International J.Math. Combin. Vol.1(2014), 13-18

Existence Results of Unique Fixed Point in 2-Banach Spaces

G.S.Saluja

(Department of Mathematics, Govt.Nagarjuna P.G. College of Science, Raipur - 492010 (C.G.), India)

E-mail: saluja1963@gmail.com

Abstract: In this paper, we establish some fixed point theorems in the setup of 2-Banach spaces. The results obtained are the 2-Banach space extension of the result of Zhao [9].

Key Words: Fixed point, 2-Banach space, 2-Banach contraction, 2-Kannan contraction, 2-Chatterjea contraction, 2-Zamfirescu operator.

AMS(2010): 47H10

§1. Introduction

The concept of 2-Banach space and some basic fixed point results in such spaces are initially given by Gahler ([3], [4]) during $1960's$. Later on some fixed point results have been obtained in such spaces by Iseki [5], Khan et al. [6], Rhoades [7] and many others extending the fixed point results for non expansive mappings from Banach space to 2-Banach space. In 2011, Choudhury and Som [2] (J. Indian Acad. Math. 33(2) (2011), 411-418) have established common fixed point and coincidence fixed point results for a pair of non-linear mappings in 2-Banach space which generalize the results of Som [8], Cho et al. [1] and Zhao [9] in turn. In this paper we establish some fixed point theorems satisfying the contractive type condition in 2-Banach spaces.

§2. Preliminaries

Here we give some preliminary definitions related to 2-Banach spaces which are needed in the sequel.

Definition 2.1 (See [1]) *Let X be a linear space and $\|.,.\|$ be a real valued function defined on X satisfying the following conditions:*

(i) $\|x, y\| = 0$ *if and only if x and y are linearly dependent;*
(ii) $\|x, y\| = \|y, x\|$ *for all $x, y \in X$;*
(iii) $\|x, ay\| = |a|\,\|x, y\|$ *for all $x, y \in X$ and real a;*
(iv) $\|x, y + z\| = \|x, y\| + \|x, z\|$ *for all $x, y, z \in X$.*

[1]Received June 14, 2013, Accepted February 10, 2014.

Then, $\|.,\,.\|$ *is called a* 2*-norm and the pair* $(X, \|.,\,.\|)$ *is called a linear* 2*-normed space.*

Some of the basic properties of the 2-norms are that they are non negative and

$$\|x,\, y + ax\| = \|x,\, y\|$$

for all $x, y \in X$ and all real number a.

Definition 2.2(See [1]) *A sequence* $\{x_n\}$ *in a linear* 2*-normed space* $(X, \|.,\,.\|)$ *is called a Cauchy sequence if* $\lim_{m,\,n\to\infty} \|x_m - x_n,\, y\| = 0$ *for all* $y \in X$*.*

Definition 2.3(See [1]) *A sequence* $\{x_n\}$ *in a linear* 2*-normed space* $(X, \|.,\,.\|)$ *is said to be convergent to a point* x *in* X *if* $\lim_{n\to\infty} \|x_n - x,\, y\| = 0$ *for all* $y \in X$*.*

Definition 2.4(See [1]) *A linear* 2*-normed space* $(X, \|.,\,.\|)$ *in which every Cauchy sequence is convergent is called a* 2*-Banach space.*

Definition 2.5(See [1]) *Let* X *be a* 2*-Banach space and* T *be a self mapping of* X*.* T *is said to be continuous at* x *if for any sequence* $\{x_n\}$ *in* X *with* $x_n \to x$ *implies that* $Tx_n \to Tx$*.*

Definition 2.6 *Let* $(X, \|.,\,.\|)$ *be a linear* 2*-normed space and* T *be a self mapping of* X*. A mapping* T *is said to be* 2*-Banach contraction if there is* $a \in [0, 1)$ *such that*

$$\|Tx - Ty,\, u\| \le a\ \|x - y,\, u\|$$

for all $x, y, u \in X$*.*

Definition 2.7 *Let* $(X, \|.,\,.\|)$ *be a linear* 2*-normed space and* T *be a self mapping of* X*. A mapping* T *is said to be* 2*-Kannan contraction if there is* $b \in [0, \frac{1}{2})$ *such that*

$$\|Tx - Ty,\, u\| \le b\,\big[\, \|x - Tx,\, u\| + \|y - Ty,\, u\| \,\big]$$

for all $x, y, u \in X$*.*

Definition 2.8 *Let* $(X, \|.,\,.\|)$ *be a linear* 2*-normed space and* T *be a self mapping of* X*. A mapping* T *is said to be* 2*-Chatterjea contraction if there is* $c \in [0, \frac{1}{2})$ *such that*

$$\|Tx - Ty,\, u\| \le c\,\big[\, \|x - Ty,\, u\| + \|y - Tx,\, u\| \,\big]$$

for all $x, y, u \in X$*.*

Definition 2.9 *Let* $(X, \|.,\,.\|)$ *be a linear* 2*-normed space and* T *be a self mapping of* X*. A mapping* T *is said to be* 2*-Zamfirescu operator if there are real numbers* $0 \le a < 1$, $0 \le b < 1/2$, $0 \le c < 1/2$ *such that for all* $x, y, u \in X$ *at least one of the conditions is true:*

(z_1) $\|Tx - Ty,\, u\| \le a\ \|x - y,\, u\|$*;*
(z_2) $\|Tx - Ty,\, u\| \le b\ (\|x - Tx,\, u\| + \|y - Ty,\, u\|)$*;*
(z_3) $\|Tx - Ty,\, u\| \le c\ (\|x - Ty,\, u\| + \|y - Tx,\, u\|)$*.*

Condition 2.1 *Let X be a 2-Banach space (with $\dim X \geq 2$) and let T be a self mapping of X such that for all x, y, u in X satisfying the condition:*

$$\begin{aligned} \|Tx - Ty,\, u\| \quad \leq \quad & h \max\Big\{ \|x - y,\, u\|\,, \frac{(\|x - Tx,\, u\| + \|y - Ty,\, u\|)}{2}, \\ & \frac{(\|x - Ty,\, u\| + \|y - Tx,\, u\|)}{2}\Big\} \end{aligned} \tag{2.1}$$

where $0 < h < 1$.

Remark 2.1 It is obvious that each of the conditions $(z_1) - (z_3)$ implies (2.1).

§3. Main Results

In this section we shall prove a fixed point theorem using condition (2.1) in the setting of 2-Banach spaces.

Theorem 3.1 *Let X be a 2-Banach space (with $\dim X \geq 2$) and let T be a continuous self mapping of X satisfying the condition (2.1), then T has a unique fixed point in X.*

Proof For given each $x_0 \in X$ and $n \geq 1$, we choose $x_1, x_2 \in X$ such that $x_1 = Tx_0$ and $x_2 = Tx_1$. In general we define sequence of elements of X such that $x_{n+1} = Tx_n = T^{n+1}x_0$. Now for all $u \in X$, using (2.1), we have

$$\begin{aligned} \|x_n - x_{n+1},\, u\| \quad = \quad & \|Tx_{n-1} - Tx_n,\, u\| \\ \leq \quad & h \max\Big\{ \|x_{n-1} - x_n,\, u\|\,, \frac{(\|x_{n-1} - Tx_{n-1},\, u\| + \|x_n - Tx_n,\, u\|)}{2}, \\ & \frac{(\|x_{n-1} - Tx_n,\, u\| + \|x_n - Tx_{n-1},\, u\|)}{2}\Big\} \\ = \quad & h \max\Big\{ \|x_{n-1} - x_n,\, u\|\,, \frac{(\|x_{n-1} - x_n,\, u\| + \|x_n - x_{n+1},\, u\|)}{2}, \\ & \frac{(\|x_{n-1} - x_{n+1},\, u\| + \|x_n - x_n,\, u\|)}{2}\Big\} \\ = \quad & h \max\Big\{ \|x_{n-1} - x_n,\, u\|\,, \frac{(\|x_{n-1} - x_n,\, u\| + \|x_n - x_{n+1},\, u\|)}{2}, \\ & \frac{\|x_{n-1} - x_{n+1},\, u\|}{2}\Big\} \\ \leq \quad & h \max\Big\{ \|x_{n-1} - x_n,\, u\|\,, \frac{(\|x_{n-1} - x_n,\, u\| + \|x_n - x_{n+1},\, u\|)}{2}, \\ & \frac{(\|x_{n-1} - x_n,\, u\| + \|x_n - x_{n+1},\, u\|)}{2}\Big\}. \end{aligned} \tag{3.1}$$

But

$$\frac{(\|x_{n-1} - x_n,\, u\| + \|x_n - x_{n+1},\, u\|)}{2} \leq \max\Big\{ \|x_{n-1} - x_n,\, u\|\,, \|x_n - x_{n+1},\, u\| \Big\} \tag{3.2}$$

From (3.1) and (3.2), we get

$$\begin{aligned} \|x_n - x_{n+1}, u\| &\leq h \max\Big\{ \|x_{n-1} - x_n, u\|, \|x_{n-1} - x_n, u\|, \|x_n - x_{n+1}, u\|, \\ &\quad \|x_{n-1} - x_n, u\|, \|x_n - x_{n+1}, u\| \Big\} \\ &\leq h \|x_{n-1} - x_n, u\|. \qquad (3.3) \end{aligned}$$

Similarly, we have

$$\|x_{n-1} - x_n, u\| \leq h \|x_{n-2} - x_{n-1}, u\|. \tag{3.4}$$

Hence form (3.3) and (3.4), we have

$$\|x_n - x_{n+1}, u\| \leq h^2 \|x_{n-2} - x_{n-1}, u\|. \tag{3.5}$$

On continuing in this process, we get

$$\|x_n - x_{n+1}, u\| \leq h^n \|x_0 - x_1, u\|. \tag{3.6}$$

Also for $n > m$, we have

$$\begin{aligned} \|x_n - x_m, u\| &\leq \|x_n - x_{n-1}, u\| + \|x_{n-1} - x_{n-2}, u\| + \dots \\ &\quad + \|x_{m+1} - x_m, u\| \\ &\leq \left(h^{n-1} + h^{n-2} + \dots + h^m\right) \|x_1 - x_0, u\| \\ &\leq \left(\frac{h^m}{1-h}\right) \|x_1 - x_0, u\|. \end{aligned} \tag{3.7}$$

Since $0 < h < 1$ by condition 2.1, $\left(\frac{h^m}{1-h}\right) \to 0$ as $m \to \infty$. Hence $\|x_n - x_m, u\| \to 0$ as $n, m \to \infty$. This shows that $\{x_n\}$ is a Cauchy sequence in X. Hence there exist a point z in X such that $x_n \to z$ as $n \to \infty$. It follows from the continuity of T that $Tz = z$. Thus z is a fixed point of T.

For the uniqueness, let $Tv = v$ be another fixed point of the mapping T. Then, we have

$$\begin{aligned} \|z - v, u\| &= \|Tz - Tv, u\| \\ &\leq h \max\Big\{ \|z - v, u\|, \frac{(\|z - Tz, u\| + \|v - Tv, u\|)}{2}, \\ &\quad \frac{(\|z - Tv, u\| + \|v - Tz, u\|)}{2} \Big\} \\ &\leq h \max\Big\{ \|z - v, u\|, 0, \|z - v, u\| \Big\} \\ &\leq h \|z - v, u\| \\ &< \|z - v, u\|, \text{ since } 0 < h < 1, \end{aligned} \tag{3.8}$$

a contradiction. Hence $z = v$ and for all $u \in X$. Thus z is a unique fixed point of T. This completes the proof. □

Since Condition 2.1 includes the 2-Banach contraction condition, 2-Kannan contraction condition, 2-Chatterjea contraction condition and 2-Zamfirescu operator. Thus from Theorem 3.1, we obtain the following results as corollaries.

Corollary 3.1 *Let X be a 2-Banach space (with $\dim X \geq 2$) and let T be a self mapping of X satisfying the condition:*

$$\|Tx - Ty,\, u\| \leq a\ \|x - y,\, u\|$$

for all $x, y, u \in X$, where a is a constant in $(0,1)$. Then T has a unique fixed point in X.

Corollary 3.2 *Let X be a 2-Banach space (with $\dim X \geq 2$) and let T be a continuous self mapping of X satisfying the condition:*

$$\|Tx - Ty,\, u\| \leq b\,[\|x - Tx,\, u\| + \|y - Ty,\, u\|]$$

for all $x, y, u \in X$, where b is a constant in $(0, \frac{1}{2})$. Then T has a unique fixed point in X.

Corollary 3.3 *Let X be a 2-Banach space (with $\dim X \geq 2$) and let T be a continuous self mapping of X satisfying the condition:*

$$\|Tx - Ty,\, u\| \leq c\,[\|x - Ty,\, u\| + \|y - Tx,\, u\|]$$

for all $x, y, u \in X$, where c is a constant in $(0, \frac{1}{2})$. Then T has a unique fixed point in X.

Corollary 3.4 *Let X be a 2-Banach space (with $\dim X \geq 2$) and let T be a continuous self mapping of X satisfying 2-Zamfirescu operator, that is, satisfying at least one of the conditions in $(z_1) - (z_3)$. Then T has a unique fixed point in X.*

Remark 3.1 Our results extend the corresponding result of Zhao [9] (Acta Math. Sinica 22(1979), 459-470), Cho et al. [1] (Far East Jour. Math. Sci. 3(2)(1995), 125-133) and many others from the existing literature.

References

[1] Y.J.Cho, N.Huang and X.Long, Some fixed point theorems for linear mappings in 2-Banach spaces, *Far East Jour. Math. Sci.*, 3(2)(1995), 125-133.

[2] A.Choudhury and T.Som, 2-Banach space and some fixed point results, *Journal of Indian Acad. Math.*, 33(2) (2011), 411-418.

[3] S.Gahler, Metricsche Roume and their topologische structure, *Math. Nachr.*, 26(1963), 115-148.

[4] S.Gahler, Linear 2-Normietre Roume, *Math. Nachr.*, (1965), 281-43.

[5] K.Iseki, Fixed point theorems in Banach spaces, *Math. Sem. Notes*, Kobe Univ., (1976), 211-213.

[6] M.S.Khan and M.D.Khan, Involutions with fixed points in 2-Banach spaces, *Int. J. Math. Math. Sci.*, 16(1993), 429-434.

[7] B.E.Rhoades, Contraction type mappings on 2-metric space, *Math. Nachr.*, 91(1979), 151-155.

[8] T.Som, Some fixed point results in 2-Banach space, *International Jour. Math. Scs.*, 4(2)(2005), 323-328.

[9] H.B.Zhao, Existence theorems of fixed point of average non expansive mappings in Banach spaces, *Acta Math. Sinica* 22(1979), 459-470.