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On Finsler Spaces with Unified Main Scalar (LC) of the Form $L^2C^2 = f(y) + g(x)$

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Abstract: In the year 1979, M. Matsumoto was studying Finsler spaces with vanishing T-tensor and come to know that for such Finsler spaces L^2C^2 is a function of x only. Later on in the year 1980, Matsumoto with Numata concluded that the condition $L^2C^2 = f(x)$ is not sufficient for vanishing of T-tensor. F. Ikeda in the year 1984, studied Finsler spaces whose L^2C^2 is function of x in detail. In the present paper we shall discuss a Finsler space for which L^2C^2 is a function of x and y ($y^i = \dot{x}^i$) in the form $L^2C^2 = f(y) + g(x)$.

Key Words: Unified main scalar, Berwald space, Landsberg space, T-tensor, C-Reducible Finsler space, Two-dimensional Finsler space.

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§1. Introduction

In the paper [7] it has been shown that if a Finsler space satisfies the T-condition, then the unified Main Scalar L^2C^2 of the Finsler space M^n is reduced to the function of the position only (i.e. $L^2C^2 = f(x)$), where L is the fundamental function and C^2 is the square of length of the torsion vector C_i. F. Ikeda in the paper [1] has worked out, whether the condition $L^2C^2 = f(x)$ is equivalent to the T-condition or not and considered the properties of such Finsler spaces in detail.

In the present paper, we shall study the T-tensor of such a Finsler space with the condition $L^2C^2 = f(y) + g(x)$. As F. Ikeda in the paper [1], we have also obtained the condition for such a Finsler space ($L^2C^2 = f(y) + g(x)$) to be a Landsberg or Berwald space so that a Landsberg space (resp. Berwald space) satisfying the condition $L^2C^2 = f(y) + g(x)$ reduces to a Berwald space.

The terminology and notation are referred to the Matsumoto's monograph [5].

§2. The Condition $L^2C^2 = f(y) + g(x)$

Let l_i, h_{ij} and C_{ijk} denote the unit vector (i.e. $l_i = \dfrac{y_i}{L}$), the angular metric tensor and the (h)hv-torsion tensor (the Cartan torsion tensor), respectively. The T-tensor T_{ijkl} is defined by

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[5] [pp. 188, equ. (28.20)] $T_{ijkl} = LC_{ijk}|_l + C_{ijk}l_l + C_{ijl}l_k + C_{ilk}l_j + C_{ljk}l_i$ and the torsion vector C_i is given by $C_i = g^{jk}C_{ijk}$, where the symbol $|_l$ denotes the v-covariant differentiation and g^{jk} is the reciprocal tensor of g_{jk}.

Assume that the function L^2C^2 is a non-zero function of position and direction s.t. $L^2C^2 = f(y) + g(x)$. Differentiation of this equation by y^i yields

$$L^2C^2|_i + 2C^2y_i = f_i \tag{1}$$

where the symbol $|_i$ denotes the differentiation by $y^i and f_i = \dfrac{\partial f}{\partial y^i}$. Since, $C^2 = g^{ij}C_iC_j$ and $T_{ij}(= g^{kl}T_{ijkl}) = LC_i|_j + C_il_j + C_jl_i$. Since, $C^2 = g^{ij}C_iC_j$, then

$$C^2|_h = 2g^{ij}C_i|_hC_j = 2C^iC_i|_h \tag{2}$$

From (1) and (2), we have

$$2C^2Ll_i + 2L^2C^hC_h|_i = f_i \tag{3}$$

Since $T_{ij} = LC_i|_j + l_iC_j + l_jC_i$

$$C^iT_{ih} = LC^iC_i|_h + C^il_iC_h + C^il_hC_i$$

$$C^iT_{ih} = LC^iC_i|_h + C^2l_h \tag{4}$$

From (4) and (3), we get,

$$2LC^iT_{ih} = f_h \tag{5}$$

Conversely, let $2LC^iT_{ih} = f_h$

$$2LC^i(LC_i|_h + l_iC_h + l_hC_i) = f_h$$

$$2LC^iC_i|_hC^i + 2LC^2l_h = f_h$$

$$(L^2C^2)|_h = f_h$$

Integrating, we get

$$L^2C^2 = f(y) + g(x) \tag{6}$$

where, g(x) is a arbitrary function of x only. Thus, we have,

Theorem 2.1 *If for a n-dimensional Finsler space unified scalar L^2C^2 is of the form $L^2C^2 = f(y) + g(x)$, if and only if the T-tensor satisfies the condition $T_{ij}C^j = \dfrac{f_i}{2L}$.*

Again, for a two-dimensional Finsler space the T-tensor [4,5,7,8] can be written as

$$T_{hijk} = I_{;2}m_hm_im_jm_k \text{ and } LC_{ijk} = Im_im_jm_k \text{ this implies } LC = I$$

Since, $(L^2C^2)|_i = f_i$, this implies

$$2LC(LC)|_i = f_i$$

Thus, we have

$$T_{hijk} = \frac{f_r m^r}{2LC} m_h m_i m_j m_k$$

Corollary 2.1 *In a two-dimensional Finsler space with the unified scalar $L^2C^2(L^2C^2 = f(y) + g(x))$ satisfies T-condition if and only if f_i is parallel to l_i i.e. $f_i = \lambda l_i$ for some scalar function λ.*

Now, $f_i = \lambda l_i$ Differentiating above equation with respect to y^j, we get

$$\frac{\partial f_i}{\partial y^j} = \frac{\partial \lambda}{\partial y^j} l_i + \lambda L^{-1}(h_{ij})$$

contracting above equation with respect to y^i, we get

$$\frac{\partial f_i}{\partial y^j} y^i = L \frac{\partial \lambda}{\partial y^j}$$

this implies

$$L\frac{\partial \lambda}{\partial y^j} = -f_j \qquad [\text{Since } f_i y^i = \frac{\partial f}{\partial y^i} y^i = 0 \Longrightarrow \frac{\partial (f_i y^i)}{\partial y^j} = \frac{\partial f_i}{\partial y^j} y^i + f_i \delta^i_j]$$

this implies

$$L\frac{\partial \lambda}{\partial y^j} = -f_j = -\lambda l_j$$

Integrating we get

$$\lambda = \frac{h(x)}{L}$$

where, $h(x)$ is any arbitrary function of x. Again,

$$f_i = \frac{h(x)}{L} l_i = h(x) \frac{\partial L}{L \partial y^i} \text{ or}$$

$$\frac{\partial f}{\partial y^i} = \frac{h(x)}{h(x)L} \frac{\partial (Lh(x))}{\partial y^i}$$

Integrating above equation, we get

$$f(y) = h(x) log(Lh(x)) + p(x) \tag{7}$$

where, p(x) is also any arbitrary function of x. From (7), we have

$$L = \frac{1}{h(x)} e^{\frac{f(y)-p(x)}{h(x)}}$$

Thus, we have

Theorem 2.2 *If a two-dimensional Finsler space with $L^2C^2 = f(y) + g(x)$ satisfies T-condition then the metric function L is given by $L = \frac{1}{h} e^{\frac{f-p}{h}}$, where scalars h and p are arbitrary function of x only.*

In C-reducible Finsler space the T-tensor [5] can be written as,

$$T_{hijk} = \frac{LC^*}{n^2-1}\pi_{hijk}(h_{hi}h_{jk}) \tag{8}$$

where, $C^* = g^{ij}C_i|_j$ and π_{hijk} represents cyclic permutation of the indices h,i,j,k. contracting (8) by g^{jk}, we get

$$T_{hi} = \frac{LC^*}{n-1}h_{hi}$$

this implies

$$f_h = \frac{2L^2C^*}{n-1}C_h$$

Thus,

$$f_hC^h = \frac{2L^2C^2C^*}{n-1} \tag{9}$$

Corollary 2.2 *For a n-dimensional C-reducible Finsler space with unified scalar $L^2C^2 = f(y)+g(x)$ satisfies T-condition if f_i is perpendicular to C^i.*

§3. Landsberg and Berwald Spaces Satisfying Condition $L^2C^2 = f(y)+g(x)$

Hereafter, assume that a Finsler space M^n satisfies the condition $L^2C^2 = f(y)+g(x)$. From the equation (1), the important tensors which will be used later are given by,

$$g_{ij} = \frac{f_{ij} - L^2C^2|_i|_j}{2C^2} - \frac{1}{LC^2}(f_il_j + f_jl_i) + yl_il_j \tag{10}$$

$$\begin{aligned} C_{ijk} \quad = \quad & \frac{f_{ijk} - L^2C^2|_i|_j|_k}{4C^2} + \frac{2}{L}(h_{ij}l_k + h_{jk}l_i + h_{ki}l_j - 3l_il_jl_k) - \\ & \frac{1}{2LC^2}(f_{ik}l_j + f_{jk}l_i + f_{ij}l_k) + \frac{1}{2L^2C^2}\pi_{(ijk)}(f_i(h_{jk} - f_{jk})) - \\ & \frac{1}{2L^2C^2}(f_il_jl_k + f_jl_il_k + f_kl_il_j) - \frac{1}{L^3C^2}(f_if_jl_k + f_jf_kl_i + f_kf_il_j) \end{aligned} \tag{11}$$

$$\begin{aligned} C_{ijk|h} \quad = \quad & \frac{f_{ijk|h} - L^2C^2|_i|_j|_{k|h}}{4C^2} - \frac{C^2_{|h}}{C^2}C_{ijk} - \frac{1}{2LC^2}(f_{ik|h}l_j + \\ & f_{jk|h}l_i + f_{ij|h}l_k) + \frac{1}{2L^2C^2}\pi_{(ijk)}(f_{i|h}h_{jk}) - \frac{1}{2L^2C^2}(f_{ij|h}f_k + \\ & f_{ij}f_{k|h} + f_{jk|h}f_i + f_{jk}f_{i|h} + f_{ik|h}f_j + f_{ik}f_{j|h}) - \frac{1}{2L^2C^2}(f_{i|h}l_jl_k + \\ & f_{j|h}l_il_k + f_{k|h}l_il_j) - \frac{1}{L^3C^2}(f_{i|h}f_jl_k + f_if_{j|h}l_k + f_{j|h}f_kl_i + \\ & f_jf_{k|h}l_i + f_{k|h}f_il_j + f_kf_{i|h}l_j) \end{aligned} \tag{12}$$

Contracting above equation by y^h, we get

$$\begin{aligned} P_{ijk} \;=\; & \frac{f_{ijk|0} - L^2C^2|_i|_j|_{k|0}}{4C^2} - \frac{C^2_{|0}}{C^2}C_{ijk} - \frac{1}{2LC^2}(f_{ik|0}l_j + \\ & f_{jk|0}l_i + f_{ij|0}l_k) + \frac{1}{2L^2C^2}\pi_{(ijk)}(f_{i|0}h_{jk}) - \frac{1}{2L^2C^2}(f_{ij|0}f_k + \\ & f_{ij}f_{k|0} + f_{jk|0}f_i + f_{jk}f_{i|0} + f_{ik|0}f_j + f_{ik}f_{j|0}) - \frac{1}{2L^2C^2}(f_{i|0}l_jl_k + \\ & f_{j|0}l_il_k + f_{k|0}l_il_j) - \frac{1}{L^3C^2}(f_{i|0}f_jl_k + f_if_{j|0}l_k + f_{j|0}f_kl_i + \\ & f_jf_{k|0}l_i + f_{k|0}f_il_j + f_kf_{i|0}l_j) \end{aligned} \tag{13}$$

where, P_{ijk} is the (v)hv-torsion tensor, the symbol $_{|i}$ denotes the h-covariant differentiation and the index '0' means the contraction by y^i.

The above equation (12) (resp. 13) gives the result that the condition $C_{ijk|l} = 0$ $(res.P_{ijk} = 0)$ is equivalent to equation (14) (resp.15).

$$\begin{aligned} & \frac{f_{ijk|h} - L^2C^2|_i|_j|_{k|h}}{4C^2} - \frac{C^2_{|h}}{C^2}C_{ijk} - \frac{1}{2LC^2}(f_{ik|h}l_j + f_{jk|h}l_i + f_{ij|h}l_k) \\ & + \frac{1}{2L^2C^2}\pi_{(ijk)}(f_{i|h}h_{jk}) - \frac{1}{2L^2C^2}(f_{ij|h}f_k + f_{ij}f_{k|h} + f_{jk|h}f_i + f_{jk}f_{i|h} \\ & + f_{ik|h}f_j + f_{ik}f_{j|h}) - \frac{1}{2L^2C^2}(f_{i|h}l_jl_k + f_{j|h}l_il_k + f_{k|h}l_il_j) \\ & - \frac{1}{L^3C^2}(f_{i|h}f_jl_k + f_if_{j|h}l_k + f_{j|h}f_kl_i + f_jf_{k|h}l_i + f_{k|h}f_il_j + f_kf_{i|h}l_j) = 0 \end{aligned} \tag{14}$$

$$\begin{aligned} & \frac{f_{ijk|0} - L^2C^2|_i|_j|_{k|0}}{4C^2} - \frac{C^2_{|0}}{C^2}C_{ijk} - \frac{1}{2LC^2}(f_{ik|0}l_j + f_{jk|0}l_i + f_{ij|0}l_k) \\ & + \frac{1}{2L^2C^2}\pi_{(ijk)}(f_{i|0}h_{jk}) - \frac{1}{2L^2C^2}(f_{ij|0}f_k + f_{ij}f_{k|0} + f_{jk|0}f_i \\ & + f_{jk}f_{i|0} + f_{ik|0}f_j + f_{ik}f_{j|0}) - \frac{1}{2L^2C^2}(f_{i|0}l_jl_k + f_{j|0}l_il_k + f_{k|0}l_il_j) \\ & - \frac{1}{L^3C^2}(f_{i|0}f_jl_k + f_if_{j|0}l_k + f_{j|0}f_kl_i + f_jf_{k|0}l_i + f_{k|0}f_il_j + f_kf_{i|0}l_j) = 0 \end{aligned} \tag{15}$$

So, we have

Theorem 3.1 *If an n-dimensional Finsler space M^n satisfies the condition $L^2C^2 = f(y)+g(x)$, then the necessary and sufficient condition for M^n to be a Berwald space is that equation (14) holds good.*

Theorem 3.2 *If an n-dimensional Finsler space M^n satisfies the condition $L^2C^2 = f(y)+g(x)$, then the necessary and sufficient condition for M^n to be a Landsberg space is that equation (15) holds good.*

If h-covariant differentiation of f_i is vanishes then the theorem 3 and theorem 4 gives the result that the condition $C_{ijk|h} = 0$ (resp. $P_{ijk} = 0$) is equivalent to $C^2|_i|_j|_{k|h} = 0$ (resp. $C^2|_i|_j|_{k|0} = 0$). So, we have

Corollary 3.1 *If an n-dimensional Finsler space M^n satisfies the condition $L^2C^2 = f(y)+g(x)$, then the necessary and sufficient condition for M^n to be a Berwald space is that $C^2|_i|_j|_{k|h} = 0$ holds good.*

Corollary 3.2 *If an n-dimensional Finsler space M^n satisfies the condition $L^2C^2 = f(y)+g(x)$, then the necessary and sufficient condition for M^n to be a Berwald space is that $C^2|_i|_j|_{k|0} = 0$ holds good.*

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