International J.Math. Combin. Vol.1(2014), 102-108

Bounds for the Harmonious Coloring of Myceilskians

Vernold Vivin.J

Department of Mathematics, University College of Engineering Nagercoil, Anna University, Tirunelveli Region
Nagercoil - 629 004, Tamil Nadu, India

E-mail: vernoldvivin@yahoo.in

Abstract: In this paper, we find the harmonious chromatic number on Mycielskian graph of cycle, path, complete graph and complete bipartite graph.

Key Words: Harmonious coloring, harmonious chromatic number, Myceilskian graph.

AMS(2010): 05C15

§1. Introduction

The first paper on harmonious graph coloring was published in 1982 by Frank Harary and M.J.Plantholt [2]. However, the proper definition of this notion is due to J.E.Hopcroft and M.S. Krishnamoorthy [5] in 1983. It was shown by Hopcroft and Krishnamoorthy that the problem of determining the harmonious chromatic number of a graph is NP-hard.

A harmonious coloring [1, 2, 3, 5, 6, 9] of a simple graph G is proper vertex coloring such that each pair of colors appears together on at most one edge. The harmonious chromatic number $\chi_H(G)$ is the least number of colors in such a coloring.

The concept of harmonious coloring of graphs has been studied extensively by several authors; see [8, 11] for surveys. If G has m edges and G has a harmonious coloring with k colors, then clearly, $\binom{k}{2} \geq m$. Let $k(G)$ be the smallest integer satisfying the inequality. This number can be expressed as a function of m, namely

$$k(G) = \left\lceil \frac{1+\sqrt{8m+1}}{2} \right\rceil .$$

Paths are among the first graphs whose harmonious chromatic numbers have been established. Let P_n denote the path of order n. The following fact has been proved [2].

If $k(P_n)$ is odd or if $k(P_n)$ is even and $n-1 = k(k-1)/2 - j, j = k/2-1, k/2, \cdots, k-2$, where $k = k(P_n)$, then $\chi_H(P_n) = k(P_n)$. Otherwise, $\chi_H(P_n) = k(P_n)+1$.

In this present paper, we find the harmonious chromatic number on Mycielskian graph of cycle, path, complete graph and complete bipartite graph.

[1]Received November 21, 2013, Accepted March 8, 2014.

§2. Mycielskian Graph

We consider only finite, loopless graphs without multiple edges. For a given graph G on the vertex set $V(G) = \{v_1, \ldots, v_n\}$, we define its Mycielskian $\mu(G)$ [4, 7, 10] as follows:

The vertex set of $\mu(G)$ is $V(\mu(G)) = \{X, Y, z\} = \{x_1, \cdots, x_n, y_1, \cdots, y_n, z\}$ for a total of $2n + 1$ vertices. As for adjacency, we put

- $x_i \sim x_j$ in $\mu(G)$ if and only if $v_i \sim v_j$ in G,
- $x_i \sim y_j$ in $\mu(G)$ if and only if $v_i \sim v_j$ in G,
- and $y_i \sim z$ in $\mu(G)$ for all $i \in \{1, 2, \cdots, n\}$.

§3. Harmonious Coloring on Myceilskian Graph of Cycles

Theorem 3.1 *Let n be a positive integer, then*

$$\chi_H(\mu(C_n)) = 2n + 1.$$

Proof For any cycle C_n with the vertex set $V(C_n) = \{v_1, \ldots, v_n\}$, we define its Mycielskian $\mu(C_n)$ as follows. The vertex set of $\mu(C_n)$ is $V(\mu(C_n)) = \{X, Y, z\} = \{x_1, \ldots, x_n, y_1, \ldots, y_n, z\}$ for a total of $2n + 1$ vertices. As for adjacency, we put

- $x_i \sim x_j$ in $\mu(C_n)$ if and only if $v_i \sim v_j$ in C_n,
- $x_i \sim y_j$ in $\mu(C_n)$ if and only if $v_i \sim v_j$ in C_n,
- and $y_i \sim z$ in $\mu(C_n)$ for all $i \in \{1, 2, \ldots, n\}$.

The number of edges in $\mu(C_n)$ is $4n$ and all the vertices z, x_i, y_i are mutually at a distance at least 2 and $deg(z) = n$, $deg(x_i) = 4$, $deg(y_i) = 3$, and so all must have distinct colors. Thus we have, $\chi_H(\mu(C_n)) \geq 2n + 1$.

Now consider the vertex set $V(\mu(C_n))$ and assign a proper harmonious coloring to $V(\mu(C_n))$ as follows:

For $(1 \leq i \leq n)$, assign the color c_{i+1} for y_i and assign the color c_1 to z. For $(1 \leq i \leq n)$, assign the color c_{n+1+i} for x_i. Therefore, $\chi_H(\mu(C_n)) \leq 2n + 1$. Hence, $\chi_H(\mu(C_n)) = 2n + 1$. □

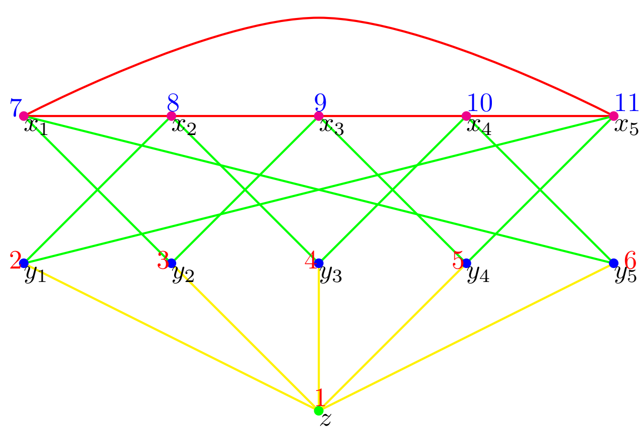

Figure 1 Mycielskian graph of C_5 with $\chi_H(\mu(C_5)) = 11$

§4. Harmonious Coloring on Myceilskian Graph of Paths

Theorem 4.1 *Let n be a positive integer, then*

$$\chi_H(\mu(P_n)) = 2n - 1, \forall n > 2.$$

Proof For any path P_n with the vertex set $V(P_n) = \{v_1, \ldots, v_n\}$, we define its Mycielskian $\mu(P_n)$ as follows. The vertex set of $\mu(P_n)$ is $V(\mu(P_n)) = \{X, Y, z\} = \{x_1, \ldots, x_n, y_1, \ldots, y_n, z\}$ for a total of $2n + 1$ vertices. As for adjacency, we put

- $x_i \sim x_j$ in $\mu(P_n)$ if and only if $v_i \sim v_j$ in P_n,
- $x_i \sim y_j$ in $\mu(P_n)$ if and only if $v_i \sim v_j$ in P_n,
- and $y_i \sim z$ in $\mu(P_n)$ for all $i \in \{1, 2, \ldots, n\}$.

The number of edges in $\mu(P_n)$ is $4n - 3$ and all the vertices z, x_i, y_i are mutually at a distance at least 2 and $deg(z) = n$, $2 \leq deg(x_i) \leq 4$, $deg(y_i) = 3$, and so all must have distinct colors. Thus we have, $\chi_H(\mu(P_n)) \geq 2n - 1, \forall n > 2$.

Now consider the vertex set $V(\mu(P_n))$ and assign a proper harmonious coloring to $V(\mu(P_n))$ as follows:

For $(1 \leq i \leq n)$, assign the color c_{i+1} for y_i and assign the color c_1 to z. For $(1 \leq i \leq n)$, assign the color c_{n+i} for x_i. Therefore, $\chi_H(\mu(P_n)) \leq 2n - 1, \forall n > 2$. Hence, $\chi_H(\mu(P_n)) = 2n - 1, \forall n > 2$. □

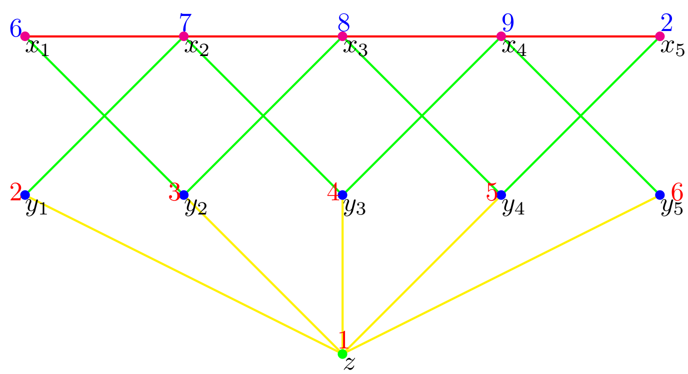

Figure 2 Mycielskian graph of P_5 with $\chi_H(\mu(P_5)) = 9$

§5. Harmonious Coloring on Myceilskian Graph of Complete Graphs

Theorem 5.1 *Let n be a positive integer, then*

$$\chi_H(\mu(K_n)) = 2n + 1 \ for \ n \neq 2.$$

Proof For any complete graph K_n with the vertex set $V(K_n) = \{v_1, \ldots, v_n\}$, we define its Mycielskian $\mu(K_n)$ as follows. The vertex set of $\mu(K_n)$ is $V(\mu(K_n)) = \{X, Y, z\} = \{x_1, \ldots, x_n, y_1, \ldots, y_n, z\}$ for a total of $2n + 1$ vertices. As for adjacency, we put

- $x_i \sim x_j$ in $\mu(K_n)$ if and only if $v_i \sim v_j$ in K_n,
- $x_i \sim y_j$ in $\mu(K_n)$ if and only if $v_i \sim v_j$ in K_n,
- and $y_i \sim z$ in $\mu(K_n)$ for all $i \in \{1, 2, \ldots, n\}$.

The number of edges in $\mu(K_n)$ is $\dfrac{3n^2 - n}{2}$ and all the vertices z, x_i, y_i are mutually at a distance at least 2 and $deg(z) = n$, $deg(x_i) = n + 1$, $deg(y_i) = 3$, and so all must have distinct colors. Thus we have, $\chi_H(\mu(K_n)) \geq 2n + 1$, for $n \neq 2$.

Now consider the vertex set $V(\mu(K_n))$ and assign a proper harmonious coloring to $V(\mu(K_n))$ as follows:

For $(1 \leq i \leq n)$, assign the color c_{i+1} for y_i and assign the color c_1 to z. For $(1 \leq i \leq n)$, assign the color c_{n+1+i} for x_i. Therefore, $\chi_H(\mu(K_n)) \leq 2n+1$, for $n \neq 2$. Hence, $\chi_H(\mu(K_n)) = 2n + 1$, for $n \neq 2$. □

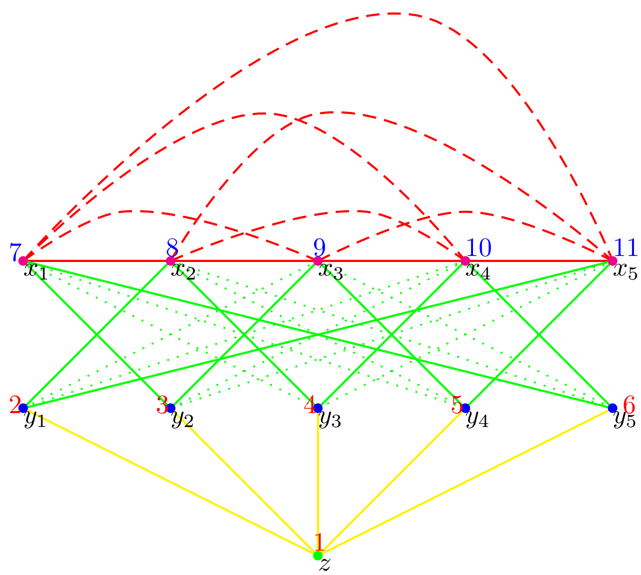

Figure 3 Mycielskian graph of K_5 with $\chi_H(\mu(K_5)) = 11$

§6. Harmonious Coloring on Myceilskian Graph of Complete Bipartite Graphs

Theorem 6.1 *Let n and m be a positive integers, then*

$$\chi_H(\mu(K_{m,n})) = 2(m + n) + 1.$$

Proof For any complete bipartite graph $K_{m,n}$ with the vertex set $V(K_{m,n}) = \{v_1, \ldots, v_n\}\cup$

$\{u_1, \cdots, u_n\}$, we define its Mycielskian $\mu(K_{m,n})$ as follows. The vertex set of $\mu(K_{m,n})$ is

$$V(\mu(K_{m,n})) = \{X, X', Y, Y', z\} = \{x_1, \ldots, x_n, x'_1, \cdots, x'_m, y_1, \ldots, y_n, y'_1, \cdots, y'_m, z\}$$

for a total of $2n + 2m + 1$ vertices. As for adjacency, we put

- $x_i \sim x_j$ in $\mu(K_{m,n})$ if and only if $v_i \sim v_j$ in $K_{m,n}$,
- $x'_i \sim x'_j$ in $\mu(K_{m,n})$ if and only if $u_i \sim u_j$ in $K_{m,n}$,
- $x_i \sim y_j$ in $\mu(K_{m,n})$ if and only if $v_i \sim v_j$ in $K_{m,n}$,
- $x'_i \sim y'_j$ in $\mu(K_{m,n})$ if and only if $u_i \sim u_j$ in $K_{m,n}$,
- and $y_i \sim z$ in $\mu(K_{m,n})$ for all $i \in \{1, 2, \ldots, n\}$.

The number of edges in $\mu(K_{m,n})$ is $m^2 + n^2 + mn + m + n$ and all the vertices z, x_i, x'_i, y_i, y'_i are mutually at a distance at least 2 and $deg(z) = n$, $deg(x_i) = 2m$,$deg(x'_i) = 2n$, $deg(y_i) = 4$,$deg(y'_i) = 4$ and so all must have distinct colors. Thus we have $\chi_H(\mu(K_{m,n})) \geq 2(m+n)+1$.

Now consider the vertex set $V(\mu(K_{m,n}))$ and assign a proper harmonious coloring to $V(\mu(K_{m,n}))$ as follows: For $(1 \leq i \leq n)$, assign the color c_{i+1} for y_i and assign the color c_1 to z. For $(1 \leq i \leq m)$, assign the color c_{n+1+i} for y'_i. For $(1 \leq i \leq m)$, assign the color $c_{n+m+1+i}$ for y'_i. For $(1 \leq i \leq n)$, assign the color $c_{2m+n+1+i}$ for x_i Therefore, $\chi_H(\mu(K_{m,n})) \leq 2(m+n)+1$. Hence, $\chi_H(\mu(K_{m,n})) = 2(m+n)+1$. □

Case 1

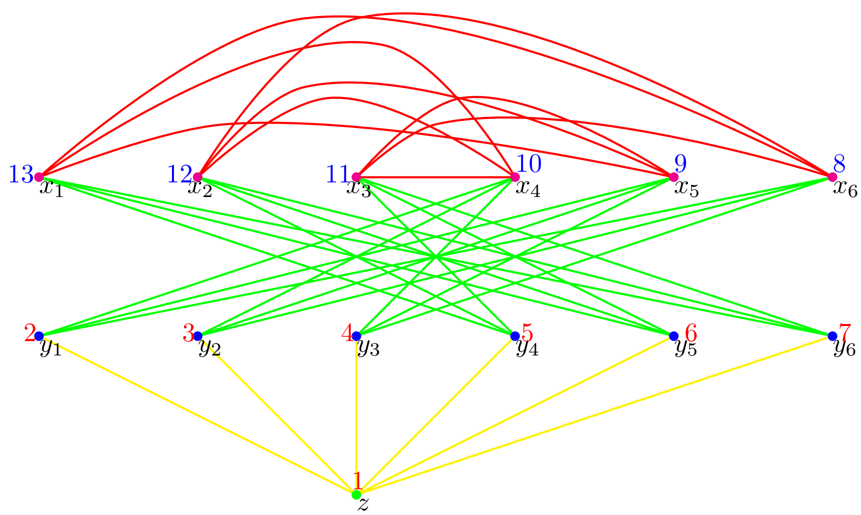

Figure 4 Mycielskian Graph of $K_{3,3}$ with $\chi_H(\mu(K_{3,3})) = 13$

Case 2

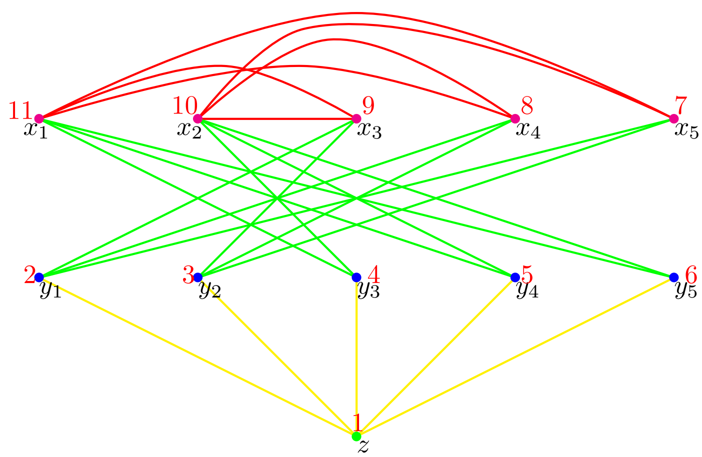

Figure 5 Mycielskian Graph of $K_{2,3}$ with $\chi_H(\mu(K_{2,3})) = 11$

§7. Main Theorem

Theorem 7.1 *Let G be any graph without pendant vertices, then*

$$\chi_H(\mu(G)) = 2(V(\mu(G))) + 1.$$

References

[1] A.Aflaki, S.Akbari, K.J.Edwards, D.S.Eskandani, M.Jamaali and H. Ravanbod, On harmonious colouring of trees, *The Electronic Journal of Combinatorics* 19 (2012), #P3

[2] O.Frank, F.Harary, M.Plantholt, The line distinguishing chromatic number of a graph, *Ars Combin.*, 14(1982), 241–252.

[3] F.Harary, M. Plantholt, Graphs with the line distinguishing chromatic number equal to the usual one, *Utilitas Math.*, 23(1983), 201–207.

[4] Gerard J. Chang, Lingling Huang, Xuding Zhu, Circular chromatic numbers of Mycielski's graphs, *Discrete Mathematics* 205 (1999), 23–37

[5] J.Hopcroft, M.S.Krishnamoorthy, On the harmonious colouring of graphs, *SIAM J. Algebra Discrete Math.*, 4(1983), 306-311.

[6] Jensen, Tommy R, Toft, Bjarne, *Graph Coloring Problems*, New York, Wiley-Interscience 1995.

[7] Jozef Miškuf, Riste Škrekovski, Martin Tancer, Backbone colorings and generalized Mycielski graphs, *SIAM Journal of Discrete Mathematics* 23(2), (2009), 1063-1070.

[8] K.J.Edwards, The harmonious chromatic number and the achromatic number, In: R.A.Bailey, ed., *Surveys in Combinatorics 1997* (Invited papers for 16th British Combinatorial Conference) (Cambridge University Press, Cambridge, 1997) 13-47.

[9] Marek Kubale, *Graph Colourings*, American Mathematical Society Providence, Rhode Island-2004.

[10] J.Mycielski, Sur le coloriage des graphes, *Colloq. Math.* 3 (1955), 161–162.

[11] B.Wilson, *Line Distinguishing and Harmonious Colourings, Graph Colouring*, (eds.R. Nelson and R. J. Wilson) Pitman Research Notes in Mathematics 218, Longman Scientific and Technical, Essex (1990) 115-133.