International J.Math. Combin. Vol.1(2012), 31-40

On Square Difference Graphs

Ajitha V.

(Department of Mathematics, M. G. College, Iritty-670703, India)

K.L.Princy

(Department of Mathematics, Bharata Matha College, Thrikkakara, Kochi-682021, India)

V.Lokesha

(Department of Mathematics, Acharya Institute of Technology, Bengaluru-90, India)

P.S.Ranjini

(Department of Mathematics, Don Bosco Institute of Technology, Bengaluru-61, India)

E-mail: mohanrajvm@yahoo.co.in, srjayarose@yahoo.co.in, lokeshav@acharya.ac.in, ranjini_p_s@yahoo.com

Abstract: In graph theory number labeling problems play vital role. Let $G = (V, E)$ be a (p, q)-graph with vertex set V and edge set E. Let f be a vertex valued bijective function from $V(G) \to \{0, 1, ..., p-1\}$. An edge valued function f^* can be defined on G as a function of squares of vertex values. Graphs which satisfy the injectivity of this type of edge valued functions are called square graphs. Square graphs have two major divisions: they are square sum graphs and square difference graphs. In this paper we concentrate on square difference or SD graphs. An edge labeling f^* on $E(G)$ can be defined as follows. $f^*(uv) = \left|(f(u))^2 - (f(v))^2\right|$ for every uv in $E(G)$. If f^* is injective, then the labeling is said to be a SD labeling. A graph which satisfies SD labeling is known as a SD graph. We illuminate some of the results on number theory into the structure of SD graphs. Also, established some classes of SD graphs and established that every graph can be embedded into a SD graph.

Key Words: SD labeling, SD graph, strongly SD graph, perfect SD graph.

AMS(2010): 05C20

§1. Introduction

The research in graph enumeration and graph labeling started way back in 1857 by Arthur Cayley. Graph labeling and enumeration finds the application in chemical graph theory, social networking and computer networking and channel assignment problem. Abundant literature exists as of today concerning the structure of graphs admitting a variety of functions assigning real numbers to their elements so that certain given conditions are satisfied. Here we are

[1]Received September 9, 2011. Accepted February 16, 2012.

interested the study of vertex functions $f : V(G) \to \{0, 1, \cdots, p-1\}$ for which an edge valued injective function f^* can be defined on G as function of squares of vertex values. Graphs which satisfy this type of labeling are called square graphs. Square graphs have two major divisions: they are square sum graphs and square difference graphs.

In this paper, we concentrate on square difference or SD graphs. This new type of labeling of graphs is closely related to the equation $x^2 - y^2 = n$. It is important to note that certain numbers like $6, 10, 14$ etc., cannot be written as the difference of two squares. Hence it is very interesting to study those graphs which takes the first consecutive numbers that can be expressed as the difference of two squares. Here, consider only a finite undirected graph without loops or multiple edges. Terms not specifically defined in this paper may be found in Harrary (1969), [6] and all the number theoretic results used here found in [3,4] and [2,8].

Here we recall some results of number theory, which are essential for our study.

Definition 1.1 *An integer is said to be representable if it can be represent as difference of two squares.*

Theorem 1.2([7]) *The product of any number of representable integers is also representable.*

Theorem 1.3([7]) *Every square integer is of the form following*

(i) $4q$ *or* $4q + 1$;

(ii) $5q$, $5q + 1$ *or* $5q - 1$.

Theorem 1.4([7]) *If* $n = x^2 - y^2$, *then* $n \equiv 0, 1, 3 (\mathrm{mod} 4)$.

Corollary 1.5([7]) *An odd number is a difference of two successive squares.*

Theorem 1.6([7]) *The difference of squares of consecutive numbers is equal to the sum of the numbers.*

§2. SD Graphs

Definition 2.1 *Let* $G = (V, E)$ *be a* (p, q)*-graph with vertex set* V *and edge set* E. *Let* f *be a vertex valued bijective function from* $V(G) \to \{0, 1, \cdots, p-1\}$. *An edge valued function* f^* *can be defined on* G *as* $f^*(uv) = \left|(f(u))^2 - (f(v))^2\right|$ *for every* uv *in* $E(G)$. *If* f^* *is injective, then the labeling is said to be a* SD *labeling. A graph which satisfies* SD *labeling is known as an* SD *graph.*

Example 2.2 An example of SD graphs is given below.

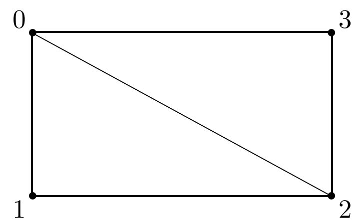

Fig.1

The following are some simple observations obtained immediately from the definition of SD graph.

Observation 2.3 *For every edge* $e = uv \in E(G)$ *then* $f^*(e) \equiv 0, 1, 3(\text{mod}4)$.

Observation 2.4 *If* $e = uv \in E(G)$ *with* $f(u) = 0$ *then* $f^*(e) \equiv 0, 1(\text{mod}4)$.

Observation 2.5 *For a SD graph the product of edge values is a representable number.*

Observation 2.6 *Let* G *be a* (p, q)*-graph with a square difference labeling* f *then* $[f(u)]^2$ *occurs* $d(u)$ *times in the sum* $\sum f^*(e)$. *Take the* p *vertices of* G *as* $u_1, u_2, ...u_p$ *and assign values to them in such a way that* $f(u_1) < f(u_2) < ... < f(u_p)$, *then for all* $i = 0,\ 1, \cdots,\ p-1$ *we have* $\sum f^*(e) = d(u_p)\,[f(u_p)]^2 + \sum\limits_{i=1}^{p-1}(k_i - l_i)\,[f(u_i)]^2$, *where for each edge* u_iu_j,

$k_i =$ number of vertices $u_j$with $f(u_i) > f(u_j), i \neq j$

$l_i =$ number of vertices $u_j$with $f(u_i) < f(u_j), i \neq j$

In particular if $k_i = l_i$ *for all* i, *we get the above result as* $\sum f^*(e) = d(u_p)\,[f(u_p)]^2$.

Theorem 2.7 *Let* G *be a connected* SD *graph with an* SD *labeling* f. *Then* $f^*(e) \equiv 1(mod\ 2)$ *for at least one edge* $e \in E(G)$. *Further if* $f^*(e) \equiv 1(mod\ 2)$, $\forall e \in E(G)$, *then* G *is bipartite.*

Proof Let $X = \{u : f(u) \text{ is even }\}$ and $Y = \{v : f(v) \text{ is odd }\}$. Since G is connected there exists at least one edge $e = uv$ such that $u \in X$ and $v \in Y$. Hence $f^*(e) \equiv 1(mod\ 2)$. If $f^*(e) \equiv 1(mod\ 2)$, $\forall e \in E(G)$, it follows that $f(u)$ and $f(v)$ are of opposite parity and X and Y form a bipartition of G and G is a bipartite graph. □

§3. Some classes of SD graphs

Theorem 3.1 *The graph* $G = K_2 + mK_1$ *is an* SD *graph.*

Proof Let $V(G) = \{u_1, u_2, ...u_{m+2}\}$ where $V(K_2) = \{u_1, u_2\}$. Define $f : V(G) \to \{0, 1, ..., m+1\}$ by $f(u_i) = i - 1, 1 \leqslant i \leqslant m+2$. Clearly, the induced function f^* is injective, for if $f^*(u_1u_i) = f^*(u_2u_j)$ then, $|[f(u_1)]^2 - [f(u_i)]^2| = [|f(u_2)]^2 - [f(u_j)]^2|$. Since $f(u_1) = 0$ and $f(u_2) = 1$, we get $(f(u_i))^2 = (f(u_j))^2 - 1$ so that either $f(u_i) = 0$ or $f(u_j) = 0$, which is a contradiction. Hence f^* is injective and G is an SD graph. □

Theorem 3.2 *Every star is an SD graph.*

Proof Let $V(K_{1,n}) = \{u_1, u_2, \cdots, u_n, u_{n+1}\}$ where $e_i = u_1u_i$ for $2 \leqslant i \leqslant n$. Define $f : V(K_{1n}) \to \{0, 1, \cdots, n-1\}$ as $f(u_1) = 0$ and $f(u_i) = i - 1$. Then $f^*(E(K_{1,n})) = \{1^2, 2^2, \cdots, (n-1)^2\}$ and hence f^* is infective and f is a SD labeling on K_{1n}. Hence every star is an SD graph. □

Theorem 3.3 *Every path is an SD graph.*

Proof Let $P_n = (u_1, u_2, \cdots, u_n)$ where $e_i = u_iu_{i+1}$ for $1 \leqslant i \leqslant n$. Define $f : V(G) \to \{0, 1, \cdots, n-1\}$ as $f(u_i) = i - 1$. Then by the Theorem **??** $f^*(E(P_n)) = \{1, 3, \cdots, 2n-3\}$ and hence f^* is infective and f is a SD labeling on P_n. Hence every path is an SD graph. □

Theorem 3.4 *A complete graph K_n is SD if and only if $n \leqslant 5$.*

Proof The SD labeling of the complete graph K_n for $n \leqslant 5$ is given in figures 2 and 5. Further since $5^2 - 4^2 = 3^2 - 0^2$ it follows that $K_n,\ n \geqslant 6$ is not an SD graph. □

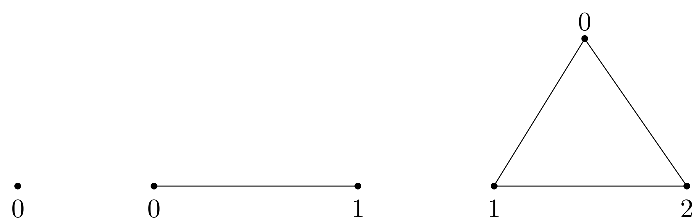

Fig.2

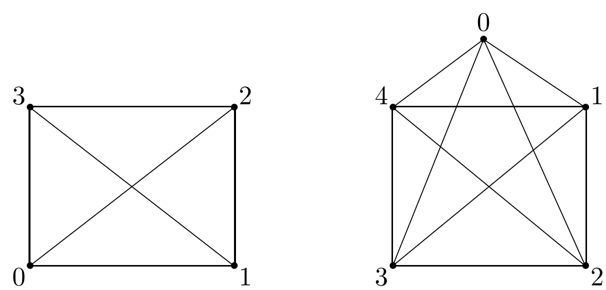

Fig.3

Theorem 3.5 *Every cycle is an SD graph.*

Proof Let $C_n = (u_1, u_2, \cdots, u_n)$ where $e_i = u_iu_{i+1}$ for $1 \leqslant i \leqslant n-1$ and $e_n = u_nu_1$. Define $f : V(G) \to \{0, 1, \cdots, n-1\}$ as $f(u_i) = i - 1$ for $1 \leqslant i \leqslant n$. Also $f^*(E(C_n)) = \{1, 3, \cdots, 2n-3, (n-1)^2\}$. If n is odd then $(n-1)^2$ is even, so f^* is infective and f is a SD labeling on C_n. If n is even, since $(n-1)^2 > 2n-3$ for $n \geqslant 3$, then also f^* is injective and f is a SD labeling on C_n. Hence cycles are SD graphs. □

Theorem 3.6 *Every friendship graph $C_3^{(n)}$ is an SD graph.*

Proof Let $G_1, G_2, \cdots G_n$ be the n copies of C_3, all concatenated at exactly one vertex say, z. Let $G_i = (z, u_{i1}, u_{i2})$, $1 \leqslant i \leqslant n$. Define $f : V(C_3^{(n)}) \to \{0, 1, \cdots, 2n\}$ as $f(z) = 0, f(u_{i1}) = i$ and $f(u_{i2}) = n + i$.
Then $f^*\left(E(C_3^{(n)})\right) = \left\{1^2, 2^2, \cdots, (2n)^2, n^2 + 2n, n^2 + 4n, n^2 + 6n, \cdots, 3n^2\right\}$ and the induced edge function f^* is injective and f is an SD labeling on $C_3^{(n)}$. Hence the proof. □

Definition 3.7([1]) *A cycle-cactus is such a graph that consisting of n copies of C_k, $k \geq 3$ concatenated at exactly one vertex is denoted as $C_k^{(n)}$.*

Theorem 3.8 *A complete bipartite graphs $K_{m,n}$ is SD if $m \leq 4$ for any integer $n \geq 1$.*

Proof Let $X = \{x_1, x_2, \cdots, x_m\}$,$Y = \{y_1, y_2, \cdots, y_n\}$ be the partition of $K_{m,n}$. Define a vertex labeling as follows:

(1) If $m = 1$, then $f(x_1) = 0$ and $f(y_i) = i, 1 \leqslant i \leqslant n$;

(2) If $m = 2$, then $f(x_1) = 0, f(x_2) = 1$ and $f(y_i) = i + 1, 1 \leqslant i \leqslant n$;

(3) If $m = 3$, then $f(x_1) = 0, f(x_2) = 1, f(x_3) = 2$ and $f(y_i) = i + 2, 1 \leqslant i \leqslant n$;

(4) If $m = 4$, then $f(x_1) = 0, f(x_2) = 1, f(x_3) = 3, f(x_4) = 5$ and $f(y_i) = 2i$, if $i = 1, 2, 3$, $f(y_i) = 2i - 1$, if $i = 4$, and $f(y_i) = i + 3$, if $i > 4$.

It is easy to verify that the edge valued function f^* is injective on $E(K_{m,n})$ and hence the theorem. □

Theorem 3.9 *Every triangle snake is an SD graph.*

Proof Let G be a triangle snake with $2n + 1$ vertices and let the vertex set $V(G)$ be $\{u_1, u_2, \cdots, u_n, v_1, v_2, \cdots, v_{n+1}\}$ and the edge set $E(G)$ be $\{u_iv_i, u_iv_{i+1}, v_iv_{i+1}\}$, where $1 \leqslant i \leqslant n$. We can define a vertex labeling $f : V(G) \to \{0, 1, 2, \cdots, 2n\}$ as $f(u_i) = 2i - 1, 1 \leqslant i \leqslant n$ and $f(v_i) = 2(i - 1), 1 \leqslant i \leqslant n + 1$. Then the corresponding edge function f^* is injective and hence all triangle snakes are SD graphs. □

Example 3.10 The following example gives an illustration for the above theorem.

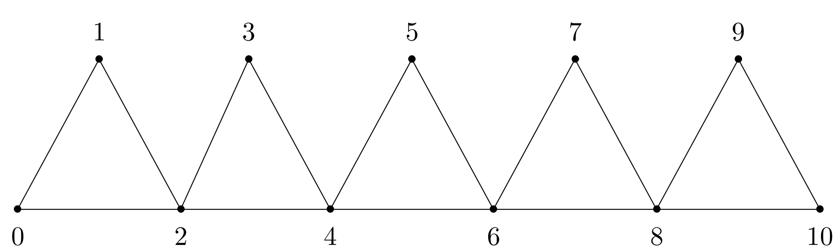

Fig.4

Observation 3.11 P_4 is a connected SD graph with prime edge labels.

Proof A prime number is a representable number if and only if it can be represent as difference of two consecutive integers. Hence there exist a SD labelling on P_4 with prime edge values as follows

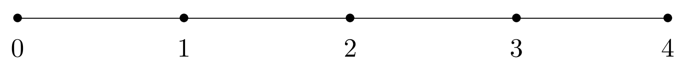

Fig.5

This completes the proof. □

Corollary 3.12 *A connected* 5 *vertexed graph with prime edge values is isomorphic to* P_4.

Corollary 3.13 *There is not exist a connected* (p, q) - SD *graph with prime edge values if* $p > 5$.

Conjecture 3.14 *Every tree is an* SD *graph.*

Conjecture 3.15 *Every cycle-cactus* $C_k^{(n)}$ *is an* SD *graph.*

Conjecture 3.16 *Every complete bipartite graph is an* SD *graph.*

§4. Strongly SD Graphs

Definition 4.1 *An* SD *graph is said to be a* strongly SD *if the edge values are consecutive representable numbers.*

In other words, an SD graph $G = (V, E)$ is said to be a strongly SD if $f^*(E(G))$ consists the first q consecutive numbers of the form $a^2 - b^2, a \leqslant p-1, b \leqslant p-1, a \neq b$ and the corresponding labeling is said to be a strongly SD labeling of G.

Example 4.2 An example of a strongly SD graph is given below.

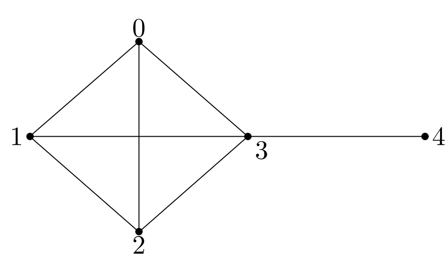

Fig.6

Theorem 4.3 *Every strongly* SD *graph except* K_1, K_2 *and* $K_{1,2}$ *contain at least one triangle.*

Proof Clearly K_1, K_2 and $K_{1,2}$ are triangle free strongly SD graphs. For a strongly square difference graph with three edges, the possible edge values are $1, 3$ and 4. To obtain the edge value 3, the vertices which label 1 and 2 should be adjacent. Similarly to obtain the edge value 4, the vertices which label 0 and 2 should be adjacent. In this case the graph should contains a triangle $(uvwu)$ with $f(u) = 0, f(v) = 1$ and $f(w) = 2$. □

Corollary 4.4 *All cycles* $C_n, n \geqslant 4$ *are not strongly* SD *graphs.*

Corollary 4.5 *A complete bipartite graph* $K_{m,n}$ *is a strongly* SD *graph if and only if* $m = 1,\ n \leqslant 2$.

Theorem 4.6 *A unicyclic graph is strongly* SD *if and only if it is either* C_3 *or* C_3 *with one pendant edge or* C_3 *with a path of length* 2.

Proof Clearly, C_3, C_3 with one pendant edge and are strongly SD graphs. Suppose G is a unicyclic graph which is strongly SD. Then $f^*(E(G)) = \{1, 3, 4, 5, 7, 8, \cdots\}$.

If $f^*(E(G)) = \{1, 3, 4\}$, then $G \cong C_3$;

If $f^*(E(G)) = \{1, 3, 4, 5\}$, then G is isomorphic to C_3 along with one pendant edge;

If $f^*(E(G)) = \{1, 3, 4, 5, 7\}$, then G is isomorphic to C_3 along with a path of length 2;

If $f^*(E(G)) = \{1, 3, 4, 5, 7, 8\}$, then to obtain the edge value 8, the vertices with labels 3 and 1 should be adjacent. In this case one more triangle will be formed, a contradiction since G is unicyclic.

Therefore the unicyclic graphs which admits a strongly SD labeling are either C_3 or C_3 with one pendant edge or C_3 with a path of length 2. □

Conjecture 4.7 *Every cycle* $C_n, n \geqslant 4$ *can be embedded as an induced subgraph of a strongly* SD *graph.*

Problem 4.8 *Find the number of strongly* SD *graphs for a given number of edges.*

Problem 4.9 *Characterize the strongly* SD *graphs.*

By the definition of SD graphs it is clear that a big family of graphs are not SD. Hence on SD graphs embedding theorems have an important role to play. In the following section we proved an embedding theorem.

§5. Embeddings on SD Graphs

Theorem 5.1 *Every* $(p, q)-$ *graph* G *can be embedded into a connected* SD *graph.*

Proof Let G be a graph with vertex set $V(G) = \{u_1, u_2, \cdots, u_p\}$. We shall establish an embedding of G in H, where H is a graph with $|V(H)| = 5^{p-1} + 1$ and $|E(H)| = 5^{p-1} + q - p$. Label the vertices of G by $f(u_1) = 0, f(u_2) = 1$ and $f(u_i) = 5^{i-1}$, $3 \leqslant i \leqslant p$. Let $v_1,\ v_2, \cdots,\ v_n$ be the isolated vertices where $n = 5^{p-1} + 1 - p$. Let $X = \left\{1, 2, \cdots, 5^{p-1} - 1\right\}$. Label the vertices v_i with the numbers from the set $X - \{5, 5^2, \cdots, 5^{p-1}\}$. The graph H obtained from G as follows. Join $u_1$to v_k if $f(v_k)$ is odd for $1 \leqslant k \leqslant n$ and $u_2$to v_l if $f(v_l)$ is even for $1 \leqslant l \leqslant n$. Clearly f is a bijection from $V(H) \to \{0, 1, \cdots, 5^{p-1}\}$. Now we prove that the induced edge labeling f^* is injective. Assume $f^*(e_1) = f^*(e_2)$ for some $e_{1,}e_2 \in E(G)$. Since f is injective it follows that e_1 and e_2 are non adjacent. Suppose $e_1 = u_k u_l$ and $e_2 = u_i u_j$ where $1 \leqslant i \leqslant j, k \leqslant l \leqslant p$. Then $(5^j)^2 - (5^l)^2 = (5^l)^2 + (5^k)^2$ from which we obtain $(5^i)^2 + (5^k)^2 = (5^l)^2 + (5^j)^2$. If $u_i = u_1$ then $(5^k)^2 = (5^l)^2 + (5^j)^2$ which is a contradiction. Otherwise dividing both sides of the equation by $(5a)^2$, where $a = \min(i, j, k, l)$, we get an edge where one side is congruent to 1(mod)5 and the other side is congruent to 0(mod)5, which is a contradiction. If $e_1 = u_k u_l$ and $e_2 = u_1 v_i$ then $f^*(e_1) = f^*(e_2) \Rightarrow (5^l)^2 - (5^k)^2 = (f(v_i))^2$ if $f(v_i)$ is odd and $(5^l)^2 - (5^k)^2 = (f(v_i))^2 - 1$ if $f(v_i)$ is even. In both case we get contradictions. Hence the induced function f^* is a SD labeling and the graph H is a SD graph which contains G as a subgraph. □

§6. Perfect SD Graphs

Definition 6.1 *An SD graph is said to be a perfect SD graph if the edge values are consecutive perfect squares. The corresponding SD labelling is said to be a perfect SD labelling.*

Example 6.2

Fig.7

Observation 6.3 *Every star is a perfect SD graph.*

Observation 6.4 *The cycles C_3, C_4 and C_5 are not perfect SD graphs.*

Observation 6.5 *Every friendship graph $C_3^{(n)}$ is not a perfect SD graph.*

Conjecture 6.6 *Every cycle is not a perfect SD graph.*

Theorem 6.7 *These graphs C_3, C_4 and C_5 can be embedded into perfect SD graph.*

Proof The following embeddings give the proof. □

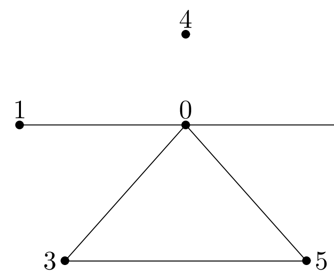

Fig.8 **The embedding of** C_3

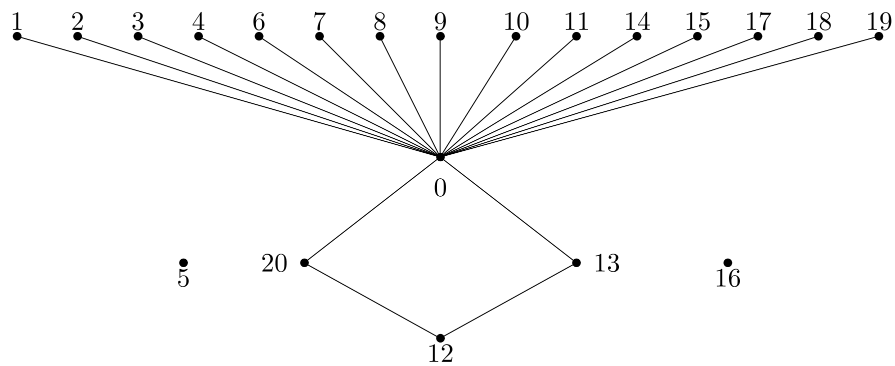

Fig.9 **The embedding of** C_4

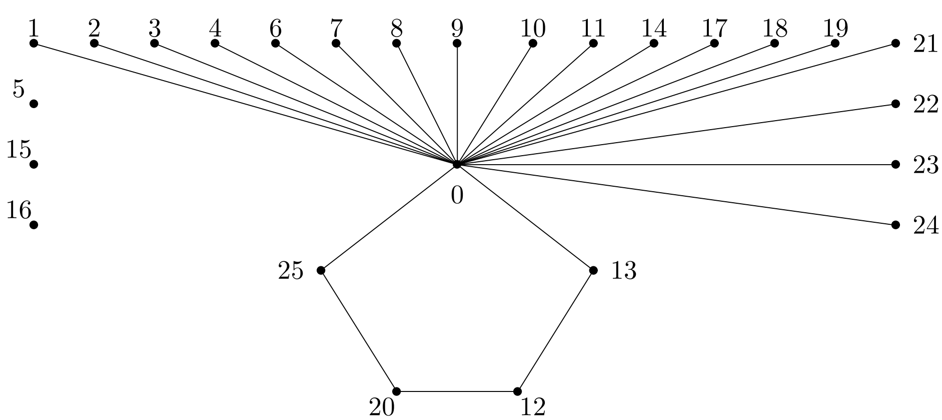

Fig.10 **The embedding of** C_5

Conjecture 6.8 *Every cycle can be embedded into a perfect SD graph.*

Theorem 6.9 *The friendship graph* $C_3^{(n)}$ *can be embedded into a perfect SD graph.*

Proof Let $G_1, G_2, \cdots G_n$ be the n copies of C_3, all concatenated at exactly one vertex say, z. Let $G_i = (z, u_{i1}, u_{i2}), 1 \leqslant i \leqslant n$. Define $f : V(C_3^{(n)}) \to \{0, 1, \cdots, 2n\}$ as $f(z) = 0$, $f(u_{i1}) = 2i^2+2i+1$ and $f(u_{i2}) = 2i+1$. Introduce $2n^2+1$ new vertices V_k where $1 \leqslant k \leqslant 2n^2+1$. Join the vertex z to V_k where $k \neq 2i(i+1)$, $1 \leqslant i \leqslant n$.
Then f is a perfect SD labeling on the embedded graph H and hence H is a perfect SD graph. □

Problem 6.10 *Every cycle-cactus $C_k^{(n)}$ can be embedded into a perfect SD graph.*

Acknowledgement

The first author's participation in this research work is supported under the minor project No.MRP(S)- 432/08-09 /KLMG 039 / UGC-SWRO of the University Grants Commission (UGC), awarded to her, which is gratefully acknowledged. The second author is indebted to the University Grants Commission (UGC) for enabling her to participate in this research work under the minor project No.MRP(S)-069/07-08/KLKA016/UGC-SWRO.

References

[1] B.D.Acharya, K.A.Germina and Ajitha V, Multiplicatively indexable graphs, *Labelings of Discrete Structures and Applications*, Eds. B.D.Acharya, S.Arumugam and A. Rosa, Narosa Publishing House, New Delhi, 29-40.

[2] S.Arumugam, K.A.Germina and Ajitha V, On square sum graphs, *AKCE International Journal of Graphs and Combinatorics*, 1-10 (2008).

[3] David M.Burton, *Elementary Number Theory*,(Sixth edition), TATA McGRAW-HILL(2006).

[4] J.A.Gallian, A dynamic survey of graph labelling, *The Electronic J. Combinatorics*, **5**(2005), #DS6.

[5] M.C.Golumbic, *Algorithemic Graph Theory and Perfect Graphs*, Academic press, New York, 1980.

[6] F.Harary, *Graph Theory*, Addison Wesley, Reading, Massachusetts, 1969.

[7] S.G.Telang, *Number Theory*, (Sixth edition), TATA McGRAW-HILL(1996).

[8] M.A.Rajan, V.Lokesha and K.M.Niranjan, On study of vertex labeling graph operations, *J. Sci. Res.*, 3 (2), 291-301 (2011).