3)'$	-0.42	-0.02	0.26
$(sp_4)'$	-0.02	0.42	0.26
$(sp_5)'$	0	0	0.44

	Vertex Coordinates of sP'		
	x	y	z
sp'_1	0.38	-0.02	0.34
sp'_2	-0.02	-0.38	0.34
sp'_3	-0.42	-0.02	0.26
sp'_4	-0.02	0.42	0.26
sp'_5	0	0	0.45

Hence for this pentahedron the transformation is scale invariant. Now, if we use the method of apex transformation (3) on a pentahedron then one can verify from the above table that the transformation is scale invariant. Next we give an example using the method of centroid transformation of a pentahedron.

Example 2 In this example, we use the transformation (4) and then try to investigate the property of $(sP)' = sP'$. We use the same pentahedron which is used in example (1) with $s = 0.5$ and then applying the transformation (4) both on $(sP)'$ and sP' and after calculations we get the following table.

	Vertex Coordinates of $(sP)'$		
	x	y	z
$(sp_1)'$	-0.15	-0.15	0.403
$(sp_2)'$	-0.15	0.15	0.403
$(sp_3)'$	0.12	0.12	0.33
$(sp_4)'$	0.12	-0.12	0.33
$(sp_5)'$	0	0	0.357

	Vertex Coordinates of sP'		
	x	y	z
sp'_1	-0.15	-0.15	0.403
sp'_2	-0.15	0.15	0.403
sp'_3	0.12	0.12	0.33
sp'_4	0.12	-0.12	0.33
sp'_5	0	0	0.356

One can also show that the transformation (5) is scale invariant.

5.2 Transformations (2), (3), (4) and (5) do not preserve the centroid of the pentahedron

It should be noted, that the transformations given by (2), (3), (4) and (5) do not preserve the centroid of the pentahedron, that is $\frac{1}{5}\Sigma_{i=1}^{5} p_i \neq \frac{1}{5}\Sigma_{i=1}^{5} p'_i$, where p_1, p_2, p_3, p_4, p_5 are the vertex coordinates of original pentahedron and p'_1, p'_2, p'_3, p'_4, p'_5 are the vertex coordinates of the transformed pentahedron. As the scale normals $n_i/\sqrt{|n_i|}$ have been used to ensure the scale invariance of the transformation, so the transformations (2) and (3) does not preserve the centroid of the pentahedron. We verify it by an example.

Example 3 Let $P := (p_1, p_2, p_3, p_4, p_5)^t$ denote a pentahedron with $p_1 \equiv (1,0,0)$, $p_2 \equiv (1,1,0)$, $p_3 \equiv (0,1,0)$, $p_4 \equiv (0,0,0)$ and $p_5 \equiv (0,0,2)$. Then using the transformation (3) we get $p_1' \equiv (1,0,0)$, $p_2' \equiv (1,1,0)$, $p_3' \equiv (0,1,0)$, $p_4' \equiv (0,0,0)$ and $p_5' \equiv (0,0,2.10)$.

Centroid of the pentahedron	
Before transformation ($\frac{1}{5}\Sigma_{i=1}^{5} p_i$)	After transformation ($\frac{1}{5}\Sigma_{i=1}^{5} p_i'$)
(0.4,0.4,0.4)	(0.4,0.4,0.42)

Hence from the above we can say that the transformations (3) does not preserve the centroid of the pentahedron. Now, if we use transformation (2), then we can show that it also does not preserve the centroid of the pentahedron, provided after transformation the base of the pentahedron must also be coplanar.

Example 4 In this example, we have shown that after using transformation (4) on a pentahedron, it does not satisfy the preserving property of centroid of the pentahedron. Although in [12], the transformation given by (4) preserve the centroid of the initial hexahedron. In this case, we use the same pentahedron as used in the above example 1. Then using the transformation (4) we get $p_1' \equiv (0.33, 0.53, 0.60)$, $p_2' \equiv (0.14, 0.33, 0.67)$, $p_3' \equiv (0.33, -0.14, 0.67)$, $p_4' \equiv (0.80, 0.33, 0.73)$ and $p_5' \equiv (0.50, 0.50, 0.10)$.

Centroid of the pentahedron	
Before transformation ($\frac{1}{5}\Sigma_{i=1}^{5} p_i$)	After transformation ($\frac{1}{5}\Sigma_{i=1}^{5} p_i'$)
(0.4,0.4,0.4)	(0.42,0.31,0.55)

Hence from the above we can say that the transformations (4) does not preserve the centroid of the pentahedron. One can also easily see that the transformation (5) also does not preserve the centroid of the pentahedron.

5.3 Characterization of mean ratio quality of a pentahedron

To define mean ratio quality for pentahedron, first we use BDAMe to get four tetrahedra and then choose any tetrahedron. Let $T := (p_1, p_2, p_3, p_4)$ denote a tetrahedron with the four pairwise disjoint nodes $p_i \in R^3$, $i \in \{1, \cdots, 4\}$, which is positively oriented. That is $det(A) > 0$ with $A := (p_2 - p_1, p_3 - p_1, p_4 - p_1)$ representing the (3×3) Jacobian matrix of the difference vectors, which span the tetrahedron. In [8,11,12,13] authors have discussed how to get mean ratio quality of a tetrahedron and using this procedure we define the mean ratio quality for pentahedron, as

$$q(P) := \frac{1}{4} \sum_{k=1}^{4} \frac{3det(S_k)^{2/3}}{\|S\|_F},$$

with $\|S\| := \sqrt{t}race(S^tS)$ denoting the Frobenius norm of the matrix $S_k := A_kW^{-1}$ where

$$W = \begin{pmatrix} 1 & 1/2 & 1/2 \\ 0 & \sqrt{3}/2 & \sqrt{3}/6 \\ 0 & 0 & \sqrt{2}/\sqrt{3} \end{pmatrix}$$

denotes the difference matrix of a regular reference tetrahedron. Now in the case of pentahedron, the criterion of $q(P)$ is not same as in [8,12]. In that case, if P is regular then $q(P) \in [0,1]$, where very small values indicate nearly degenerated elements and large values element good quality. Now, if the transformation is applied iteratively, the resulting pentahedron became more and more regular. In order to assess the regularity of a pentahedron P numerically, the mean ratio quality criterion will be used. Now, next we give an example of a pentahedron which is regular square pyramid but $q(P) \neq 1$.

Example 5 Let $P := (p_1, p_2, p_3, p_4, p_5)^t$ denote a pentahedron with $p_1 \equiv (0,0,0)$, $p_2 \equiv (1,0,0)$, $p_3 \equiv (1,1,0)$, $p_4 \equiv (0,1,0)$ and $p_5 \equiv (0.5, 0.5, 1)$. Here, $q(p) = 0.797$.

5.4 Significance of the scaling factor σ

The resulting iteration numbers are totally depended upon the scaling factor σ. This can be used in order to control the regularization speed by a quality depended choice of the scaling factor. For a given pentahedron, there is most important thing to choice the scaling factor when we try to regularize it. For this, if the transformation is applied iteratively, the resulting pentahedron more and more regular. Now, depending upon the choice of the scaling factor σ, the size of the pentahedron might also change significantly.

Also the important fact is that there is no specific choice of σ, for which the transformation given exactly once to any arbitrary pentahedron results a regular one. To show this, we give an example.

Example 6 Let us choose the pentahedron with the same coordinate as given in article (5.2) Example 1. According to (2), the nodes of the transformed pentahedron P' are given by $p'_1 \equiv (1+\sigma(0), 0+\sigma(-1.3), 0+\sigma(-0.7))$, $p'_2 \equiv (1+\sigma(1.4), 1+\sigma(0), 0+\sigma(0))$, $p'_3 \equiv (0+\sigma(0), 1+\sigma(-1.4), 0+\sigma(0))$, $p'_4 \equiv (0+\sigma(1.3), 0+\sigma(0), 0+\sigma(0.7))$ and $p'_5 \equiv (0+\sigma(0), 0+\sigma(0), 0+\sigma(-1))$ using an arbitrary scaling factor $\sigma \in R_0^+$. In order to be regular, all edge lengths of the base of the transformed pentahedron have to be equal if the base is square and in that case our example is square base pyramid but not regular. Since the equation $|p'_2 - p'_3| = |p'_3 - p'_4|$ has only valid solution $\sigma = 25.45$ and on the other way $|p'_1 - p'_2| = |p'_2 - p'_3|$ has only valid solution if $\sigma = 0.91$. Therefore there is a contradiction that there is no $\sigma \in R_0^+$ for which the pentahedron P' obtained by one step of the transformation is regular.

5.5 Uniqueness of the circumsphere and volume of the pentahedron

Now for any 3-simplex, we can always draw a sphere through the four vertices of the 3-simplex, but for pentahedron and other 3D-figures we always do not get a sphere through the all vertices of the 3D-figure except tetrahedron. But if we choose a pentahedron so that its all vertices satisfy some particular sphere equation and then we use the transformation (2), (3), (4) and (5) we can show that after transformation the transformed pentahedron may not satisfy some particular sphere equation. Let us give an example using transformation (2) and (3) and one can also verify show using transformation (4) and (5).

Example 7 Let $P := (p_1, p_2, p_3, p_4, p_5)^t$ denote a pentahedron with $p_1 \equiv (-0.8, 0, 0.6)$, $p_2 \equiv (0, -0.8, 0.6)$, $p_3 \equiv (-0.8, 0, 0.6)$, $p_4 \equiv (0, 0.8, 0.6)$ and $p_5 \equiv (0, 0, 1)$. Here all vertices of the pentahedron P satisfy the sphere equation $x^2 + y^2 + z^2 = 1$. After transformation using formula (2) the transformed pentahedron does not satisfy any sphere equation. If we use apex transformation then after transformation always we get a pentahedron and this pentahedron does not satisfy any particular sphere equation. For the given example after one step (using method (3)) we get, $p'_1 \equiv (-0.8, 0, 0.6)$, $p'_2 \equiv (0, -0.8, 0.6)$, $p'_3 \equiv (-0.8, 0, 0.6)$, $p'_4 \equiv (0, 0.8, 0.6)$ and $p'_5 \equiv (0, 0, 2.13)$ and p'_i's do not satisfy any sphere equation.

On the other hand the volume of the pentahedron will also be changed and does not depend on whether the pentahedron is regular or not. The volume of the transformed pentahedron will decrease or increase that depends upon the choice of the pentahedron.

Example 8 Let $P := (p_1, p_2, p_3, p_4, p_5)^t$ denote a pentahedron with $p_1 \equiv (4, 3, 0)$, $p_2 \equiv (4, 7, 0)$, $p_3 \equiv (0, 7, 0)$, $p_4 \equiv (0, 3, 0)$ and $p_5 \equiv (3, 6, 10)$. Now the volume of the pentahedron is 73.34 cube unit and using the formula (3) to the pentahedron P we get the volume of the transformed pentahedron 76.26 cube unit.

§6. Regularization of a Pyramid (Pentahedron)

Here we shall consider a process to regularize a pentahedron. In the case of pentahedron regular means that the base of the pentahedron is regular (square, rectangle, $\cdots$, etc.) and the upper all edge lengths are equal. So, when we consider an arbitrary pentahedron it is quite difficult to regularize the pentahedron, but if we take the base of the pentahedron is regular and upper portion of the pentahedron is irregular then we can regularize the pentahedron using apex transformations (3) and (5). One can also use transformations (2) and (4) provided after transformations the base points are coplanar. Here we furnish an example where transformation (3) and transformation (5) are used to regularize the pentahedron whose base is regular.

Example 9 Let $P := (p_1, p_2, p_3, p_4, p_5)^t$ denote a pentahedron with $p_1 \equiv (4, 3, 4)$, $p_2 \equiv (4, 7, 4)$, $p_3 \equiv (0, 7, 4)$, $p_4 \equiv (0, 3, 4)$ and $p_5 \equiv (3, 8, 10)$. This is square based pyramid but not regular because the base edges length $p_1p_2 = p_2p_3 = p_3p_4 = p_4p_1 = 4$ and for the upper portion length of the edges are $p_1p_5 = 7.87$, $p_2p_5 = 6.16$, $p_3p_5 = 6.78$, $p_4p_5 = 8.37$ which all are not equal. Now, if we use the transformation (3) on the given pentahedron, then in first step, length of the edges (upper portion) are $p_1p'_5 = 8.18$, $p_2p'_5 = 6.44$, $p_3p'_5 = 7.14$, $p_4p'_5 = 8.66$ and in third step, length of the edges are $p_1p''_5 = 8.50$, $p_2p''_5 = 6.95$, $p_3p''_5 = 7.50$, $p_4p''_5 = 8.96$. Here we observe that the pyramid tends to regularize but slow. The speed of the regularization depends upon the choice of the scaling factor σ. In this case we take the scaling factor $\sigma = 0.1$.

Next we use the apex transformation (5) to the given pentahedron and we get $p'_5 \equiv$

$(2, 5, 4.40)$. After calculations we see that length of the edges (upper portion) are $p_1p_5' = p_2p_5' = p_3p_5' = p_4p_5' = 2.86$. Hence the given pyramid converge to regularize and it turns to a regular square pyramid.

§7. Characterization of Energy Function of a Particular Type of Pentahedron Using GETMe and BDAMe

The changing cost function, after transforming the pentahedron by GETMe, is given by

$$f(\sigma_p) = |f_{p_k}(\sigma^n) \sim f_{p_{k+1}}(\sigma^n)|.$$

Now using BDAMe we find the numerical values of changing cost function. We can calculate the changing cost function of the pyramid provided after transformation the base points are coplanar. Let us consider $p_1(4, 3, 0)$, $p_2(4, 7, 0)$, $p_3(0, 7, 0)$, and $p_4(0, 3, 0)$ are the base points and $p_5(2, 5, 10)$ be the apex of the pyramid and $p(2, 5, 0)$ be the intersection point of the base diagonals. In this case, it is regular square base pyramid.

After calculation, we get

	Apex Transformations			
	Initial Step	2nd Step	3rd Step	4th step
$f_p(\sigma^n)$	0.944	0.934	0.939	0.941

Now, this example is similar to the example of section (6) and it can be regularized by using the method (3). Here we see that the values of the changing cost function $f(\sigma_p)$ are 0.01, 0.005 and 0.002. Therefore for this example we see that when it converges to regularize the changing cost function also decreases. One can also calculate the changing cost function for any arbitrary pyramid (using the method (2) and (4)) provided after transformations the base points are coplanar.

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