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The Skew Energy of Cayley Digraphs of Cyclic Groups and Dihedral Groups

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Abstract: This paper is motivated by the skew energy of a digraph as vitiated by C.Adiga, R.Balakrishnan and Wasin So [1]. We introduce and investigate the skew energy of a Cayley digraphs of cyclic groups and dihedral groups and establish sharp upper bound for the same.

Key Words: Cayley graph, dihedral graph, skew energy.

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§1. Introduction

Let G be a non trivial finite group and S be an non-empty subset of G such that for $x \in S, x^{-1} \notin S$ and $I_G \notin S$, then the Cayley digraph $\Gamma = Cay(G,S)$ of G with respect to S is defined as a simple directed graph with vertex set G and arc set $E(\Gamma) = \{(g,h)|hg^{-1} \in S\}$. If S is inverse closed and doesn't contain identity then $Cay(G,S)$ is viewed as undirected graph and is simply the Cayley graph of G with respect to S. It easily follows that valency of $Cay(G,S)$ is $|S|$ and $Cay(G,S)$ is connected if and only if $\langle S\rangle = G$. For an elaborate literature on Cayley graphs one may refer [5]. A dihedral group D_{2n} is a group with $2n$ elements such that it contains an element $'a'$ of order 2 and an element $'b'$ of order n with $a^{-1}ba = b^{-1}$. Thus $D_{2n} = \langle a,b|a^2 = b^n = 1, a^{-1}ba = b^{-1}\rangle = \langle a,b|a^2 = b^n = 1, a^{-1}ba = b^{\alpha}, \alpha \not\equiv 1(mod \quad n), \alpha^2 \equiv 1(mod \quad n)\rangle$.

If $n = 2$, then D_4 is Abelian; for $n \geq 3$, D_{2n} is not abelian. The elements of diheral group can be explicitly listed as

$$D_{2n} = \{1, a, ab, ab^2, \cdots, ab^{n-1}, b, b^2, \cdots, b^{n-1}\}.$$

In short, its elements can be listed as a^ib^k where $i = 0,1$ and $k = 0,1,\cdots,(n-1)$. It is easy to explicitly describe the product of any two elements $a^ib^ka^jb^l = a^rb^s$ as follows:

1. If $j = 0$ then $r = i$ and s equals the remainder of $k + l$ modulo n.
2. If $j = 1$, then r is the remainder of $i + j$ modulo 2 and s is the remainder of $k\alpha + l$ modulo n.

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The orders of the elements in the Dihedral group D_{2n} are: $o(1) = 1$, $o(ab^i) = 2$, where; $0 \leq i \leq n-1$, $o(b^i) = n$, where; $0 < i \leq n-1$ and if n is even than $o(b^{\frac{n}{2}}) = 2$.

Let Γ be a digraph of order n with vertex set $V(\Gamma) = \{v_1, \cdots, v_n\}$,and arc set $\Lambda(\Gamma) \subset V(\Gamma) \times V(\Gamma)$. We assume that Γ does not have loops and multiple arcs, i.e.,$(v_i, v_i) \notin \Lambda(\Gamma)$ for all i , and $(v_i, v_j) \in \Lambda(\Gamma)$ implies that $(v_j, v_i) \notin \Lambda(\Gamma)$. Hence the underlying undirected graph G_Γ of Γ is a simple graph. The skew-adjacency matrix of Γ is the $n \times n$ matrix $S(\Gamma) = [s_{ij}]$, where $s_{ij} = 1$ whenever $(v_i, v_j) \in \Lambda(\Gamma)$,$s_{ij} = -1$ whenever $(v_j, v_i) \in \Lambda(\Gamma)$, and $s_{ij} = 0$ otherwise. Because of the assumptions on Γ, $S(\Gamma)$ is indeed a skew-symmetric matrix. Hence the eigenvalues $\{\lambda_1, \cdots, \lambda_n\}$ of $S(\Gamma)$ are all purely imaginary numbers, and the singular values of $S(\Gamma)$ coincide with the absolute values $\{|\lambda_1|, \cdots, |\lambda_n|\}$, of its eigenvalues. Consequently, the energy of $S(\Gamma)$, which is defined as the sum of its singular values [6], is also the sum of the absolute values of its eigenvalues. For the sake of convenience, we simply refer the energy of $S(\Gamma)$ as the skew energy of the digraph Γ. If we denote the skew energy of Γ by $\varepsilon_s(\Gamma)$ then, $\varepsilon_s(\Gamma) = \sum_{i=1}^{n} |\lambda_i|$.

The degree of a vertex in a digraph Γ is the degree of the corresponding vertex of the underlying graph of Γ. Let $D(\Gamma) = diag(d_1, d_2, \cdots, d_n)$, the diagonal matrix with vertex degrees $d_1, d_2, \cdots, d_n$ of $v_1, v_2, \cdots, v_n$ and $S(\Gamma)$ be the skew adjacency matrix of a simple digraph Γ, possessing n vertices and m edges. Then $L(\Gamma) = D(\Gamma) - S(\Gamma)$ is called the Laplacian matrix of the digraph Γ. If $\lambda_i, i = 1, 2, \cdots, n$ are the eigenvalues of the Laplacian matrix $L(\Gamma)$ then the skew Laplacian energy of the digraph Γ is defined as $SLE(\Gamma) = \sum_{i=1}^{n} |\lambda_i - \frac{2m}{n}|$.

An $n \times n$ matrix S is said to be a circulant matrix if its entries satisfy $s_{ij} = s_{1,j-i+1}$, where the subscripts are reduced modulo n and lie in the set $\{1, 2, ..., n\}$. In other words, ith row of S is obtained from the first row of S by a cyclic shift of $i-1$ steps, and so any circulant matrix is determined by its first row. It is easy to see that the eigenvalues of S are $\lambda_k = \sum_{j=1}^{n} s_{1j}\omega^{(j-1)k}$, $k = 0, 1, \cdots, n-1$. For any positive integer n, let $\tau_n = \{\omega^k : 0 \leq k < n\}$ be the set of all nth roots of unity, where $\omega = e^{\frac{2\pi i}{n}} = \cos(\frac{2\pi}{n}) + i\sin(\frac{2\pi}{n})$ that $i^2 = -1$. τ_n is an abelian group with respect to multiplication. A circulant graph is a graph Γ whose adjacency matrix $A(\Gamma)$ is a circulant matrix. More details about circulant graphs can be found in [3].

Ever since the concept of the energy of simple undirected graphs was introduced by Gutman in [7], there has been a constant stream of papers devoted to this topic. In [1], Adiga, et al. have studied the skew energy of digraphs. In [4], Gui-Xian Tian, gave the skew energy of orientations of hypercubes. In this paper we introduce and investigate the skew energy of a Cayley digraphs of cyclic groups and dihedral groups and establish sharp upper bound for the same.

§2. Main Results

First we present some facts that are needed to prove our main results.

Lemma 2.1([2]) *Let Γ is disconnected graph into the λ components $\Gamma_1, \Gamma_2, \cdots, \Gamma_\lambda$, then*

$$Spec(\Gamma) = \bigcup_{i=1}^{\lambda} Spec(\Gamma_i).$$

Lemma 2.2 *Let $\omega = e^{\frac{2k\pi i}{n}} = \cos(\frac{2k\pi}{n}) + i\sin(\frac{2k\pi}{n})$ for $1 \le k \le n$, where n is a positive integer and $i^2 = -1$. Then*

(i) $\omega^t + \omega^{n-t} = 2\cos(\frac{2kt\pi}{n})$ *for* $1 \le k \le n$,

(ii) $\omega^t - \omega^{n-t} = 2i\sin(\frac{2kt\pi}{n})$ *for* $1 \le k \le n$.

Lemma 2.3([1]) *Let n be a positive integer. Then*

$$(i)\quad \sum_{k=1}^{\frac{n-1}{2}} \sin\frac{2k\pi}{n} = \frac{1}{2}cot\frac{\pi}{2n},\ n \equiv 1(mod2),$$

$$(ii)\quad \sum_{k=1}^{\frac{n-2}{2}} \sin\frac{2k\pi}{n} = \cot\frac{\pi}{n},\ n \equiv 0(mod2),$$

$$(iii)\quad \sum_{k=1}^{\frac{n-1}{2}} |\cos\frac{2k\pi}{n}| = \frac{1}{2}\csc\frac{\pi}{2n} - \frac{1}{2},\ n \equiv 1(mod2),$$

$$(iv)\quad \sum_{k=0}^{n-1} |\cos\frac{2k\pi}{n}| = 2\cot\frac{\pi}{n},\ n \equiv 0(mod4),$$

$$(iv)\quad \sum_{k=0}^{n-1} |\cos\frac{2k\pi}{n}| = 2\csc\frac{\pi}{n},\ n \equiv 2(mod4),$$

$$(vi)\quad \sum_{k=1}^{\frac{n}{2}} \sin\frac{(2k-1)\pi}{n} = \csc\frac{\pi}{n},\ n \equiv 0(mod2),$$

$$(vii)\quad \sum_{k=1}^{n-1} \sin\frac{k\pi}{n} = \cot\frac{\pi}{2n},\ n \equiv 1(mod2),$$

$$(viii)\quad \sum_{k=1}^{n-1} |\cos\frac{2k\pi}{n}| = \csc\frac{\pi}{2n} - 1,\ n \equiv 1(mod2).$$

Lemma 2.4 *Let n be a positive integer. Then*

$$(i)\quad \sum_{k=1}^{\frac{n-2}{2}} |\cos\frac{2k\pi}{n}| = \cot\frac{\pi}{n} - 1,\ n \equiv 0(mod4),$$

$$(ii)\quad \sum_{k=1}^{\frac{n-2}{2}} |\cos\frac{2k\pi}{n}| = \csc\frac{\pi}{n} - 1,\ n \equiv 2(mod4).$$

Proof The proof of (i) follows directly from Lemma 2.3(iv), and (ii) is a consequence of Lemma 2.3(v). $\square$

Lemma 2.5 *Let n be a positive integer. Then*

$$(i)\ \sum_{k=0}^{\frac{n-1}{2}} \sin\frac{2k\pi}{n} = \sum_{k=0}^{\frac{n-1}{2}} |\sin\frac{4k\pi}{n}|,\ n \equiv 1(mod2),$$

$$(ii)\ \sum_{k=0}^{\frac{n-2}{2}} \sin\frac{2k\pi}{n} = \sum_{k=0}^{\frac{n-2}{2}} |\sin\frac{4k\pi}{n}|,\ n \equiv 2(mod4),$$

$$(iii)\ \sum_{k=1}^{\frac{n-2}{2}} \sin\frac{2k\pi}{n} = 1 + 2\sum_{k=1}^{\frac{n-4}{4}} \sin\frac{2k\pi}{n},\ n \equiv 0(mod4),$$

$$(iv)\ \sum_{k=1}^{\frac{n-2}{2}} |\sin\frac{4k\pi}{n}| = 2\sum_{k=1}^{\frac{n-4}{4}} |\sin\frac{4k\pi}{n}|,\ n \equiv 0(mod4),$$

$$(v)\ \sum_{k=1}^{\frac{n-4}{4}} \sin\frac{4k\pi}{n} = \csc\frac{4\pi}{n} + \cot\frac{4\pi}{n},\ n \equiv 0(mod8),$$

$$(vi)\ \sum_{k=1}^{\frac{n-4}{4}} \sin\frac{4k\pi}{n} = \cot\frac{2\pi}{n},\ n \equiv 4(mod8).$$

Proof (i) Let $n \equiv 1(mod2)$, $f(k) = \sin\dfrac{2k\pi}{n}$ and $g(k) = \sin\dfrac{4k\pi}{n}$, where $k \in \{0, 1, 2, \cdots, \dfrac{n-1}{2}\}$. Then it is easy to check that

$$g(k) = \begin{cases} f(2k) & \text{if} \quad 0 \le k \le \lfloor\frac{n-1}{4}\rfloor, \\ -f(n-2k) & \text{if} \quad \lfloor\frac{n-1}{4}\rfloor < k \le \frac{n-1}{2}. \end{cases}$$

This implies (i).

(ii) Let $n \equiv 2(mod4)$, $f(k) = \sin\dfrac{2k\pi}{n}$ and $g(k) = \sin\dfrac{4k\pi}{n}$, where $k \in \{0, 1, 2, \cdots, \dfrac{n-2}{2}\}$. Then it follows that

$$g(k) = \begin{cases} f(2k) & \text{if} \quad 1 \le k \le \frac{n-2}{4}, \\ -f(-\frac{n}{2}+2k) & \text{if} \quad \frac{n-2}{4} < k \le \frac{n-2}{2}. \end{cases}$$

This implies (ii). Proofs of (iii) and (iv) are similar to that of (i) and (ii).

(v) Let $n \equiv 0(mod8)$. Then $n = 8m$, $m \in \mathbb{N}$ and

$$\begin{aligned}\sum_{k=1}^{\frac{n-4}{4}} \sin\frac{4k\pi}{n} &= \sum_{k=1}^{2m-1} \sin\frac{k\pi}{2m} = \left(\sin\frac{\pi}{2m} + \sin\frac{3\pi}{2m} + \cdots + \sin\frac{(2m-1)\pi}{2m}\right)\\ &\quad + \left(\sin\frac{2\pi}{2m} + sin\frac{4\pi}{2m} + \cdots + \sin\frac{(2m-2)\pi}{2m}\right)\\ &= \csc\frac{\pi}{2m} + \cot\frac{\pi}{2m}\\ &= \csc\frac{4\pi}{n} + \cot\frac{4\pi}{n} \quad \text{(Using Lemma2.3}(vi) \text{ and } (ii)).\end{aligned}$$

(vi) Let $n \equiv 4(mod8)$. Then $n = 8m + 4$, $m \in \mathbb{N}$ and

$$\begin{aligned}\sum_{k=1}^{\frac{n-4}{4}} \sin\frac{4k\pi}{n} &= \sum_{k=1}^{2m} \sin\frac{k\pi}{2m+1}\\ &= \cot\frac{\pi}{2(2m+1)} = \cot\frac{2\pi}{n}. \quad \text{(using Lemma 2.3}(vii)) \qquad \square\end{aligned}$$

Lemma 2.6 *Let n be a positive integer. Then*

(i) $\displaystyle\sum_{k=1}^{n-1} |\sin\frac{2k\pi}{n}| = \cot\frac{\pi}{2n},\ n \equiv 1(mod2),$

(ii) $\displaystyle\sum_{k=1}^{n-1} |\sin\frac{2k\pi}{n}| = 2\cot\frac{\pi}{n},\ n \equiv 0(mod2).$

Now we compute skew energy of some Cayley digraphs.

Theorem 2.7 *Let $G = \{v_1 = e, v_2, \cdots, v_n\}$ be a group, $S = \{v_i\} \subset G$ with $v_i \neq v_i^{-1}$, $v_i \neq e$ and $\Gamma = Cay(G, S)$ be a Cayley digraph on G with respect to S. Suppose $H = \langle S\rangle$, $|H| = m$, $|G : H| = \lambda$. Then*

$$\varepsilon_s(\Gamma) = \begin{cases} 2\lambda cot\frac{\pi}{2m} & if \quad m \equiv 1(mod2),\\ 4\lambda cot\frac{\pi}{m} & if \quad m \equiv 0(mod2).\end{cases}$$

Proof Let $G = \{v_1 = e, v_2, v_3, \cdots, v_n\}$, $S = \{v_i\}$, $v_i \in G$ with $v_i \neq v_i^{-1}$, $v_i \neq e$ and suppose $H = \langle S\rangle$, $|H| = m$, $|G : H| = \lambda$. If $\lambda = 1$ then $G = \{e, v_i, v_i^2, \cdots, v_i^{n-1}\}$ and hence the the skew-adjacency matrix of $\Gamma = Cay(G, S)$ is a circulant matrix. Its first row is $[0, 1, 0, \cdots, 0, -1]$. So all eigenvalues of Γ are $\lambda_k = \omega^k - \omega^{kn-k} = \omega^k - \omega^{-k} = 2isin\frac{2k\pi}{n}$, $k = 0, 1, \cdots, n-1$ where $\omega = e^{\frac{2\pi i}{n}}$ and $i^2 = -1$. Applying Lemma 2.2(ii), we obtain $\lambda_k = 2isin\frac{2k\pi}{n}$, $k = 0, 1, \cdots, n-1$. Now by Lemma 2.6 we have

$$\varepsilon_s(\Gamma) = \sum_{k=0}^{n-1} |2isin\frac{2k\pi}{n}| = 2\sum_{k=1}^{n-1} |sin\frac{2k\pi}{n}| = \begin{cases} 2cot\frac{\pi}{2n} & \text{if} \quad n \equiv 1(mod2),\\ 4cot\frac{\pi}{n} & \text{if} \quad n \equiv 0(mod2).\end{cases}$$

If $\lambda > 1$, then Γ is disconnected graph in to the $\Gamma_i, i = 1, \cdots, \lambda$ components and all components are isomorphic with Cayley digraph $\Gamma_m = Cay(H, S)$ where $H = \langle v_i :\ v_i^m = 1\rangle$

and $m|n$, $S=\{v_i\}$. Since Γ is not connected, by Lemma 2.1, its energy is the sum of the energies of its connected components. Thus

$$\varepsilon_s(\Gamma)=\sum_{i=1}^{\lambda}\varepsilon_s(\Gamma_i)=\lambda\varepsilon_s(Cay(H,S))=\begin{cases}2\lambda cot\frac{\pi}{2m} & \text{if} \quad m\equiv 1(mod2),\\ 4\lambda cot\frac{\pi}{m} & \text{if} \quad m\equiv 0(mod2).\end{cases}$$

This completes the proof. □

Theorem 2.8 *Let $G=\{e,b,b^2,\cdots,b^{n-1}\}$ be a cyclic group of order n, and $\Gamma=Cay(G,S)$ be a Cayley digraph on G with respect to $S=\{b^i,b^j\}, 0<i,j\le n-1, i\ne j$ $H=\langle S\rangle$, $|H|=m$, and $|G:H|=\lambda$. Then*

(i) $\varepsilon_s(\Gamma)\le 4\lambda\cot\frac{\pi}{2m}$ *if* $m\equiv 1(mod2)$,

(ii) $\varepsilon_s(\Gamma)\le 8\lambda\cot\frac{\pi}{m}$ *if* $m\equiv 2(mod4)$,

(iii) $\varepsilon_s(\Gamma)\le 4\lambda(\cot\frac{\pi}{m}+2\csc\frac{4\pi}{m}+2\cot\frac{4\pi}{m})$ *if* $m\equiv 0(mod8)$,

(iv) $\varepsilon_s(\Gamma)\le 4\lambda(\cot\frac{\pi}{m}+2\cot\frac{2\pi}{m}))$ *if* $m\equiv 4(mod8)$.

Proof Let $G=\{e,b,b^2,\cdots,b^{n-1}\}$ be a cyclic group of order n and $\Gamma=Cay(G,S)$ be a Cayley digraph on G with respect to $S=\{b^i,b^j\}, 0<i,j\le n-1, i\ne j$, $H=\langle S\rangle$, $|H|=m$, and $|G:H|=\lambda$. If $\lambda=1$, then $G=H$ and hence the the skew-adjacency matrix of $\Gamma=Cay(G,S)$ is a circulant matrix. So all eigenvalues of Γ are$\lambda_k=\omega^k-\omega^{-k}+\omega^{2k}-\omega^{-2k}$, $k=0,1,\cdots,n-1$, where $\omega=e^{\frac{2\pi i}{n}}$ and $i^2=-1$. Hence

$$\lambda_k=\omega^k-\omega^{-k}+\omega^{2k}-\omega^{-2k}=2isin\frac{2k\pi}{n}+2isin\frac{4k\pi}{n}=2i(sin\frac{2k\pi}{n}+sin\frac{4k\pi}{n})$$

for $k=0,1,\cdots,n-1$.

(i) Suppose $n\equiv 1(mod2)$. Then

$$\begin{aligned}\varepsilon_s(\Gamma) &= \sum_{k=0}^{n-1}|\lambda_k|=\sum_{k=0}^{n-1}|2i(sin\frac{2k\pi}{n}+sin\frac{4k\pi}{n})|\\ &= \sum_{k=1}^{n-1}|2i(sin\frac{2k\pi}{n}+sin\frac{4k\pi}{n})|\\ &= 4\sum_{k=1}^{\frac{n-1}{2}}|sin\frac{2k\pi}{n}+sin\frac{4k\pi}{n}|\\ &\le 4(\sum_{k=1}^{\frac{n-1}{2}}|sin\frac{2k\pi}{n}|+\sum_{k=1}^{\frac{n-1}{2}}|sin\frac{4k\pi}{n}|)\\ &= 4(\sum_{k=1}^{\frac{n-1}{2}}sin\frac{2k\pi}{n}+\sum_{k=1}^{\frac{n-1}{2}}sin\frac{2k\pi}{n}) \quad \text{(using Lemma 2.5}(i))\\ &= 4cot\frac{\pi}{2n} \quad \text{(applying Lemma 2.3}(i)).\end{aligned}$$

Thus (i) holds for $\lambda = 1$.

Suppose $\lambda > 1$. Then Γ is disconnected graph in to the $\Gamma_i, i = 1, \cdots, \lambda$ components and all components are isomorphic with Cayley digraph $\Gamma_m = Cay(H, S)$ where $H = \langle v_i : v_i^m = 1\rangle$ and $m|n$, $S = \{v_i\}$. Since Γ is not connected, its energy is the sum of the energies of its connected components. Thus

$$\varepsilon_s(\Gamma) = \sum_{i=1}^{\lambda} \varepsilon_s(\Gamma_i) \lambda \varepsilon_s(Cay(H,S)) \leq 4\lambda \cot \frac{\pi}{2m}.$$

Now we shall prove (ii),(iii) and (iv) only for $\lambda = 1$. For $\lambda > 1$, proofs are similar to that of (i).

(ii) If $n \equiv 2(mod4)$, then

$$\begin{aligned}
\varepsilon_s(\Gamma) &= \sum_{k=0}^{n-1} |\lambda_k| = \sum_{k=0}^{n-1} |2i(sin\frac{2k\pi}{n} + sin\frac{4k\pi}{n})| \\
&= \sum_{k=1}^{n-1} |2i(sin\frac{2k\pi}{n} + sin\frac{4k\pi}{n})| \\
&= 4\sum_{k=1}^{\frac{n-2}{2}} |sin\frac{2k\pi}{n} + sin\frac{4k\pi}{n}| \\
&\leq 4(\sum_{k=1}^{\frac{n-2}{2}} |sin\frac{2k\pi}{n}| + \sum_{k=1}^{\frac{n-2}{2}} |sin\frac{4k\pi}{n}|) \\
&= 4(\sum_{k=1}^{\frac{n-2}{2}} sin\frac{2k\pi}{n} + \sum_{k=1}^{\frac{n-2}{2}} sin\frac{2k\pi}{n}) \quad \text{(using Lemma 2.5}(ii)) \\
&= 8cot\frac{\pi}{n}.
\end{aligned}$$

Here we used the Lemma 2.3(ii).

(iii) If $n \equiv 0(mod8)$, then

$$\begin{aligned}
\varepsilon_s(\Gamma) &= 4\sum_{k=1}^{\frac{n-2}{2}} |sin\frac{2k\pi}{n} + sin\frac{4k\pi}{n}| \\
&\leq 4(\sum_{k=1}^{\frac{n-2}{2}} |sin\frac{2k\pi}{n}| + \sum_{k=1}^{\frac{n-2}{2}} |sin\frac{4k\pi}{n}|) \\
&= 4(\sum_{k=1}^{\frac{n-2}{2}} sin\frac{2k\pi}{n} + 2\sum_{k=1}^{\frac{n-4}{4}} sin\frac{4k\pi}{n}) \quad \text{(using Lemma 2.5}(iv)) \\
&= 4cot\frac{\pi}{n} + 8csc\frac{4\pi}{n} + 8cot\frac{4\pi}{n}.
\end{aligned}$$

To get the last equality we have used Lemma 2.3(ii) and 2.5(v).

(iv) If $n \equiv 4(mod8)$, then

$$\begin{aligned}\varepsilon_s(\Gamma) &\leq 4(\sum_{k=1}^{\frac{n-2}{2}}|sin\frac{2k\pi}{n}|+\sum_{k=1}^{\frac{n-2}{2}}|sin\frac{4k\pi}{n}|)\\ &= 4(\sum_{k=1}^{\frac{n-2}{2}}sin\frac{2k\pi}{n}+2\sum_{k=1}^{\frac{n-4}{4}}sin\frac{4k\pi}{n}) \text{ (using Lemma 2.5}(iv))\\ &= 4(cot\frac{\pi}{n}+2cot\frac{2\pi}{n})).\end{aligned}$$

To get the last equality we have used Lemma 2.3(ii) and 2.5(vi). □

Lemma 2.9 *Let $G = \langle b : b^n = 1\rangle$ be a cyclic group and $\Gamma = Cay(G, S_t)$, $t \in \{1, \cdots, \lfloor\frac{n-1}{2}\rfloor\}$,be a Cayley digraph on G with respect to $S_t = \{b^l, b^{2l}, \cdots, b^{tl}\}$, where $l \in U(n) = \{r : 1 \leq r < n, gcd(n, r) = 1\}$. Then the eigenvalues of Γ are*

$$\lambda_k = \sum_{j=0}^{|S_t|} 2isin\frac{2kj\pi}{n}, \quad k = 0, 1, \cdots, n-1,$$

where $i^2 = -1$.

Proof The proof directly follows from the definition of cyclic group and is similar to that of Theorem 2.7. □

Lemma 2.10 *Let $G = \langle b : b^n = 1\rangle$ be a cyclic group and $\Gamma = Cay(G, S_t)$, $t \in \{1, \cdots, \lfloor\frac{n-1}{2}\rfloor\}$,be a Cayley digraph on G with respect to $S_t = \{b^l, b^{2l}, \cdots, b^{tl}\}$ where $l \in U(n) = \{r : 1 \leq r < n, gcd(n, r) = 1\}$. Also suppose $\varepsilon_s(\Gamma), SLE(\Gamma)$ denote the skew energy and the skew Laplacian energy of Γ respectively. Then $\varepsilon_s(\Gamma) = SLE(\Gamma)$.*

Proof The proof directly follows from the definition of the skew energy and the skew Laplacian energy. □

Lemma 2.11 *Let n be a positive integer. Then*

$$(i)\ \sum_{k=1}^{\frac{n-1}{2}}\cos\frac{4k\pi}{n} = \sum_{k=1}^{\frac{n-1}{2}}\cos\frac{2k\pi}{n},\ n \equiv 1(mod2),$$

$$(ii)\ \sum_{k=1}^{\frac{n-2}{2}}|\cos\frac{4k\pi}{n}| = \csc\frac{\pi}{n} - 1,\ n \equiv 2(mod4),$$

$$(iii)\ \sum_{k=1}^{\frac{n-2}{2}}\cos\frac{4k\pi}{n} = -1,\ n \equiv 0(mod4).$$

Proof (i) Let $n \equiv 1(mod2)$, $f(k) = \cos\frac{2k\pi}{n}$, $g(k) = \cos\frac{4k\pi}{n}$, where $k \in \{1, 2, \cdots, \frac{n-1}{2}\}$.

It is easy to verify that

$$g(k) = \begin{cases} f(2k) & \text{if} \quad 1 \le k \le \lfloor \frac{n-1}{4} \rfloor, \\ f(n-2k) & \text{if} \quad \lfloor \frac{n-1}{4} \rfloor < k \le \frac{n-1}{2}. \end{cases}$$

This implies (i).

(ii) Let $n \equiv 2(mod 4)$. Then $n = 4m+2$ for some $m \in \mathbb{N}$. We have

$$\begin{aligned} \sum_{k=1}^{\frac{n-2}{2}} |\cos\frac{4k\pi}{n}| &= \sum_{k=1}^{\frac{n-2}{2}} |\cos\frac{4k\pi}{n}| = \sum_{k=1}^{2m} |\cos\frac{4k\pi}{4m+2}| \\ &= \sum_{k=1}^{2m} |\cos\frac{2k\pi}{2m+1}| = \csc\frac{\pi}{2(2m+1)} - 1 \quad \text{(using Lemma 2.3}(viii)) \\ &= \csc\frac{\pi}{n} - 1. \end{aligned}$$

(iii) Suppose $n = 4m, m \in \mathbb{N}$. Then

$$\sum_{k=1}^{\frac{n-2}{2}} \cos\frac{4k\pi}{n} = \sum_{k=1}^{2m-1} cos\frac{k\pi}{m} = \cos\frac{m\pi}{m} + \sum_{k=1}^{m-1} cos\frac{k\pi}{m} + \sum_{k=m+1}^{2m-1} cos\frac{k\pi}{m}.$$

Changing k to $k+m$ in the last summation we get $\sum_{k=1}^{\frac{n-2}{2}} \cos\frac{4k\pi}{n} = -1.$ □

Theorem 2.12 *Let $G = \langle b : b^n = 1\rangle$ be a cyclic group and $\Gamma = Cay(G, S)$,be a Cayley digraph on G with respect to $S = \{b^l\}$ where $l \in U(n) = \{r : 1 \le r < n, gcd(n, r) = 1\}$ and $C_s(\Gamma)$ be the skew-adjacency matrix of Γ , $D(\Gamma) = diag(d_1, d_2, \cdots, d_n)$, the diagonal matrix with vertex degrees $d_1, d_2, \cdots, d_n$ of $e, b, b^2, \cdots, b^{n-1}$. Suppose $L(\Gamma) = D(\Gamma) - C_s(\Gamma)$ and $\mu_1, \cdots, \mu_n$ are eigenvalues of $L(\Gamma)$. We define $\alpha(\Gamma) = \sum_{i=1}^{n} \mu_i^2$. Then*

(i) $\alpha(\Gamma) \le 2n + 2csc\frac{\pi}{2n}$ *if* $n \equiv 1(mod2)$,

(ii) $\alpha(\Gamma) \le 2(n-1) + 4\csc\frac{\pi}{n}$ *if* $n \equiv 2(mod4)$,

(iii) $\alpha(\Gamma) = 2(n-2)$ *if* $n \equiv 4(mod0)$.

Proof Let $G = \langle b : b^n = 1\rangle$ be a cyclic group and $\Gamma = Cay(G, S)$,be a Cayley digraph on G with respect to $S = \{b^l\}$ where $l \in U(n) = \{r : 1 \le r < n, gcd(n, r) = 1\}$ and $C_s(\Gamma)$ be the skew-adjacency matrix of Γ. Note that underlying graph of Γ is a $2-$regular graph. Hence $D(\Gamma) = diag(2, 2, \cdots, 2)$. Suppose $L(\Gamma) = D(\Gamma) - C_s(\Gamma)$ then $L(\Gamma)$ is a circulant matrix and its first row is $[2, -1, 0, \cdots, 0, 1]$. This implies that the eigenvalues of $L(\Gamma)$ are $\mu_k = 2 - \omega^k + \omega^{kn-k} = 2 - \omega^k + \omega^{-k} = 2 - (\omega^k - \omega^{-k}) = 2 - 2i\sin\frac{2k\pi}{n}$, $k = 0, 1, \cdots, n-1$ where $\omega = e^{\frac{2\pi i}{n}}$ and $i^2 = -1$.

If $n \equiv 1(mod2)$, then

$$
\begin{aligned}
\alpha(\Gamma) &= \sum_{k=0}^{n-1} \mu_k^2 \\
&= \sum_{k=0}^{n-1} (2-2i\sin\frac{2k\pi}{n})^2 = 4+\sum_{k=1}^{n-1}(2-2i\sin\frac{2k\pi}{n})^2 \\
&= 4+\sum_{k=1}^{\frac{n-1}{2}}(2-2i\sin\frac{2k\pi}{n})^2+\sum_{k=\frac{n+1}{2}}^{n-1}(2-2i\sin\frac{2k\pi}{n})^2 \\
&= 4+\sum_{k=1}^{\frac{n-1}{2}}(2-2i\sin\frac{2k\pi}{n})^2+\sum_{k=1}^{\frac{n-1}{2}}(2+2i\sin\frac{2k\pi}{n})^2 \\
&= 4+\sum_{k=1}^{\frac{n-1}{2}} 2(4-4\sin^2\frac{2k\pi}{n}) = 4+8(\frac{n-1}{2})-8\sum_{k=1}^{\frac{n-1}{2}}\sin^2\frac{2k\pi}{n} \\
&= 4+8(\frac{n-1}{2})-8\sum_{k=1}^{\frac{n-1}{2}}(\frac{1}{2}-\frac{1}{2}\cos\frac{4k\pi}{n}) = 4+4(\frac{n-1}{2})+4\sum_{k=1}^{\frac{n-1}{2}}\cos\frac{4k\pi}{n} \\
&= 4+4(\frac{n-1}{2})+4\sum_{k=1}^{\frac{n-1}{2}}\cos\frac{2k\pi}{n} \quad \text{(using Lemma 2.11}(i)\text{)} \\
&\leq 4+4(\frac{n-1}{2})+4\sum_{k=1}^{\frac{n-1}{2}}|\cos\frac{2k\pi}{n}| \\
&= 4+4(\frac{n-1}{2})+4(\frac{1}{2}\csc\frac{\pi}{2n}-\frac{1}{2}) \quad \text{(using Lemma 2.3}(iii)\text{)} \\
&= 2n+2\csc\frac{\pi}{2n}.
\end{aligned}
$$

If $n \equiv 0(mod2)$, then

$$
\begin{aligned}
\alpha(\Gamma) &= \sum_{k=0}^{n-1} \mu_k^2 \\
&= \sum_{k=0}^{n-1} (2-2i\sin\frac{2k\pi}{n})^2 = 4+\sum_{k=1}^{n-1}(2-2i\sin\frac{2k\pi}{n})^2 \\
&= 4+\sum_{k=1}^{\frac{n-2}{2}}(2-2i\sin\frac{2k\pi}{n})^2+(2-2i\sin\frac{2(\frac{n}{2})\pi}{n})^2+\sum_{k=\frac{n+2}{2}}^{n-1}(2-2i\sin\frac{2k\pi}{n})^2 \\
&= 8+\sum_{k=1}^{\frac{n-2}{2}}(2-2i\sin\frac{2k\pi}{n})^2+\sum_{k=1}^{\frac{n-2}{2}}(2+2i\sin\frac{2k\pi}{n})^2 \\
&= 8+\sum_{k=1}^{\frac{n-2}{2}} 2(4-4\sin^2\frac{2k\pi}{n}) = 4+8(\frac{n-2}{2})-8\sum_{k=1}^{\frac{n-2}{2}}\sin^2\frac{2k\pi}{n} \\
&= 4+8(\frac{n-2}{2})-8\sum_{k=1}^{\frac{n-2}{2}}(\frac{1}{2}-\frac{1}{2}\cos\frac{4k\pi}{n}) = 4+4(\frac{n-2}{2})+4\sum_{k=1}^{\frac{n-2}{2}}\cos\frac{4k\pi}{n}.
\end{aligned}
$$

If $n\equiv 2(mod4)$, then

$$\begin{aligned}\alpha(\Gamma) &\leq 4+4(\frac{n-2}{2})+4\sum_{k=1}^{\frac{n-2}{2}}|\cos\frac{4k\pi}{n}|\\ &= 4+4(\frac{n-1}{2})+4(\csc\frac{\pi}{n}-1)\ \ \text{(using Lemma2.11}(ii))\\ &= 2(n-1)+4\csc\frac{\pi}{n}.\end{aligned}$$

This completes the proof of (ii).

If $n\equiv 0(mod4)$, then

$$\begin{aligned}\alpha(\Gamma) &= 4+4(\frac{n-2}{2})+4\sum_{k=1}^{\frac{n-2}{2}}\cos\frac{4k\pi}{n}\\ &= 4+4(\frac{n-2}{2})+4(-1)\ \ \text{(using Lemma2.11}(iii))\\ &= 2(n-2).\end{aligned}$$

This completes the proof of (iii). □

Lemma 2.13 *Let n be a positive integer. Then*

(i) $\sum_{k=1}^{\frac{n-1}{2}}|\cos\frac{6k\pi}{n}|=\frac{3}{2}\csc\frac{3\pi}{2n}+\frac{1}{2}$, $n\equiv 3(mod6)$,

(ii) $\sum_{k=1}^{\frac{n-1}{2}}|\cos\frac{6k\pi}{n}|=\sum_{k=1}^{\frac{n-1}{2}}|\cos\frac{2k\pi}{n}|$, $n\equiv 1(mod6)$,

(iii) $\sum_{k=1}^{\frac{n-1}{2}}|\cos\frac{6k\pi}{n}|=\sum_{k=1}^{\frac{n-1}{2}}|\cos\frac{2k\pi}{n}|$, $n\equiv 5(mod6)$,

(iv) $\sum_{k=1}^{\frac{n-2}{2}}\cos\frac{6k\pi}{n}=0$, $n\equiv 0(mod6)$,

(v) $\sum_{k=1}^{\frac{n-2}{2}}\cos\frac{6k\pi}{n}=\sum_{k=1}^{\frac{n-2}{2}}\cos\frac{2k\pi}{n}$, $n\equiv 2(mod6)$,

(vi) $\sum_{k=1}^{\frac{n-2}{2}}\cos\frac{6k\pi}{n}=\sum_{k=1}^{\frac{n-2}{2}}\cos\frac{2k\pi}{n}$, $n\equiv 4(mod6)$,

vii) $\sum_{k=1}^{\frac{n-1}{2}}\cos\frac{8k\pi}{n}=\sum_{k=1}^{\frac{n-1}{2}}\cos\frac{4k\pi}{n}$, $n\equiv 1(mod2)$,

(vii) $\sum_{k=1}^{\frac{n-2}{2}}\cos\frac{8k\pi}{n}=\sum_{k=1}^{\frac{n-2}{2}}\cos\frac{4k\pi}{n}$, $n\equiv 2(mod4)$,

$(viii)$ $\sum_{k=1}^{\frac{n-2}{2}} |\cos\frac{8k\pi}{n}| = -1 + 2\csc\frac{2\pi}{n}$, $n \equiv 4(mod8)$,

(ix) $\sum_{k=1}^{\frac{n-2}{2}} |\cos\frac{8k\pi}{n}| = -1 + 4\cot\frac{4\pi}{n}$, $n \equiv 0(mod16)$,

(x) $\sum_{k=1}^{\frac{n-2}{2}} |\cos\frac{8k\pi}{n}| = -1 + 4\csc\frac{4\pi}{n}$, $n \equiv 8(mod16)$ and $n \equiv 2$ or $4(mod6)$.

Proof (i) Suppose $n = 6m + 3$, then

$$\begin{aligned}
\sum_{k=1}^{\frac{n-1}{2}} |\cos\frac{6k\pi}{n}| &= \sum_{k=1}^{3m+1} |\cos\frac{2k\pi}{2m+1}| \\
&= \sum_{k=1}^{2m} |\cos\frac{2k\pi}{2m+1}| + \sum_{k=2m+1}^{3m+1} |\cos\frac{2k\pi}{2m+1}| \\
&= \sum_{k=1}^{2m} |\cos\frac{2k\pi}{2m+1}| + \sum_{k=0}^{m} |\cos\frac{2k\pi}{2m+1}| \\
&\quad \text{(changing k to k + (2m + 1) in the last summation)} \\
&= \frac{3}{2}\csc\frac{3\pi}{2n} + \frac{1}{2} \quad \text{(using Lemma 2.3}(viii), (iii)).
\end{aligned}$$

(ii) Let $n = 6m + 1$, $f(k) = \cos\frac{2k\pi}{n}$, $g(k) = \cos\frac{6k\pi}{n}$, where $k \in \{1, 2, \cdots, \frac{n-1}{2}\}$. Then we have

$$g(k) = \begin{cases} f(3k) & \text{if} \quad 1 \le k \le m, \\ f(n-3k) & \text{if} \quad m < k \le 2m, \\ f(3k-n) & \text{if} \quad 2m < k \le 3m. \end{cases}$$

This implies (ii).

The proofs of (iii), (iv), (v), (vi), (vii) and $(viii)$ are similar to the proof of (ii).

Suppose $n = 4m$ then

$$\begin{aligned}
\sum_{k=1}^{\frac{n-2}{2}} |\cos\frac{8k\pi}{n}| &= \sum_{k=1}^{2m-1} |\cos\frac{2k\pi}{m}| \\
&= 1 + \sum_{k=1}^{m-1} |\cos\frac{2k\pi}{m}| + \sum_{k=m+1}^{2m-1} |\cos\frac{2k\pi}{m}| \\
&= 1 + 2\sum_{k=1}^{m-1} |\cos\frac{2k\pi}{m}| = -1 + 2\sum_{k=0}^{m-1} |\cos\frac{2k\pi}{m}|
\end{aligned}$$

$$= \begin{cases} -1+2\csc\frac{\pi}{2m} \text{ if } & m\equiv 1(mod2) \\ -1+4\cot\frac{\pi}{m} \text{ if } & m\equiv 0(mod4) \\ -1+4\csc\frac{\pi}{m} \text{ if } & m\equiv 2(mod4) \end{cases}$$

$$= \begin{cases} -1+2\csc\frac{2\pi}{n} \text{ if } & n\equiv 4(mod8), \\ -1+4\cot\frac{4\pi}{n} \text{ if } & n\equiv 0(mod16), \\ -1+4\csc\frac{4\pi}{n} \text{ if } & n\equiv 8(mod16). \end{cases}$$

This completes the proof of $(viii)$,(ix),(x). □

Theorem 2.14 *Let $G=\langle b : b^n=1\rangle$ be a cyclic group and $\Gamma=Cay(G,S)$,be a Cayley digraph on G with respect to $S=\{b^l,b^{2l}\}$ where $l\in U(n)=\{r:1\le r<n, gcd(n,r)=1\}$ and $C_s(\Gamma)$ be the skew-adjacency matrix of Γ , $D(\Gamma)=diag(d_1,d_2,\cdots,d_n)$, the diagonal matrix with vertex degrees $d_1,d_2,\cdots,d_n$ of $e,b,b^2,\cdots,b^{n-1}$. Suppose $L(\Gamma)=D(\Gamma)-C_s(\Gamma)$ and $\mu_1,\cdots,\mu_n$ are eigenvalues of $L(\Gamma)$. Define $\alpha(\Gamma)=\sum_{i=1}^{n}\mu_i^2$. Then*

(i) $\alpha(\Gamma)\le 4(3n+2)-12\csc\frac{3\pi}{2n}$ *if* $n\equiv 3(mod6)$.

(ii) $\alpha(\Gamma)\le 4(3n+2)-4\csc\frac{\pi}{2n}$ *if* $n\equiv 1$ *or* $5(mod6)$.

(iii) $\alpha(\Gamma)\le 4(3n-2)+16\csc\frac{\pi}{n}$ *if* $n\equiv 2(mod4)$ *and* $n\equiv 0(mod6)$.

(iv) $\alpha(\Gamma)\le 4(3n-2)+24\csc\frac{\pi}{n}$ *if* $n\equiv 2(mod4)$*and* $n\equiv 2$ *or* $4(mod6)$.

(v) $\alpha(\Gamma)\le 4(3n-2)+8\cot\frac{\pi}{n}+8\csc\frac{2\pi}{n}$ *if* $n\equiv 4(mod8)$ *and* $n\equiv 0(mod6)$.

(vi) $\alpha(\Gamma)\le 4(3n-4)+16\cot\frac{\pi}{n}+8\csc\frac{2\pi}{n}$ *if* $n\equiv 4(mod8)$ *and* $n\equiv 2$ *or* $4(mod6)$.

(vii) $\alpha(\Gamma)\le 4(3n-2)+8\cot\frac{\pi}{n}+16\cot\frac{4\pi}{n}$ *if* $n\equiv 0(mod16)$, *and* $n\equiv 0(mod6)$.

$(viii)$ $\alpha(\Gamma)\le 2(n-8)+16\cot\frac{\pi}{n}+16\cot\frac{4\pi}{n}$ *if* $n\equiv 0(mod16)$ *and* $n\equiv 2$ *or* $4(mod6)$.

(ix) $\alpha(\Gamma)\le 4(3n-2)+8\cot\frac{\pi}{n}+16\csc\frac{4\pi}{n}$ *if* $n\equiv 8(mod16)$ *and* $n\equiv 0(mod6)$.

(x) $\alpha(\Gamma)\le 4(3n-4)+16\cot\frac{\pi}{n}+16\csc\frac{4\pi}{n}$ *if* $n\equiv 8(mod16)$ *and* $n\equiv 2$ *or* $4(mod6)$.

Proof Let $G=\langle b : b^n=1\rangle$ be a cyclic group and $\Gamma=Cay(G,S)$,be a Cayley digraph on G with respect to $S=\{b^l,b^{2l}\}$ where $l\in U(n)=\{r:1\le r<n, gcd(n,r)=1\}$ and $C_s(\Gamma)$ be the skew-adjacency matrix of Γ. Note that underlying graph of Γ is a $4-$regular graph. Hence $D(\Gamma)=diag(4,4,\cdots,4)$. Suppose $L(\Gamma)=D(\Gamma)-C_s(\Gamma)$ then $L(\Gamma)$ is circulant matrix and its first row is $[4,-1,-1,\cdots,0,1,1]$. This implies that the eigenvalues of $L(\Gamma)$ are

$$\mu_k=4-\omega^k-\omega^{2k}+\omega^{-2k}+\omega^{-k}=4-2i(\sin\frac{2k\pi}{n}+\sin\frac{4k\pi}{n}),\quad k=0,1,\cdots,n-1,$$

where $\omega=e^{\frac{2\pi i}{n}}$ and $i^2=-1$. It is clear that

$$\mu_{n-k}=4+2i(\sin\frac{2k\pi}{n}+\sin\frac{4k\pi}{n}) \text{ and } \mu_k+\mu_{n-k}=8$$

for $k = 1, 2, \cdots, \lfloor\frac{n-1}{2}\rfloor$. So

$$\begin{aligned}
\mu_k^2 + \mu_{n-k}^2 &= 64 - 2\mu_k\mu_{n-k} \\
&= 64 - 2(4 - 2i(\sin\frac{2k\pi}{n} + \sin\frac{4k\pi}{n}))(4 + 2i(\sin\frac{2k\pi}{n} + \sin\frac{4k\pi}{n})) \\
&= 64 - 2(16 + 4(\sin\frac{2k\pi}{n} + \sin\frac{4k\pi}{n})^2) \\
&= 24 - 8\cos\frac{2k\pi}{n} + 4\cos\frac{4k\pi}{n} - 8\cos\frac{6k\pi}{n} + 4\cos\frac{8k\pi}{n}
\end{aligned}$$

for $k = 1, 2, \cdots, \lfloor\frac{n-1}{2}\rfloor$. Let $n \equiv 1(mod 2)$, then

$$\begin{aligned}
\alpha(\Gamma) &= \sum_{k=0}^{n-1}\mu_k^2 \\
&= \mu_0^2 + \sum_{k=1}^{n-1}\lambda_k^2 \\
&= 16 + \sum_{k=1}^{\frac{n-1}{2}}(\mu_k^2 + \mu_{n-k}^2) \\
&= 16 + \sum_{k=1}^{\frac{n-1}{2}}(24 - 8\cos\frac{2k\pi}{n} + 4\cos\frac{4k\pi}{n} - 8\cos\frac{6k\pi}{n} + 4\cos\frac{8k\pi}{n}) \\
&= 4(3n+1) - 8\sum_{k=1}^{\frac{n-1}{2}}\cos\frac{2k\pi}{n} + 4\sum_{k=1}^{\frac{n-1}{2}}\cos\frac{4k\pi}{n} - 8\sum_{k=1}^{\frac{n-1}{2}}\cos\frac{6k\pi}{n} + 4\sum_{k=1}^{\frac{n-1}{2}}\cos\frac{8k\pi}{n} \\
&= 4(3n+1) - 8\sum_{k=1}^{\frac{n-1}{2}}\cos\frac{2k\pi}{n} + 4\sum_{k=1}^{\frac{n-1}{2}}\cos\frac{2k\pi}{n} - 8\sum_{k=1}^{\frac{n-1}{2}}\cos\frac{6k\pi}{n} + 4\sum_{k=1}^{\frac{n-1}{2}}\cos\frac{2k\pi}{n} \\
&\qquad (\text{using Lemma2.11}(i), 2.13(vii)) \\
&= 4(3n+1) - 8\sum_{k=1}^{\frac{n-1}{2}}\cos\frac{6k\pi}{n} \le 4(3n+1) - 8\sum_{k=1}^{\frac{n-1}{2}}|\cos\frac{6k\pi}{n}|. \qquad (2.1)
\end{aligned}$$

(i) If $n \equiv 3(mod 6)$ then using Lemma 2.13(i) in above inequelity, we get

$$\alpha(\Gamma) \le 4(3n+1) - 8(\frac{3}{2}\csc\frac{3\pi}{2n} - \frac{1}{2}) = 4(3n+2) - 12\csc\frac{3\pi}{2n}.$$

This completes the proof of (i).

(ii) If $n \equiv 1$ or $5(mod 6)$ and using Lemma 2.13(ii) and (iii), we get

$$\begin{aligned}
\alpha(\Gamma) &\le 4(3n+1) - 8\sum_{k=1}^{\frac{n-1}{2}}|\cos\frac{2k\pi}{n}| \\
&= 4(3n+1) - 8(\frac{1}{2}\csc\frac{\pi}{2n} - \frac{1}{2}) = 4(3n+2) - 4\csc\frac{\pi}{2n}.
\end{aligned}$$

Let $n \equiv 0(mod 2)$. Then

$$\begin{aligned}
\alpha(\Gamma) &= \sum_{k=0}^{n-1} \mu_k^2 = \mu_0^2 + \mu_{\frac{n}{2}}^2 + \sum_{k=1, k\neq \frac{n}{2}}^{n-1} \mu_k^2 = 16 + 16 + \sum_{k=1}^{\frac{n-2}{2}} (\mu_k^2 + \mu_{n-k}^2) \\
&= 32 + \sum_{k=1}^{\frac{n-2}{2}} (24 - 8\cos\frac{2k\pi}{n} + 4\cos\frac{4k\pi}{n} - 8\cos\frac{6k\pi}{n} + 4\cos\frac{8k\pi}{n}) \\
&= 4(3n+2) - 8\sum_{k=1}^{\frac{n-2}{2}} \cos\frac{2k\pi}{n} + 4\sum_{k=1}^{\frac{n-2}{2}} \cos\frac{4k\pi}{n} - 8\sum_{k=1}^{\frac{n-2}{2}} \cos\frac{6k\pi}{n} + 4\sum_{k=1}^{\frac{n-2}{2}} \cos\frac{8k\pi}{n}. \quad (2.2)
\end{aligned}$$

If $n \equiv 2(\text{mod } 4)$, then employing Lemma 2.13(viii) in (2.2), we get

$$\begin{aligned}
\alpha(\Gamma) &= 4(3n+2) - 8\sum_{k=1}^{\frac{n-2}{2}} \cos\frac{2k\pi}{n} + 4\sum_{k=1}^{\frac{n-2}{2}} \cos\frac{4k\pi}{n} - 8\sum_{k=1}^{\frac{n-2}{2}} \cos\frac{6k\pi}{n} + 4\sum_{k=1}^{\frac{n-2}{2}} \cos\frac{4k\pi}{n} \\
&= 4(3n+2) - 8\sum_{k=1}^{\frac{n-2}{2}} \cos\frac{2k\pi}{n} + 8\sum_{k=1}^{\frac{n-2}{2}} \cos\frac{4k\pi}{n} - 8\sum_{k=1}^{\frac{n-2}{2}} \cos\frac{6k\pi}{n}. \quad (2.3)
\end{aligned}$$

(iii) If $n \equiv 2(mod 4)$ and $n \equiv 0(\text{mod } 6)$, then using Lemma 2.13$(iv)$ in (2.3) we deduce that

$$\begin{aligned}
\alpha(\Gamma) &\leq 4(3n+2) + 8\sum_{k=1}^{\frac{n-2}{2}} |\cos\frac{2k\pi}{n}| + 8\sum_{k=1}^{\frac{n-2}{2}} |\cos\frac{4k\pi}{n}| \\
&= 4(3n+2) + 16(\csc\frac{\pi}{n} - 1) = 4(3n-2) + 16\csc\frac{\pi}{n}
\end{aligned}$$

by using Lemma 2.4(ii) and 2.11(ii).

(iv) If $n \equiv 2(mod 4)$ and $n \equiv 2$ or $4(mod 6)$, then using Lemma 2.13(v) and (vi) in (2.3) we see that

$$\begin{aligned}
\alpha(\Gamma) &= 4(3n+2) - 8\sum_{k=1}^{\frac{n-2}{2}} \cos\frac{2k\pi}{n} + 8\sum_{k=1}^{\frac{n-2}{2}} \cos\frac{4k\pi}{n} - 8\sum_{k=1}^{\frac{n-2}{2}} \cos\frac{2k\pi}{n} \\
&\leq 4(3n+2) + 16\sum_{k=1}^{\frac{n-2}{2}} |\cos\frac{2k\pi}{n}| + 8\sum_{k=1}^{\frac{n-2}{2}} |\cos\frac{4k\pi}{n}| \\
&\leq 4(3n+2) + 24(\csc\frac{\pi}{n} - 1) = 4(3n-2) + 24\csc\frac{\pi}{n}.
\end{aligned}$$

Similarly we can prove (v) to (x). □

We give few interesting results on the skew energy of Cayley digraphs on dihedral groups D_{2n}.

Theorem 2.15 *Let $D_{2n} = \langle a, b | a^2 = b^n = 1, a^{-1}ba = b^{-1}\rangle$ the dihedral group of order $2n$ and $\Gamma = Cay(D_{2n}, S)$ be a Cayley digraph on D_{2n} with respect to $S = \{b^i\}$, $1 \leq i \leq n-1$, and*

$H = \langle S\rangle$, $|H| = m$, $|D'_{2n} : H| = \lambda$ that,D'_{2n} is the commutator subgroup of D_{2n}. Then

$$\varepsilon_s(\Gamma) = \begin{cases} 4\lambda cot\frac{\pi}{2m} & if \quad m \equiv 1(mod 2), \\ 8\lambda cot\frac{\pi}{m} & if \quad m \equiv 0(mod 2). \end{cases}$$

Proof The proof of Theorem 2.15 directly follows from the definition of dihedral group and Theorem 2.7. □

Theorem 2.16 *Let $D_{2n} = \langle a, b|a^2 = b^n = 1, a^{-1}ba = b^{-1}\rangle$ the dihedral group of order $2n$ and $\Gamma = Cay(D_{2n}, S)$ be a Cayley digraph on D_{2n} with respect to $S = \{b^i, b^j\}$, $1 \le i, j \le n-1, i \ne j$, and $H = \langle S\rangle$, $|H| = m$, $|D'_{2n} : H| = \lambda$ Then $\Gamma = Cay(D_{2n}, S)$ is a circulant digraph and its skew energy*

(i) $\varepsilon_s(\Gamma) \le 8\lambda cot\frac{\pi}{2m}$ *if* $m \equiv 1(mod 2)$,

(ii) $\varepsilon_s(\Gamma) \le 16\lambda cot\frac{\pi}{m}$ *if* $m \equiv 2(mod 4)$,

(iii) $\varepsilon_s(\Gamma) \le 8\lambda(cot\frac{\pi}{m} + 2csc\frac{4\pi}{m} + 2cot\frac{4\pi}{m})$ *if* $m \equiv 0(mod 8)$,

(iv) $\varepsilon_s(\Gamma) \le 8\lambda(cot\frac{\pi}{m} + 2cot\frac{2\pi}{m}))$ *if* $m \equiv 4(mod 8)$.

Proof The proof of Theorem 2.16 directly follows from the definition of dihedral group and Theorem 2.8. □

Theorem 2.17 *Let $D_{2n} = \langle a, b|a^2 = b^n = 1, a^{-1}ba = b^{-1}\rangle$ the dihedral group of order $2n$ and $\Gamma = Cay(D_{2n}, S)$ be a Cayley digraph on D_{2n} with respect to $S = \{b^l\}$ where $l \in U(n) = \{r : 1 \le r < n, gcd(n, r) = 1\}$ and $C_s(\Gamma)$ be the skew-adjacency matrix of Γ, $D(\Gamma)$ is the $n \times n$ matrix such that $d_{ij} = 2$ whenever $i = j$ otherwise $d_{ij} = 0$. Suppose $L(\Gamma) = D(\Gamma) - C_s(\Gamma)$ and $\lambda_1, \cdots, \lambda_n$ are eigenvalues of $L(\Gamma)$. Define $\alpha(\Gamma) = \sum_{i=1}^{n} \lambda_i^2$. Then*

(i) $\alpha(\Gamma) \le 4n + 4\csc\frac{\pi}{2n}$ *if* $n \equiv 1(\text{mod } 2)$,

(ii) $\alpha(\Gamma) \le 4(n-1) + 8\csc\frac{\pi}{n}$ *if* $n \equiv 2(\text{mod } 4)$

(iii) $\alpha(\Gamma) = 4(n-2)$ *if* $n \equiv 0(\text{mod } 4)$

Proof The proof of Theorem 2.17 directly follows from the definition of dihedral group and Theorem 2.12. □

Theorem 2.18 *Let $D_{2n} = \langle a, b|a^2 = b^n = 1, a^{-1}ba = b^{-1}\rangle$ the dihedral group of order $2n$ and $\Gamma = Cay(D_{2n}, S)$ be a Cayley digraph on D_{2n} with respect to $S = \{b^l, b^{2l}\}$ where $l \in U(n) = \{r : 1 \le r < n, gcd(n, r) = 1\}$ and $C_s(\Gamma)$ be the skew-adjacency matrix of Γ , $D(\Gamma)$ is the $n \times n$ matrix such that $d_{ij} = 4$ whenever $i = j$ otherwise $d_{ij} = 0$. Suppose $L(\Gamma) = D(\Gamma) - C_s(\Gamma)$ and $\lambda_1, \cdots, \lambda_n$ are eigenvalues of $L(\Gamma)$. Define $\alpha(\Gamma) = \sum_{i=1}^{n} \lambda_i^2$. Then*

(i) $\alpha(\Gamma) \leq 8(3n+2) - 24\csc\frac{3\pi}{2n}$ *if* $n \equiv 3(mod6)$.

(ii) $\alpha(\Gamma) \leq 8(3n+2) - 8\csc\frac{\pi}{2n}$ *if* $n \equiv 1$ *or* $5(mod6)$.

(iii) $\alpha(\Gamma) \leq 8(3n-2) + 32\csc\frac{\pi}{n}$ *if* $n \equiv 2(mod4)$ *and* $n \equiv 0(mod6)$.

(iv) $\alpha(\Gamma) \leq 8(3n-2) + 48\csc\frac{\pi}{n}$ *if* $n \equiv 2(mod4)$*and* $n \equiv 2$ *or* $4(mod6)$.

(v) $\alpha(\Gamma) \leq 8(3n-2) + 16\cot\frac{\pi}{n} + 16\csc\frac{2\pi}{n}$ *if* $n \equiv 4(mod8)$ *and* $n \equiv 0(mod6)$.

(vi) $\alpha(\Gamma) \leq 8(3n-4) + 32\cot\frac{\pi}{n} + 16\csc\frac{2\pi}{n}$ *if* $n \equiv 4(mod8)$ *and* $n \equiv 2$ *or* $4(mod6)$.

(vii) $\alpha(\Gamma) \leq 8(3n-2) + 16\cot\frac{\pi}{n} + 32\cot\frac{4\pi}{n}$ *if* $n \equiv 0(mod16)$, *and* $n \equiv 0(mod6)$.

(viii) $\alpha(\Gamma) \leq 4(n-8) + 32\cot\frac{\pi}{n} + 32\cot\frac{4\pi}{n}$ *if* $n \equiv 0(mod16)$ *and* $n \equiv 2$ *or* $4(mod6)$.

(ix) $\alpha(\Gamma) \leq 8(3n-2) + 16\cot\frac{\pi}{n} + 32\csc\frac{4\pi}{n}$ *if* $n \equiv 8(mod16)$ *and* $n \equiv 0(mod6)$.

(x) $\alpha(\Gamma) \leq 8(3n-4) + 32\cot\frac{\pi}{n} + 32\csc\frac{4\pi}{n}$ *if* $n \equiv 8(mod16)$ *and* $n \equiv 2$ *or* $4(mod6)$.

Proof The proof of Theorem 2.18 directly follows from the definition of dihedral group and Theorem 2.14. □

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