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Contributions to Differential Geometry of Partially Null Curves in Semi-Euclidean Space E_1^4

Süha Yılmaz

(Dokuz Eylül University, Buca Educational Faculty, Buca-Izmir, Turkey)

Emin Özyılmaz and Ümit Ziya Savcı

(Ege University, Faculty of Science, Dept. of Math., Bornova-Izmir, Turkey)

E-mail: suha.yilmaz@deu.edu.tr, emin.ozyilmaz@ege.edu.tr, ziyasavci@hotmail.com

Abstract: In this paper, some characterizations of partially null curves of constant breadth and inclined partially null curves in Semi-Rieamannian Space E_1^4 are presented.

Key Words: Semi-Rieamannian Space, partially null curves, curves of constant breadth.

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§1. Introduction

The partially null curves, lying fully in the Minkowski space-time are defined in [1] as space-like curves along which respectively the first binormal is null vector and second binormal is null vector. The Frenet equations of a partially null curve, lying fully in the Minkowski space-time are given in [14, 2], using those Frenet equations authors give some characterizations. Another work, in [10], authors define Frenet equations of such curves and study some of characterizations in Semi-Euclidean space.

Recently, a method has been developed by B.Y.Chen to classify curves with the solution of differential equations with constant coefficients, see [3, 4, 11]. Furthermore, classifications all space-like W curves are given in [11].

Curves of constant breadth were introduced by L. Euler,1870. Ö. Köse (1984) wrote some geometric properties of plane curves of constant breadth. And, in another work Ö. Köse (1986) extended these properties to the Euclidian3-space E^3 [6]. Morever, M. Fujivara (1914) obtained a problem to determine whether there exist space curve of constant breadth or not, and he defined "breadth" for space curves and obtained these curves on a surface of constant breadth [5]. A. Mağden and Ö. Köse (1997) studied this kind curves in four dimensional Euclidean space E^4 [7]. S. Yılmaz and M. Turgut extended the notation of curves of constant breadth to null curves in Semi-Rieamannian space E_2^4, see [13].

Inclined curves are well-known concept in the classical differential geometry [8].

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§2. Preliminaries

To meet the requirements in the next sections, here, the basic elements of the theory of curves in the space E_1^4 are briefly presented (A more complete elementary treatment can be found in [9].) Minkowski space-time E_1^4 is a Euclidean space E_1^4 provided with the standard flat metric given by

$$g = -dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2$$

where (x_1, x_2, x_3, x_4) is rectangular coordinate system in E_1^4. Since g is an definite metric, recall that a vector $\overrightarrow{\vartheta} \in E_1^4$ can have one of the three causal characters; it can be space-like if $g(\overrightarrow{\vartheta}, \overrightarrow{\vartheta}) > 0$ or $\overrightarrow{\vartheta} = 0$, timelike if $g(\overrightarrow{\vartheta}, \overrightarrow{\vartheta}) < 0$ and null (ligth-like) if $g(\overrightarrow{\vartheta}, \overrightarrow{\vartheta}) = 0$ and $\overrightarrow{\vartheta} \neq 0$. Similarly, and arbitrary curve $\overrightarrow{\alpha} = \overrightarrow{\alpha}(s)$ in E_1^4 can be locally be space-like, time-like or null (ligth-like) if all of its velocity vectors $\overrightarrow{\alpha}'(s)$ are respectively space-like, time-like or null. Also recall the norm of a vector $\overrightarrow{\vartheta}$ is given by $\left\|\overrightarrow{\vartheta}\right\| = \sqrt{\left|g(\overrightarrow{\vartheta}, \overrightarrow{\vartheta})\right|}$. Therefore, $\overrightarrow{\vartheta}$ is a unit vector if $g(\overrightarrow{\vartheta}, \overrightarrow{\vartheta}) = \pm 1$. Next vectors $\overrightarrow{\vartheta}$, $\overrightarrow{w}$ in E_1^4 are said to be orthogonal if $g(\overrightarrow{\vartheta}, \overrightarrow{w}) = 0$. The velocity of the curve α is given by$\|\overrightarrow{\alpha}'\|$. Thus, a space-like or a time-like curve $\overrightarrow{\alpha}$ is said to be parameterized by arc-length function s, if $g(\overrightarrow{\alpha}', \overrightarrow{\alpha}') = \pm 1$. The Lorentzian hypersphere of center $\overrightarrow{m} = (m_1, m_2, m_3, m_4)$ and radius $r \in R^+$ in the space E_1^4 defined by

$$S_1^3 = \left\{\overrightarrow{\alpha} = (\alpha_1, \alpha_2, \alpha_3, \alpha_4) \in E_1^4 : g(\overrightarrow{\alpha} - \overrightarrow{m}, \overrightarrow{\alpha} - \overrightarrow{m}) = r^2\right\}.$$

Denoted by $\left\{\overrightarrow{T}(s), \overrightarrow{N}(s), \overrightarrow{B}_1(s), \overrightarrow{B}_2(s)\right\}$ the moving Frenet frame along the curve $\overrightarrow{\alpha}$ in the space E_1^4.

Then $\overrightarrow{T}, \overrightarrow{N}, \overrightarrow{B}_1, \overrightarrow{B}_2$ are, respectively, the tangent, the principal normal, the first binormal and second binormal vector fields. Recall that a space-like curve with time-like principal normal $\overrightarrow{N}$ and null first and second binormal is called a partially null curve in E_1^4 [1]. For a partially null unit speed curve $\overrightarrow{\alpha}$ in E_1^4 the following Frenet equations are given in [2, 14]

$$\begin{bmatrix} \overrightarrow{T}' \\ \overrightarrow{N}' \\ \overrightarrow{B}_1' \\ \overrightarrow{B}_2' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 & 0 \\ -\kappa & 0 & \tau & 0 \\ 0 & 0 & \sigma & 0 \\ 0 & -\tau & 0 & \sigma \end{bmatrix} \cdot \begin{bmatrix} \overrightarrow{T} \\ \overrightarrow{N} \\ \overrightarrow{B}_1 \\ \overrightarrow{B}_2 \end{bmatrix}$$

where $\overrightarrow{T}, \overrightarrow{N}, \overrightarrow{B}_1$ and $\overrightarrow{B}_2$ are mutually orthogonal vectors satisfying equations

$$g(\overrightarrow{T}, \overrightarrow{T}) = g(\overrightarrow{B}_1, \overrightarrow{B}_2) = 1, \ g(\overrightarrow{N}, \overrightarrow{N}) = -1$$
$$g(\overrightarrow{B}_1, \overrightarrow{B}_1) = g(\overrightarrow{B}_2, \overrightarrow{B}_2) = 0.$$

And here, $\kappa(s), \tau(s)$ and $\sigma(s)$ are first, second and third curvature of the curve $\overrightarrow{\alpha}$, respectively.

In the same space, the authors, in [2], expressed a characterizations of partially null curves with the following theorem.

Theorem 2.1 *A partially null unit speed curve* $\overrightarrow{\alpha}=\overrightarrow{\alpha}(s)$, *in* E_1^4, *with curvatures* $\kappa\neq 0$, $\tau\neq 0$ *for each* $s\in I\subset\mathbb{R}$ *has* $\sigma=0$ *for eachs.*

In [13], S. Yılmaz and M. Turgut studied same characterizations of spherical and inclined partially null curves.

§3. Partially Null Curves of Constant Breadth in E_1^4

Let $\overrightarrow{\alpha}=\overrightarrow{\alpha}(s)$ and $\overrightarrow{\alpha}^*=\overrightarrow{\alpha}^*(s)$ be simple closed partially null curves in the space E_1^4. These curves will be denoted by C. Moreover let P and Q at points respectively curves α and α^*. The normal plane at every point P on the curve meets the curve at a single point Q other than P. We call the point Q the opposite point of P. We consider a partially null curve in the class Γ as in M. Fujivara (1914) having parallel tangents $\overrightarrow{\mathbf{T}}$ and $\overrightarrow{\mathbf{T}}^*$ in opposite directions at the opposite points α and α^* of the curve. A simple closed curve of constant breadth at opposite points can be represented with respect to Frenet frame by the equation

$$\overrightarrow{\alpha}^* = \overrightarrow{\alpha}+m_1\overrightarrow{T}+m_2\overrightarrow{N}+m_3\overrightarrow{B}_1+m_4\overrightarrow{B}_2 \tag{3.1}$$

where $m_i(s)$, $1\leq i\leq 4$ arbitrary functions of s, $\overrightarrow{\alpha}$ and $\overrightarrow{\alpha}^*$ are opposite points. The vector $d=\overrightarrow{\alpha}^*-\overrightarrow{\alpha}$ is called "the distance vector" of C. Differentiating both sides of (3.1) and considering Frenet equations, we have

$$\begin{aligned}\frac{d\alpha^*}{ds}=\overrightarrow{T}^*\frac{ds^*}{ds} &= \left(\frac{dm_1}{ds}-m_2\kappa+1\right)\overrightarrow{T}+\left(\frac{dm_2}{ds}-m_4\tau+m_1\kappa\right)\overrightarrow{N}\\ &\quad+\left(\frac{dm_3}{ds}+m_2\tau+m_3\sigma\right)\overrightarrow{B}_1+\left(\frac{dm_4}{ds}+m_4\sigma\right)\overrightarrow{B}_2\end{aligned} \tag{3.2}$$

Since $\overrightarrow{T}^*=-\overrightarrow{T}$, rewriting (3.2) we obtain following system of equations,

$$\begin{aligned}&\frac{dm_1}{ds}-m_2\kappa+1+\frac{ds^*}{ds}=0\\ &\frac{dm_2}{ds}+m_1\kappa-m_4\tau=0\\ &\frac{dm_3}{ds}+m_2\tau=0\\ &\frac{dm_4}{ds}=0\end{aligned} \tag{3.3}$$

If we call θ as the angle between the tangent of the curve C at point $\overrightarrow{\alpha}$ with a given fixed direction and s arc length parameter of $\overrightarrow{\alpha}(s)$, consider $\dfrac{d\theta}{ds} = \kappa$, we have (3.3) as following:

$$\begin{aligned} \frac{dm_1}{d\theta} &= m_2 - f(\theta) \\ \frac{dm_2}{d\theta} &= -m_1 + m_4\rho\tau \\ \frac{dm_3}{d\theta} &= -m_2\rho\tau \\ \frac{dm_4}{d\theta} &= 0 \end{aligned} \tag{3.4}$$

where $f(\theta) = \rho + \rho^*$, $\rho = \dfrac{1}{\kappa}$ and $\rho^* = \dfrac{1}{\kappa^*}$ denote the radius of curvature at $\overrightarrow{\alpha}$ and $\overrightarrow{\alpha}^*$, respectively. It is not difficult to see that $m_4 = c_4$ =constant. then, using system (3.4) we easily have following differential equations with respect to m_1 and m_2 as

$$\begin{aligned} &\frac{d^2m_1}{d\theta^2} + m_1 + \frac{df}{d\theta} - c_4\rho\tau = 0 \\ &\frac{d^2m_2}{d\theta^2} + m_2 - c_4\frac{d}{d\theta}(\rho\tau) - f(\theta) = 0 \end{aligned} \tag{3.5}$$

These equations are characterizations for the curve $\overrightarrow{\alpha}^*$. If the distance between opposite points of C and C^* is constant, then, due to null frame vectors, we can write that

$$\|\overrightarrow{\alpha}^* - \overrightarrow{\alpha}\|^2 = m_1^2 + m_2^2 + 2m_3m_4 = l^2 = \text{constant}. \tag{3.6}$$

Hence, by the differentiation we have

$$m_1\frac{dm_1}{d\theta} + m_2\frac{dm_2}{d\theta} + m_3\frac{dm_4}{d\theta} + m_4\frac{dm_3}{d\theta} = 0 \tag{3.7}$$

Considering system (3.4), we get

$$m_1(\frac{dm_1}{d\theta} - m_2) = 0 \tag{3.8}$$

Since, we arrive $m_1 = 0$ or $\dfrac{dm_1}{d\theta} = m_2$. Therefore, we shall study in the following cases.

Case 1 $m_1 = 0$. Moreover, let us suppose that $c_4 \neq 0$.

In this case $(3.5)_1$ deduce other components, respectively

$$m_2 = f(\theta) = c_4\int_0^\theta \rho\tau d\theta \tag{3.9}$$

and

$$m_3 = -\int_0^\theta (\rho + \rho^*)\rho\tau d\theta \tag{3.10}$$

If $c_4 = 0$, we have $f(\theta) = c =$constant. By this way, we know

$$\begin{aligned} &m_2 = c \\ &m_3 = -c\int_0^\theta \rho\tau d\theta \\ &\rho + \rho^* - c = 0 \end{aligned} \tag{3.11}$$

Case 2 $\dfrac{dm_1}{d\theta} = m_2$.

In this case, from (3.4), we know $f(\theta) = m_2 = 0$. And first let us suppose that $c_4 \neq 0$. Thus the equation $(3.5)_1$ has the form

$$\frac{d^2 m_1}{d\theta^2} + m_1 = c_4\rho\tau \tag{3.12}$$

By the method of variation of parameters, the solution of (3.12) yields that

$$m_1 = \cos\theta\left[-\int_0^\theta c_4\rho\tau\sin\theta d\theta + A\right] + \sin\theta\left[\int_0^\theta c_4\rho\tau\cos\theta d\theta + B\right] \tag{3.13}$$

where A, B real numbers. From $(3.4)_3$ and $(3.4)_4$ we get

$$m_3 = c_3 \tag{3.14}$$

and

$$m_4 = c_4 \tag{3.15}$$

And if $c_4 = 0$, we write that

$$\frac{d^2 m_1}{d\theta^2} + m_1 = 0 \tag{3.16}$$

We write the solution of (3.16) as

$$m_1 = l_1\cos\theta + l_2\sin\theta \tag{3.17}$$

Considering (3.4), we have other components

$$m_2 = -l_1\sin\theta + l_2\cos\theta \tag{3.18}$$

and

$$m_3 = \int_0^\theta (-l_1\sin\theta + l_2\cos\theta)\rho\tau d\theta. \tag{3.19}$$

§4. The Inclined Partially Null Curves In E_1^4

Theorem 4.1 *Let $\alpha = \alpha(s)$ be a unit speed partially null curve in E_1^4. α is an inclined curve, if and only if*

$$\frac{\kappa}{\tau} = \text{constant} \tag{4.1}$$

Proof Let $\alpha = \alpha(s)$ be a unit speed partially null curve is E_1^4 and also be an inclined curve from definition of inclined curves, we write that

$$g(\overrightarrow{T}, \overrightarrow{u}) = \cos\Psi \tag{4.2}$$

where $\overrightarrow{u}$ is a constant space-like vector and Ψ is a constant angle. Differentiating (4.2) respect s, we have

$$\kappa g(\overrightarrow{N}, \overrightarrow{u}) = 0 \tag{4.3}$$

which implies that $\overrightarrow{N} \perp \overrightarrow{u}$. And therefore we compose constant vector $\overrightarrow{u}$ as

$$\overrightarrow{u} = u_1\overrightarrow{T} + u_2\overrightarrow{B}_1 + u_3\overrightarrow{B}_2 \tag{4.4}$$

Differentiating (4.4) and considering Frenet equations we have following equation system:

$$\begin{aligned} &\frac{du_1}{ds} = 0 \\ &\frac{du_2}{ds} = 0 \\ &\frac{du_3}{ds} = 0 \\ &u_1\kappa - u_3\tau = 0 \end{aligned} \tag{4.5}$$

Solution of (4.5) yields that

$$\frac{\kappa}{\tau} = \text{constant} \tag{4.6}$$

Conversely, let us consider a vector given by

$$\overrightarrow{u} = \left\{\overrightarrow{T} + \overrightarrow{B}_1 + \frac{\kappa}{\tau}\overrightarrow{B}_2\right\}\cos\Psi \tag{4.7}$$

Where Ψ is a constant angle. Differentiating (4.7), we have

$$\frac{d\overrightarrow{u}}{ds} = 0 \tag{4.8}$$

(4.8) implies that $\overrightarrow{u}$ is a constant vector. And then considering a partially null curve $\alpha = \alpha(s)$; using inner product, we get

$$g(\overrightarrow{T}, \overrightarrow{u}) = \cos\Psi, \tag{4.9}$$

which shows that α is a inclined curve in E_1^4.

In the same space, S.Yılmaz gave a formulation about inclined curves with following theorem in [12]:

Let $\alpha = \alpha(s)$ be a space-like curve in E_1^4 parametrized by arclength. The curve α is an inclined curve if and only if

$$\frac{\kappa}{\tau} = A\cosh\left(\int_0^s \sigma ds\right) + B\sinh\left(\int_0^s \sigma ds\right) \tag{4.10}$$

where $\tau \neq 0$ and $\sigma \neq 0$, $A, B \in \mathbb{R}$.

Whence, we know that α is partially null curve, so $\sigma = 0$. Using (4.10) we have

$$\frac{\kappa}{\tau} = \text{constant.} \tag{4.11}$$

This completes the proof. □

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