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Quarter-Symmetric Metric Connection On Pseudosymmetric Lorentzian $\alpha-$Sasakian Manifolds

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Abstract: The object of this paper is to introduce a quarter-symmetric metric connection in a pseudosymmetric Lorentzian α-Sasakian manifold and to study of some properties of it. Also we shall discuss some properties of the Weyl-pseudosymmetric Lorentzian $\alpha-$Sasakian manifold and Ricci-pseudosymmetric Lorentzian $\alpha-$Sasakian manifold with respet to quarter-symmetric metric connection. We have given an example of pseudosymmetric Lorentzian α-Sasakian manifold with respect to quarter-symmetric metric connection.

Key Words: Lorentzian α-Sasakian manifold, quarter-symmetric metric connection, pseudosymmetric Lorentzian $\alpha-$Sasakian manifolds, Ricci-pseudosymmetric, Weyl-pseudosymmetric, $\eta-$Einstein manifold.

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§1. Introduction

The theory of pseudosymmetric manifold has been developed by many authors by two ways. One is the Chaki sense [8], [3] and another is Deszcz sense [2], [9], [11]. In this paper we shall study some properties of pseudosymmetric and Ricci-symmetric Lorentzian $\alpha-$ Sasakian manifolds with respect to quarter-symmetric metric connection in Deszcz sense. The notion of pseudo-symmetry is a natural generalization of semi-symmetry, along the line of spaces of constant sectional curvature and locally symmetric space.

A Riemannian manifold (M, g) of dimension n is said to be pseudosymmetric if the Riemannian curvature tensor R satisfies the conditions ([1]):

1. $(R(X,Y).R)(U,V,W) = L_R[((X \wedge Y).R)(U,V,W)]$ (1)

for all vector fields X, Y, U, V, W on M , where$L_R \in C^\infty(M)$, $R(X,Y)Z = \nabla_{[X,Y]}Z - [\nabla_X, \nabla_Y]Z$ and $X \wedge Y$ is an endomorphism defined by

$$(X \wedge Y)Z = g(Y,Z)X - g(X,Z)Y \tag{2}$$

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$$\begin{aligned}
2.\ (R(X,Y).R)(U,V,W) &= R(X,Y)(R(U,V)W) - R(R(X,Y)U,V)W \\
&\quad -R(U,R(X,Y)V)W - R(U,V)(R(X,Y)W) \qquad (3)\\
3.\ ((X\wedge Y).R)(U,V,W) &= (X\wedge Y)(R(U,V)W) - R((X\wedge Y)U,V)W \\
&\quad -R(U,(X\wedge Y)V)W - R(U,V)((X\wedge Y)W). \qquad (4)
\end{aligned}$$

M is said to be pseudosymmetric of constant type if L is constant. A Riemannian manifold (M,g) is called semi-symmetric if $R.R=0$, where $R.R$ is the derivative of R by R.

Remark 1.1 We know, the $(0,k+2)$ tensor fields $R.T$ and $Q(g,T)$ are defined by

$$\begin{aligned}
(R.T)(X_1,\cdots,X_k;X,Y) &= (R(X,Y).T)(X_1,\cdots,X_k)\\
&= -T(R(X,Y)X_1,\cdots,X_k)-\cdots-T(X_1,\cdots,R(X,Y)X_k)\\
Q(g,T)(X_1,\cdots,X_k;X,Y) &= -((X\wedge Y).T)(X_1,\cdots,X_k)\\
&= T((X\wedge Y)X_1,\cdots,X_k)+\cdots+T(X_1,\cdots,(X\wedge Y)X_k),
\end{aligned}$$

where T is a $(0,k)$ tensor field ([4],[5]).

Let S and r denote the Ricci tensor and the scalar curvature tensor of M respectively. The operator Q and the $(0,2)-$tensor S^2 are defined by

$$S(X,Y)=g(QX,Y) \qquad (5)$$

and

$$S^2(X,Y)=S(QX,Y) \qquad (6)$$

The Weyl conformal curvature operator C is defined by

$$C(X,Y)=R(X,Y)-\frac{1}{n-2}[X\wedge QY+QX\wedge Y-\frac{r}{n-1}X\wedge Y]. \qquad (7)$$

If $C=0$, $n\geq 3$ then M is called conformally flat. If the tensor $R.C$ and $Q(g,C)$ are linearly dependent then M is called Weyl-pseudosymmetric. This is equivalent to

$$R.C(U,V,W;X,Y)=L_C[((X\wedge Y).C)(U,V)W], \qquad (8)$$

holds on the set $U_C=\{x\in M : C\neq 0 \text{ at } x\}$, where L_C is defined on U_C. If $R.C=0$, then M is called Weyl-semi-symmetric. If $\nabla C=0$, then M is called conformally symmetric ([6],[10]).

§2. Preliminaries

A n-dimensional differentiable manifold M is said to be a Lorentzian $\alpha-$Sasakian manifold if it admits a $(1,1)-$tensor field ϕ, a contravariant vector field ξ, a covariant vector field η and Lorentzian metric g which satisfy the following conditions,

$$\phi^2=I+\eta\otimes\xi, \qquad (9)$$

$$\eta(\xi) = -1, \quad \phi\xi = 0, \quad \eta \circ \phi = 0, \tag{10}$$

$$g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y), \tag{11}$$

$$g(X, \xi) = \eta(X) \tag{12}$$

and

$$(\nabla_X \phi)(Y) = \alpha\{g(X, Y)\xi + \eta(Y)X\} \tag{13}$$

for $\forall X, Y \in \chi(M)$ and for smooth functions α on M, ∇ denotes covariant differentiation operator with respect to Lorentzian metric g ([6], [7]).

For a Lorentzian $\alpha-$Sasakian manifold,it can be shown that ([6],[7]):

$$\nabla_X \xi = \alpha\phi X, \tag{14}$$

$$(\nabla_X \eta)Y = \alpha g(\phi X, Y). \tag{15}$$

Further on a Lorentzian $\alpha-$Sasakian manifold, the following relations hold ([6])

$$\eta(R(X, Y)Z) = \alpha^2[g(Y, Z)\eta(X) - g(X, Z)\eta(Y)], \tag{16}$$

$$R(\xi, X)Y = \alpha^2[g(Y, Z)\xi - \eta(Y)X], \tag{17}$$

$$R(X, Y)\xi = \alpha^2[\eta(Y)X - \eta(X)Y], \tag{18}$$

$$S(\xi, X) = S(X, \xi) = (n-1)\alpha^2\eta(X), \tag{19}$$

$$S(\xi, \xi) = -(n-1)\alpha^2, \tag{20}$$

$$Q\xi = (n-1)\alpha^2\xi. \tag{21}$$

The above relations will be used in following sections.

§3. Quarter-Symmetric Metric Connection on Lorentzian $\alpha-$Sasakian Manifold

Let M be a Lorentzian α-Sasakian manifold with Levi-Civita connection ∇ and $X, Y, Z \in \chi(M)$. We define a linear connection D on M by

$$D_X Y = \nabla_X Y + \eta(Y)\phi(X) \tag{22}$$

where η is $1-$form and ϕ is a tensor field of type $(1, 1)$. D is said to be quarter-symmetric connection if $\bar{T}$, the torsion tensor with respect to the connection D, satisfies

$$\bar{T}(X, Y) = \eta(Y)\phi X - \eta(X)\phi Y. \tag{23}$$

D is said to be metric connection if

$$(D_X g)(Y, Z) = 0. \tag{24}$$

A linear connection D is said to be quarter-symmetric metric connection if it satisfies (22), (23) and (24).

Now we shall show the existence of the quarter-symmetric metric connection D on a Lorentzian $\alpha-$Sasakian manifold M.

Theorem 3.1 *Let X, Y, Z be any vectors fields on a Lorentzian $\alpha-$Sasakian manifold M and let a connection D is given by*

$$\begin{aligned} 2g(D_XY,Z) &= Xg(Y,Z)+Yg(Z,X)-Zg(X,Y)+g([X,Y],Z) \\ &-g([Y,Z],X)+g([Z,X],Y)+g(\eta(Y)\phi X-\eta(X)\phi Y,Z) \\ &+g(\eta(X)\phi Z-\eta(Z)\phi X,Y)+g(\eta(Y)\phi Z-\eta(Z)\phi Y,X). \end{aligned} \tag{25}$$

Then D is a quarter-symmetric metric connection on M.

Proof It can be verified that $D:(X,Y)\rightarrow D_XY$ satisfies the following equations:

$$D_X(Y+Z) = D_XY+D_XZ, \tag{26}$$

$$D_{X+Y}Z = D_XZ+D_YZ, \tag{27}$$

$$D_{fX}Y = fD_XY, \tag{28}$$

$$D_X(fY) = f(D_XY)+(Xf)Y \tag{29}$$

for all $X,Y,Z\in\chi(M)$ and for all f, differentiable function on M.

From (26), (27), (28) and (29), we can conclude that D is a linear connection on M. From (25) we have,

$$g(D_XY,Z)-g(D_YX,Z)=g([X,Y],Z)+\eta(Y)g(\phi X,Z)-\eta(X)g(\phi Y,Z)$$

or,

$$D_XY-D_YX-[X,Y]=\eta(Y)\phi X-\eta(X)\phi Y$$

or,

$$\bar{T}(X,Y)=\eta(Y)\phi X-\eta(X)\phi Y \tag{30}$$

Again from (25) we get,

$$2g(D_XY,Z)+2g(D_XZ,Y)=2Xg(Y,Z),\quad\text{or,}\quad (D_Xg)(Y,Z)=0.$$

This shows that D is a quarter-symmetric metric connection on M. □

§4. Curvature Tensor and Ricci Tensor with Respect to Quarter-Symmetric Metric Connection D in a Lorentzian $\alpha-$Sasakian Manifold

Let $\bar{R}(X,Y)Z$ and $R(X,Y)Z$ be the curvature tensors with respect to the quarter-symmetric metric connection D and with respect to the Riemannian connection ∇ respectively on a Lorentzian $\alpha-$ Sasakian manifold M. A relation between the curvature tensors $\bar{R}(X,Y)Z$ and $R(X,Y)Z$ on M is given by

$$\begin{aligned} \bar{R}(X,Y)Z &= R(X,Y)Z+\alpha[g(\phi X,Z)\phi Y \\ &-g(\phi Y,Z)\phi X]+\alpha\eta(Z)[\eta(Y)X-\eta(X)Y]. \end{aligned} \tag{31}$$

Also from (31), we obtain

$$\bar{S}(X,Y)=S(X,Y)+\alpha[g(X,Y)+n\eta(X)\eta(Y)], \tag{32}$$

where $\bar{S}$ and S are the Ricci tensors of the connections D and ∇ respectively.

Again

$$\begin{aligned}\bar{S}^2(X,Y) &= S^2(X,Y)-\alpha(n-2)S(X,Y)-\alpha^2(n-1)g(X,Y)\\&+\alpha^2 n(n-1)(\alpha-1)\eta(X)\eta(Y).\end{aligned}\tag{33}$$

Contracting (32), we get

$$\bar{r}=r,\tag{34}$$

where $\bar{r}$ and r are the scalar curvature with respect to the connection D and ∇ respectively.

Let $\bar{C}$ be the conformal curvature tensors on Lorentzian $\alpha-$ Sasakian manifolds with respect to the connections D. Then

$$\begin{aligned}\bar{C}(X,Y)Z &= \bar{R}(X,Y)Z-\frac{1}{n-2}[\bar{S}(Y,Z)X-g(X,Z)\bar{Q}Y+g(Y,Z)\bar{Q}X\\&-\bar{S}(X,Z)Y]+\frac{\bar{r}}{(n-1)(n-2)}[g(Y,Z)X-g(X,Z)Y],\end{aligned}\tag{35}$$

where $\bar{Q}$ is Ricci operator with the connection D on M and

$$\bar{S}(X,Y)=g(\bar{Q}X,Y),\tag{36}$$

$$\bar{S}^2(X,Y)=\bar{S}(\bar{Q}X,Y).\tag{37}$$

Now we shall prove the following theorem.

Theorem 4.1 *Let M be a Lorentzian $\alpha-$Sasakian manifold with respect to the quarter-symmetric metric connection D, then the following relations hold:*

$$\begin{aligned}\bar{R}(\xi,X)Y &= \alpha^2[g(X,Y)\xi-\eta(Y)X]+\alpha\eta(Y)[X+\eta(X)\xi], &(38)\\ \eta(\bar{R}(X,Y)Z) &= \alpha^2[g(Y,Z)\eta(X)-g(X,Z)\eta(Y)], &(39)\\ \bar{R}(X,Y)\xi &= (\alpha^2-\alpha)[\eta(Y)X-\eta(X)Y], &(40)\\ \bar{S}(X,\xi) &= \bar{S}(\xi,X)=(n-1)(\alpha^2-\alpha)\eta(X), &(41)\\ \bar{S}^2(X,\xi) &= \bar{S}^2(\xi,X)=\alpha^2(n-1)^2(\alpha-1)^2\eta(X), &(42)\\ \bar{S}(\xi,\xi) &= -(n-1)(\alpha^2-\alpha), &(43)\\ \bar{Q}X &= QX-\alpha(n-1)X, &(44)\\ \bar{Q}\xi &= (n-1)(\alpha^2-\alpha)\xi. &(45)\end{aligned}$$

Proof Since M is a Lorentzian $\alpha-$Sasakian manifold with respect to the quarter-symmetric metric connection D, then replacing $X=\xi$ in (31) and using (10) and (17) we get (38). Using (10) and (16), from (31) we get (39). To prove (40), we put $Z=\xi$ in (31) and then we use (18). Replacing $Y=\xi$ in (32) and using (19) we get (41). Putting $Y=\xi$ in (33) and using (6) and (19) we get (42). Again putting $X=Y=\xi$ in (32) and using (20) we get (43). Using (36) and (41) we get (44). Then putting $X=\xi$ in (44) we get (45). □

§5. Lorentzian $\alpha-$Sasakian Manifold with Respect to the Quarter-Symmetric Metric Connection D Satisfying the Condition $\bar{C}.\bar{S}=0$.

In this section we shall find out the characterization of Lorentzian $\alpha-$Sasakian manifold with respect to the quarter-symmetric metric connection D satisfying the condition $\bar{C}.\bar{S}=0$. We define $\bar{C}.\bar{S}=0$ on M by

$$(\bar{C}(X,Y).\bar{S})(Z,W)=-\bar{S}(\bar{C}(X,Y)Z,W)-\bar{S}(Z,\bar{C}(X,Y)W), \tag{46}$$

where $X,Y,Z,W\in\chi(M)$.

Theorem 5.1 *Let M be an $n-$dimensional Lorentzian $\alpha-$ Sasakian manifold with respect to the quarter-symmetric metric connection D. If $\bar{C}.\bar{S}=0$, then*

$$\begin{aligned}\frac{1}{n-2}\bar{S}^2(X,Y) &= [(\alpha^2-\alpha)+\frac{\bar{r}}{(n-1)(n-2)}]\bar{S}(X,Y)\\ &\quad +\frac{\alpha^2-\alpha}{n-2}[\alpha(n-1)(\alpha-n+1)-\bar{r}]g(X,Y)\\ &\quad -\alpha(n-1)(\alpha^2-\alpha)\eta(X)\eta(Y).\end{aligned} \tag{47}$$

Proof Let us consider M be an n-dimensional Lorentzian $\alpha-$Sasakian manifold with respect the quarter-symmetric metric connection D satisfying the condition $\bar{C}.\bar{S}=0$. Then from (46), we get

$$\bar{S}(\bar{C}(X,Y)Z,W)+\bar{S}(Z,\bar{C}(X,Y)W)=0, \tag{48}$$

where $X,Y,Z,W\in\chi(M)$. Now putting $X=\xi$ in (48), we get

$$\bar{S}(\bar{C}(\xi,X)Y,Z)+\bar{S}(Y,\bar{C}(\xi,X)Z)=0. \tag{49}$$

Using (35), (37), (38) and (41), we have

$$\begin{aligned}\bar{S}(\bar{C}(\xi,X)Y,Z) &= (n-1)(\alpha^2-\alpha)[\alpha^2-\frac{(n-1)(\alpha^2-\alpha)}{n-2}+\frac{\bar{r}}{(n-1)(n-2)}]\eta(Z)g(X,Y)\\ &\quad +[\alpha-\alpha^2+\frac{(n-1)(\alpha^2-\alpha)}{n-2}-\frac{\bar{r}}{(n-1)(n-2)}]\eta(Y)\bar{S}(X,Z)\\ &\quad +\alpha(\alpha^2-\alpha)(n-1)\eta(X)\eta(Y)\eta(Z)\\ &\quad -\frac{1}{n-2}[(n-1)(\alpha^2-\alpha)\eta(Z)\bar{S}(X,Y)-\bar{S}^2(X,Z)\eta(Y)]\end{aligned} \tag{50}$$

and

$$\begin{aligned}\bar{S}(Y,\bar{C}(\xi,X)Z) &= (n-1)(\alpha^2-\alpha)[\alpha^2-\frac{(n-1)(\alpha^2-\alpha)}{n-2}+\frac{\bar{r}}{(n-1)(n-2)}]\eta(Y)g(X,Z)\\ &\quad +[\alpha-\alpha^2+\frac{(n-1)(\alpha^2-\alpha)}{n-2}-\frac{\bar{r}}{(n-1)(n-2)}]\eta(Z)\bar{S}(Y,X)\\ &\quad +\alpha(\alpha^2-\alpha)(n-1)\eta(X)\eta(Y)\eta(Z)\\ &\quad -\frac{1}{n-2}[(n-1)(\alpha^2-\alpha)\eta(Y)\bar{S}(X,Z)-\bar{S}^2(X,Y)\eta(Z)].\end{aligned} \tag{51}$$

Using (50) and (51) in (49), we get

$$\begin{aligned}&(n-1)(\alpha^2-\alpha)[\alpha^2-\frac{(n-1)(\alpha^2-\alpha)}{n-2}+\frac{\bar{r}}{(n-1)(n-2)}][g(X,Y)\eta(Z)\\&+g(X,Z)\eta(Y)]+2\alpha(\alpha^2-\alpha)(n-1)\eta(X)\eta(Y)\eta(Z)\\&+[\alpha-\alpha^2+\frac{(n-1)(\alpha^2-\alpha)}{n-2}-\frac{\bar{r}}{(n-1)(n-2)}][\eta(Y)\bar{S}(X,Z)+\eta(Z)\bar{S}(Y,X)]\\&-\frac{1}{n-2}[(n-1)(\alpha^2-\alpha)\{\eta(Z)\bar{S}(X,Y)+\eta(Y)\bar{S}(X,Z)\}\\&-\{\bar{S}^2(X,Z)\eta(Y)+\bar{S}^2(X,Y)\eta(Z)\}]=0. \qquad (52)\end{aligned}$$

Replacing $Z=\xi$ in (52) and using (41) and (42), we get

$$\begin{aligned}\frac{1}{n-2}\bar{S}^2(X,Y) &= [(\alpha^2-\alpha)+\frac{\bar{r}}{(n-1)(n-2)}]\bar{S}(X,Y)\\&+\frac{\alpha^2-\alpha}{n-2}[\alpha(n-1)(\alpha-n+1)-\bar{r}]g(X,Y)\\&-\alpha(n-1)(\alpha^2-\alpha)\eta(X)\eta(Y).\end{aligned}$$

□

An $n-$dimensional Lorentzian $\alpha-$Sasakian manifold M with the quarter-symmetric metric connection D is said to be $\eta-$Einstein if its Ricci tensor $\bar{S}$ is of the form

$$\bar{S}(X,Y)=Ag(X,Y)+B\eta(X)\eta(Y), \qquad (53)$$

where A, B are smooth functions of M. Now putting $X=Y=e_i$, $i=1,2,\cdots,n$ in (53) and taking summation for $1\le i\le n$ we get

$$An-B=\bar{r}. \qquad (54)$$

Again replacing $X=Y=\xi$ in (53) we have

$$A-B=(n-1)(\alpha^2-\alpha). \qquad (55)$$

Solving (54) and (55) we obtain

$$A=\frac{\bar{r}}{n-1}-(\alpha^2-\alpha)\ \text{ and }\ B=\frac{\bar{r}}{n-1}-n(\alpha^2-\alpha).$$

Thus the Ricci tensor of an $\eta-$Einstein manifold with the quarter-symmetric metric connection D is given by

$$\bar{S}(X,Y)=[\frac{\bar{r}}{n-1}-(\alpha^2-\alpha)]g(X,Y)+[\frac{\bar{r}}{n-1}-n(\alpha^2-\alpha)]\eta(X)\eta(Y). \qquad (56)$$

§6. $\eta-$Einstein Lorentzian $\alpha-$Sasakian Manifold with Respect to the Quarter-Symmetric Metric Connection D Satisfying the Condition $\bar{C}.\bar{S}=0$.

Theorem 6.1 *Let M be an $\eta-$Einstein Lorentzian $\alpha-$Sasakian manifold of dimension. Then $\bar{C}.\bar{S}=0$ iff*

$$\frac{n\alpha-2\alpha}{n\alpha^2-2\alpha}[\eta(\bar{R}(X,Y)Z)\eta(W)+\eta(\bar{R}(X,Y)W)\eta(Z)]=0,$$

where $X, Y, Z, W \in \chi(M)$.

Proof Let M be an $\eta-$Einstein Lorentzian $\alpha-$Sasakian manifold with respect to the quarter-symmetric metric connection D satisfying $\bar{C}.\bar{S} = 0$. Using (56) in (48), we get

$$\eta(\bar{C}(X,Y)Z)\eta(W) + \eta(\bar{C}(X,Y)W)\eta(Z) = 0,$$

or,

$$\frac{n\alpha - 2\alpha}{n\alpha^2 - 2\alpha}[\eta(\bar{R}(X,Y)Z)\eta(W) + \eta(\bar{R}(X,Y)W)\eta(Z)] = 0.$$

Conversely, using (56) we have

$$\begin{aligned}(\bar{C}(X,Y).\bar{S})(Z,W) &= -[\frac{\bar{r}}{n-1} - n(\alpha^2 - \alpha)][\eta(\bar{C}(X,Y)Z)\eta(W) + \eta(\bar{C}(X,Y)W)\eta(Z)] \\ &= -\frac{n\alpha - 2\alpha}{n\alpha^2 - 2\alpha}[\eta(\bar{R}(X,Y)Z)\eta(W) + \eta(\bar{R}(X,Y)W)\eta(Z)] = 0. \quad \square\end{aligned}$$

§7. Ricci Pseudosymmetric Lorentzian $\alpha-$Sasakian Manifolds with Quarter-Symmetric Metric Connection D

Theorem 7.1 *A Ricci pseudosymmetric Lorentzian α- Sasakian manifolds M with quarter-symmetric metric connection D with restriction $Y = W = \xi$ and $L_{\bar{S}} = 1$ is an $\eta-$Einstein manifold.*

Proof Lorentzian $\alpha-$Sasakian manifolds M with quarter-symmetric metric connection D is called a Ricci pseudosymmetric Lorentzian $\alpha-$Sasakian manifolds if

$$(\bar{R}(X,Y).\bar{S})(Z,W) = L_{\bar{S}}[((X \wedge Y).\bar{S})(Z,W)], \tag{57}$$

or,

$$\bar{S}(\bar{R}(X,Y)Z,W) + \bar{S}(Z,\bar{R}(X,Y)W) = L_{\bar{S}}[\bar{S}((X \wedge Y)Z,W) + \bar{S}(Z,(X \wedge Y)W)]. \tag{58}$$

Putting $Y = W = \xi$ in (58) and using (2), (38) and (41), we have

$$\begin{aligned}&L_{\bar{S}}[\bar{S}(X,Z) - (n-1)(\alpha^2 - \alpha)g(X,Z)] \\ &= (\alpha^2 - \alpha)\bar{S}(X,Z) - \alpha^2(\alpha^2 - \alpha)(n-1)g(X,Z) - \alpha(\alpha^2 - \alpha)(n-1)\eta(X)\eta(Z).\end{aligned} \tag{59}$$

Then for $L_{\bar{S}} = 1$,

$$(\alpha^2 - \alpha - 1)\bar{S}(X,Z) = (\alpha^2 - \alpha)(n-1)[(\alpha^2 - 1)g(X,Z) + \alpha\eta(X)\eta(Z)].$$

Thus M is an $\eta-$Einstein manifold. $\square$

Corollary 7.1 *A Ricci semisymmetric Lorentzian α-Sasakian manifold M with quarter-symmetric metric connection D with restriction $Y = W = \xi$ is an $\eta-$Einstein manifold.*

Proof Sine M is Ricci semisymmetric Lorentzian α-Sasakian manifolds with quarter-symmetric metric connection D, then $L_{\bar{C}} = 0$. Putting $L_{\bar{C}} = 0$ in (59) we get

$$\bar{S}(X,Z) = \alpha^2(n-1)g(X,Z) + \alpha(n-1)\eta(X)\eta(Z). \qquad \square$$

§8. Pseudosymmetric Lorentzian $\alpha-$Sasakian Manifold and Weyl-pseudosymmetric Lorentzian $\alpha-$Sasakian Manifold with Quarter-Symmetric Metric Connection

In the present section we shall give the definition of pseudosymmetric Lorentzian $\alpha-$Sasakian manifold and Weyl-pseudosymmetric Lorentzian $\alpha-$Sasakian manifold with quarter-symmetric metric connection and discuss some properties on it.

Definition 8.1 *A Lorentzian $\alpha-$Sasakian manifold M with quarter-symmetric metric connection D is said to be pseudosymmetric Lorentzian $\alpha-$Sasakian manifold with quarter-symmetric metric connection if the curvature tensor$\bar{R}$ of M with respect to D satisfies the conditions*

$$(\bar{R}(X,Y).\bar{R})(U,V,W) = L_{\bar{R}}[((X\wedge Y).\bar{R})(U,V,W)], \tag{60}$$

where

$$\begin{aligned}(\bar{R}(X,Y).\bar{R})(U,V,W) &= \bar{R}(X,Y)(\bar{R}(U,V)W) - \bar{R}(\bar{R}(X,Y)U,V)W \\ &\quad -\bar{R}(U,\bar{R}(X,Y)V)W - \bar{R}(U,V)(R(X,Y)W),\end{aligned} \tag{61}$$

and

$$\begin{aligned}((X\wedge Y).\bar{R})(U,V,W) &= (X\wedge Y)(\bar{R}(U,V)W) - \bar{R}((X\wedge Y)U,V)W \\ &\quad -\bar{R}(U,(X\wedge Y)V)W - \bar{R}(U,V)((X\wedge Y)W).\end{aligned} \tag{62}$$

Definition 8.2 *A Lorentzian $\alpha-$Sasakian manifold M with quarter-symmetric metric connection D is said to be Weyl- pseudosymmetric Lorentzian $\alpha-$Sasakian manifold with quarter-symmetric metric connection if the curvature tensor$\bar{R}$ of M with respect to D satisfies the conditions*

$$(\bar{R}(X,Y).\bar{C})(U,V,W) = L_{\bar{C}}[((X\wedge Y).\bar{C})(U,V,W)], \tag{63}$$

where

$$\begin{aligned}(\bar{R}(X,Y).\bar{C})(U,V,W) &= \bar{R}(X,Y)(\bar{C}(U,V)W) - \bar{C}(\bar{R}(X,Y)U,V)W \\ &\quad -\bar{C}(U,\bar{R}(X,Y)V)W - \bar{C}(U,V)(R(X,Y)W)\end{aligned} \tag{64}$$

and

$$\begin{aligned}((X\wedge Y).\bar{C})(U,V,W) &= (X\wedge Y)(\bar{C}(U,V)W) - \bar{C}((X\wedge Y)U,V)W \\ &\quad -\bar{C}(U,(X\wedge Y)V)W - \bar{C}(U,V)((X\wedge Y)W).\end{aligned} \tag{65}$$

Theorem 8.1 *LetM be an n dimensional Lorentzian $\alpha-$Sasakian manifold. If M is Weyl-pseudosymmetric then M is either conformally flat and M is $\eta-$Einstein manifold or $L_{\bar{C}} = \alpha^2$.*

Proof Let M be an Weyl-pseudosymmetric Lorentzian $\alpha-$Sasakian manifold and $X, Y, U, V, W \in \chi(M)$. Then using (64) and (65) in (63), we have

$$\begin{aligned}
&\bar{R}(X,Y)(\bar{C}(U,V)W) - \bar{C}(\bar{R}(X,Y)U,V)W \\
&-\bar{C}(U,\bar{R}(X,Y)V)W - \bar{C}(U,V)(R(X,Y)W) \\
&= L_{\bar{C}}[(X\wedge Y)(\bar{C}(U,V)W) - \bar{C}((X\wedge Y)U,V)W \qquad (66)\\
&-\bar{C}(U,(X\wedge Y)V)W - \bar{C}(U,V)((X\wedge Y)W)]. \qquad (67)
\end{aligned}$$

Replacing X with ξ in (66) we obtain

$$\begin{aligned}
&\bar{R}(\xi,Y)(\bar{C}(U,V)W) - \bar{C}(\bar{R}(\xi,Y)U,V)W \\
&-\bar{C}(U,\bar{R}(\xi,Y)V)W - \bar{C}(U,V)(R(\xi,Y)W) \\
&= L_{\bar{C}}[(\xi\wedge Y)(\bar{C}(U,V)W) - \bar{C}((\xi\wedge Y)U,V)W \\
&-\bar{C}(U,(\xi\wedge Y)V)W - \bar{C}(U,V)((\xi\wedge Y)W)]. \qquad (68)
\end{aligned}$$

Using (2), (38) in (67) and taking inner product of (67) with ξ, we get

$$\begin{aligned}
&\alpha^2[-\bar{C}(U,V,W,Y) - \eta(\bar{C}(U,V)W)\eta(Y) - g(Y,U)\eta(\bar{C}(\xi,V)W) \\
&+\eta(U)\eta(\bar{C}(Y,V)W) - g(Y,V)\eta(\bar{C}(U,\xi)W) + \eta(V)\eta(\bar{C}(U,Y)W) \\
&+\eta(W)\eta(\bar{C}(U,V)Y)] - \alpha[\eta(U)\eta(\bar{C}(\phi^2Y,V)W) \\
&+\eta(V)\eta(\bar{C}(U,\phi^2Y)W) + \eta(W)\eta(\bar{C}(U,V)\phi^2Y)] \\
&= L_{\bar{C}}[-\bar{C}(Y,U,V,W) - \eta(Y)\eta(\bar{C}(U,V)W) - g(Y,U)\eta(\bar{C}(\xi,V)W) \\
&+\eta(U)\eta(\bar{C}(Y,V)W) - g(Y,V)\eta(\bar{C}(U,\xi)W) + \eta(V)\eta(\bar{C}(U,Y)W) \\
&+\eta(W)\eta(\bar{C}(U,V)Y)]. \qquad (69)
\end{aligned}$$

Putting $Y = U$, we get

$$[L_{\bar{C}} - \alpha^2][g(U,U)\eta(\bar{C}(\xi,V)W) + g(U,V)\eta(\bar{C}(U,\xi)W)] + \alpha\eta(V)\eta(\bar{C}(\phi^2U,V)W) = 0. \qquad (70)$$

Replacing $U = \xi$ in (68), we obtain

$$[L_{\bar{C}} - \alpha^2]\eta(\bar{C}(\xi,V)W) = 0. \qquad (71)$$

The formula (69) gives either $\eta(\bar{C}(\xi,V)W) = 0$ or $L_{\bar{C}} - \alpha^2 = 0$.

Now $L_{\bar{C}} - \alpha^2 \neq 0$, then $\eta(\bar{C}(\xi,V)W) = 0$, then we have M is conformally flat and which gives

$$\bar{S}(V,W) = Ag(V,W) + B\eta(V)\eta(W),$$

where

$$A = [\alpha^2 - \frac{(n-1)(\alpha^2-\alpha)}{n-2} + \frac{r}{(n-1)(n-2)}](n-2)$$

and

$$B = [\alpha^2 - \frac{2(n-1)(\alpha^2-\alpha)}{n-2} + \frac{r}{(n-1)(n-2)}](n-2),$$

which shows that M is an $\eta-$Einstein manifold. Now if $\eta(\bar{C}(\xi,V)W) \neq 0$, then $L_{\bar{C}} = \alpha^2$. □

Theorem 8.2 *LetM be an n dimensional Lorentzian $\alpha-$Sasakian manifold. If M is pseudosymmetric then eitherM is a space of constant curvature and $\alpha g(X,Y) = \eta(X)\eta(Y)$, for $\alpha \neq 0$ or $L_{\bar{R}} = \alpha^2$, for $X, Y \in \chi(M)$.*

Proof Let M be a pseudosymmetric Lorentzian $\alpha-$Sasakian manifold and $X, Y, U, V, W \in \chi(M)$. Then using (61) and (62) in (60), we have

$$\begin{aligned}&\bar{R}(X,Y)(\bar{R}(U,V)W) - \bar{R}(\bar{R}(X,Y)U,V)W\\ &-\bar{R}(U,\bar{R}(X,Y)V)W - \bar{R}(U,V)(R(X,Y)W)\\ &= L_{\bar{R}}[(X \wedge Y)(\bar{R}(U,V)W) - \bar{R}((X \wedge Y)U,V)W\\ &-\bar{R}(U,(X \wedge Y)V)W - \bar{R}(U,V)((X \wedge Y)W)].\end{aligned} \tag{72}$$

Replacing X with ξ in (70) we obtain

$$\begin{aligned}&\bar{R}(\xi,Y)(\bar{R}(U,V)W) - \bar{R}(\bar{R}(\xi,Y)U,V)W\\ &-\bar{R}(U,\bar{R}(\xi,Y)V)W - \bar{R}(U,V)(R(\xi,Y)W)\\ &= L_{\bar{R}}[(\xi \wedge Y)(\bar{R}(U,V)W) - \bar{R}((\xi \wedge Y)U,V)W\\ &-\bar{R}(U,(\xi \wedge Y)V)W - \bar{R}(U,V)((\xi \wedge Y)W)].\end{aligned} \tag{73}$$

Using (2), (38) in (71) and taking inner product of (71) with ξ, we get

$$\begin{aligned}&\alpha^2[-\bar{R}(U,V,W,Y) - \eta(\bar{R}(U,V)W)\eta(Y) - g(Y,U)\eta(\bar{R}(\xi,V)W)\\ &+\eta(U)\eta(\bar{R}(Y,V)W) - g(Y,V)\eta(\bar{R}(U,\xi)W) + \eta(V)\eta(\bar{R}(U,Y)W)\\ &+\eta(W)\eta(\bar{R}(U,V)Y)] - \alpha[\eta(U)\eta(\bar{R}(\phi^2Y,V)W)\\ &+\eta(V)\eta(\bar{R}(U,\phi^2Y)W) + \eta(W)\eta(\bar{R}(U,V)\phi^2Y)]\\ &= L_{\bar{R}}[-\bar{R}(Y,U,V,W) - \eta(Y)\eta(\bar{R}(U,V)W) - g(Y,U)\eta(\bar{R}(\xi,V)W)\\ &+\eta(U)\eta(\bar{R}(Y,V)W) - g(Y,V)\eta(\bar{R}(U,\xi)W) + \eta(V)\eta(\bar{R}(U,Y)W)\\ &+\eta(W)\eta(\bar{R}(U,V)Y)].\end{aligned}$$

Putting $Y = U$, we get

$$[L_{\bar{R}} - \alpha^2][g(U,U)\eta(\bar{R}(\xi,V)W) + g(U,V)\eta(\bar{R}(U,\xi)W)] + \alpha\eta(V)\eta(\bar{R}(\phi^2U,V)W) = 0. \tag{74}$$

Replacing $U = \xi$ in (72), we obtain

$$[L_{\bar{R}} - \alpha^2]\eta(\bar{R}(\xi,V)W) = 0. \tag{75}$$

The formula (73) gives either $\eta(\bar{R}(\xi,V)W) = 0$ or $L_{\bar{R}} - \alpha^2 = 0$. Now $L_{\bar{R}} - \alpha^2 \neq 0$, then $\eta(\bar{R}(\xi,V)W) = 0$. We have M is a space of constant curvature and $\eta(\bar{R}(\xi,V)W) = 0$ gives $\alpha g(V,W) = \eta(X)\eta(Y)$ for $\alpha \neq 0$. If $\eta(\bar{R}(\xi,V)W) \neq 0$, then we have $L_{\bar{R}} = \alpha^2$. □

§9. Examples

Let us consider the three dimensional manifold $M = \{(x_1,x_2,x_3) \in R^3 : x_1,x_2,x_3 \in R\}$, where (x_1,x_2,x_3) are the standard coordinates of R^3. We consider the vector fields

$$e_1 = e^{x_3}\frac{\partial}{\partial x_2}, \quad e_2 = e^{x_3}(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}) \ \text{ and } \ e_3 = \alpha\frac{\partial}{\partial x_3},$$

where α is a constant.

Clearly, $\{e_1, e_2, e_3\}$ is a set of linearly independent vectors for each point of M and hence a basis of $\chi(M)$. The Lorentzian metric g is defined by

$$g(e_1, e_2) = g(e_2, e_3) = g(e_1, e_3) = 0,$$
$$g(e_1, e_1) = g(e_2, e_2) = g(e_3, e_3) = -1.$$

Then the form of metric becomes

$$g = \frac{1}{(e^{x_3})^2}(dx_2)^2 - \frac{1}{\alpha^2}(dx_3)^2,$$

which is a Lorentzian metric.

Let η be the $1-$form defined by $\eta(Z) = g(Z, e_3)$ for any $Z \in \chi(M)$ and the $(1,1)-$tensor field ϕ is defined by

$$\phi e_1 = -e_1, \quad \phi e_2 = -e_2, \quad \phi e_3 = 0.$$

From the linearity of ϕ and g, we have

$$\eta(e_3) = -1,$$
$$\phi^2(X) = X + \eta(X)e_3 \quad \text{and}$$
$$g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y)$$

for any $X \in \chi(M)$. Then for $e_3 = \xi$, the structure (ϕ, ξ, η, g) defines a Lorentzian paracontact structure on M.

Let ∇ be the Levi-Civita connection with respect to the Lorentzian metric g. Then we have

$$[e_1, e_2] = 0, \quad [e_1, e_3] = -\alpha e_1, \quad [e_2, e_3] = -\alpha e_2.$$

Koszul's formula is defined by

$$\begin{aligned} 2g(\nabla_X Y, Z) &= Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) \\ &\quad -g(X, [Y, Z]) - g(Y, [X, Z]) + g(Z, [X, Y]). \end{aligned}$$

Then from above formula we can calculate the followings,

$$\nabla_{e_1} e_1 = -\alpha e_3, \quad \nabla_{e_1} e_2 = 0, \quad \nabla_{e_1} e_3 = -\alpha e_1,$$
$$\nabla_{e_2} e_1 = 0, \quad \nabla_{e_2} e_2 = -\alpha e_3, \quad \nabla_{e_2} e_3 = -\alpha e_2,$$
$$\nabla_{e_3} e_1 = 0, \quad \nabla_{e_3} e_2 = 0, \quad \nabla_{e_3} e_3 = 0.$$

Hence the structure (ϕ, ξ, η, g) is a Lorentzian $\alpha-$Sasakian manifold [7].

Using (22), we find D, the quarter-symmetric metric connection on M following:

$$D_{e_1} e_1 = -\alpha e_3, \quad D_{e_1} e_2 = 0, \quad D_{e_1} e_3 = e_1(1 - \alpha),$$
$$D_{e_2} e_1 = 0, \quad D_{e_2} e_2 = -\alpha e_3, \quad D_{e_2} e_3 = e_2(1 - \alpha),$$
$$D_{e_3} e_1 = 0, \quad D_{e_3} e_2 = 0, \quad D_{e_3} e_3 = 0.$$

Using (23), the torson tensor $\bar{T}$, with respect to quarter-symmetric metric connection D as follows:

$$\bar{T}(e_i, e_i) = 0, \ \forall i = 1, 2, 3$$
$$\bar{T}(e_1, e_2) = 0, \ \bar{T}(e_1, e_3) = e_1, \ \bar{T}(e_2, e_3) = e_2.$$

Also $(D_{e_1} g)(e_2, e_3) = (D_{e_2} g)(e_3, e_1) = (D_{e_3} g)(e_1, e_2) = 0$. Thus M is Lorentzian $\alpha-$Sasakian manifold with quarter-symmetric metric connection D.

Now we calculate curvature tensor $\bar{R}$ and Ricci tensors $\bar{S}$ as follows:

$$\bar{R}(e_1, e_2)e_3 = 0, \quad \bar{R}(e_1, e_3)e_3 = -(\alpha^2 - \alpha)e_1,$$
$$\bar{R}(e_3, e_2)e_2 = \alpha^2 e_3, \quad \bar{R}(e_3, e_1)e_1 = \alpha^2 e_3,$$
$$\bar{R}(e_2, e_1)e_1 = (\alpha^2 - \alpha)e_2, \quad \bar{R}(e_2, e_3)e_3 = -\alpha^2 e_2,$$
$$\bar{R}(e_1, e_2)e_2 = (\alpha^2 - \alpha)e_1.$$
$$\bar{S}(e_1, e_1) = \bar{S}(e_2, e_2) = -\alpha \text{ and } \bar{S}(e_3, e_3) = -2\alpha^2 + (n-1)\alpha.$$

Again using (2), we get

$$(e_1, e_2)e_3 = 0, \quad (e_i \wedge e_i)e_j = 0, \quad \forall i, j = 1, 2, 3,$$
$$(e_1 \wedge e_2)e_2 = (e_1 \wedge e_3)e_3 = -e_1, \quad (e_2 \wedge e_1)e_1 = (e_2 \wedge e_3)e_3 = -e_2,$$
$$(e_3 \wedge e_2)e_2 = (e_3 \wedge e_1)e_1 = -e_3.$$

Now,

$$\bar{R}(e_1, e_2)(\bar{R}(e_3, e_1)e_2) = 0, \quad \bar{R}(\bar{R}(e_1, e_2)e_3, e_1)e_2 = 0,$$
$$\bar{R}(e_3, \bar{R}(e_1, e_2)e_1)e_2 = -\alpha^2(\alpha^2 - \alpha)e_3,$$
$$(\bar{R}(e_3, e_1)(\bar{R}(e_1, e_2)e_2) = \alpha^2(\alpha^2 - \alpha)e_3.$$

Therefore, $(\bar{R}(e_1, e_2).\bar{R})(e_3, e_1, e_2) = 0$.

Again,

$$(e_1 \wedge e_2)(\bar{R}(e_3, e_1)e_2) = 0, \quad \bar{R}((e_1 \wedge e_2)e_3, e_1)e_2 = 0,$$
$$\bar{R}(e_3, (e_1 \wedge e_2)e_1)e_2 = \alpha^2 e_3, \quad \bar{R}(e_3, e_1)((e_1 \wedge e_2)e_2) = -\alpha^2 e_3.$$

Then $((e_1, e_2).\bar{R})(e_3, e_1, e_2) = 0$. Thus $(\bar{R}(e_1, e_2).\bar{R})(e_3, e_1, e_2) = L_{\bar{R}}[((e_1, e_2).\bar{R})(e_3, e_1, e_2)]$ for any function $= L_{\bar{R}} \in C^\infty(M)$.

Similarly, any combination of e_1, e_2 and e_3 we can show (60). Hence M is a pseudosymmetric Lorentzian $\alpha-$Sasakian manifold with quarter-symmetric metric connection.

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