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Matrix Representation of Biharmonic Curves in Terms of Exponential Maps in the Special Three-Dimensional $\phi-$Ricci Symmetric Para-Sasakian Manifold

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Abstract: In this paper, we study biharmonic curves in the special three-dimensional $\phi-$Ricci symmetric para-Sasakian manifold $\mathbb{P}$. We construct matrix representation of biharmonic curves in terms of exponential maps in the special three-dimensional $\phi-$Ricci symmetric para-Sasakian manifold $\mathbb{P}$.

Key Words: Biharmonic curve, bienergy, bitension field, para-Sasakian manifold, exponential map.

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§1. Introduction

In the theory of Lie groups the exponential map is a map from the Lie algebra of a Lie group to the group which allows one to recapture the local group structure from the Lie algebra. The existence of the exponential map is one of the primary justifications for the study of Lie groups at the level of Lie algebras.

The ordinary exponential function of mathematical analysis is a special case of the exponential map when G is the multiplicative group of non-zero real numbers (whose Lie algebra is the additive group of all real numbers). The exponential map of a Lie group satisfies many properties analogous to those of the ordinary exponential function, however, it also differs in many important respects.

The aim of this paper is to study matrix representation of exponential maps in terms of biharmonic curves in the special three-dimensional $\phi-$Ricci symmetric para-Sasakian manifold $\mathbb{P}$.

A smooth map $\phi : N \longrightarrow M$ is said to be biharmonic if it is a critical point of the bienergy functional:

$$E_2(\phi) = \int_N \frac{1}{2} \left|\mathcal{T}(\phi)\right|^2 dv_h,$$

where $\mathcal{T}(\phi) := \mathrm{tr}\nabla^{\phi} d\phi$ is the tension field of ϕ

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The Euler–Lagrange equation of the bienergy is given by $\mathcal{T}_2(\phi)=0$. Here the section $\mathcal{T}_2(\phi)$ is defined by

$$\mathcal{T}_2(\phi)=-\Delta_\phi\mathcal{T}(\phi)+\mathrm{tr}R\left(\mathcal{T}(\phi),d\phi\right)d\phi, \tag{1.1}$$

and called the bitension field of ϕ. Non-harmonic biharmonic maps are called proper biharmonic maps.

In this paper, we study biharmonic curves in the special three-dimensional $\phi-$Ricci symmetric para-Sasakian manifold $\mathbb{P}$. We construct matrix representation of biharmonic curves in terms of exponential maps in the special three-dimensional $\phi-$Ricci symmetric para-Sasakian manifold $\mathbb{P}$.

§2. Preliminaries

An n-dimensional differentiable manifold M is said to admit an almost para-contact Riemannian structure (ϕ,ξ,η,g), where ϕ is a $(1,1)$ tensor field, ξ is a vector field, η is a 1-form and g is a Riemannian metric on M such that

$$\phi\xi=0,\quad \eta\left(\xi\right)=1, g\left(X,\xi\right)=\eta\left(X\right), \tag{2.1}$$

$$\phi^2\left(X\right)=X-\ \eta\left(X\right)\xi, \tag{2.2}$$

$$g\left(\phi X,\phi Y\right)=g\left(X,Y\right)-\eta\left(X\right)\eta\left(Y\right), \tag{2.3}$$

for any vector fields X, Y on M.

In addition, if (ϕ,ξ,η,g), satisfy the equations

$$d\eta=0,\quad \nabla_X\xi=\phi X, \tag{2.4}$$

$$\left(\nabla_X\phi\right)Y=-g\left(X,Y\right)\xi-\eta\left(Y\right)X+2\eta\left(X\right)\eta\left(Y\right)\xi,\qquad X,Y\in\chi\left(M\right), \tag{2.5}$$

then M is called a para-Sasakian manifold or, briefly a $P-$Sasakian manifold [2].

Definition 2.1 *A para-Sasakian manifold M is said to be locally ϕ-symmetric if*

$$\phi^2\left(\left(\nabla_W R\right)\left(X,Y\right)Z\right)=0,$$

for all vector fields X,Y,Z,W orthogonal to ξ. This notion was introduced by Takahashi [16], for a Sasakian manifold.

Definition 2.2 *A para-Sasakian manifold M is said to be ϕ-symmetric if*

$$\phi^2\left(\left(\nabla_W R\right)\left(X,Y\right)Z\right)=0,$$

for all vector fields X,Y,Z,W on M.

Definition 2.3 *A para-Sasakian manifold M is said to be ϕ-Ricci symmetric if the Ricci operator satisfies*

$$\phi^2\left(\left(\nabla_X Q\right)\left(Y\right)\right)=0,$$

for all vector fields X and Y on M and $S(X,Y) = g(QX,Y)$.

If X,Y are orthogonal to ξ, then the manifold is said to be locally ϕ-Ricci symmetric.

We consider the three-dimensional manifold

$$\mathbb{P} = \left\{ \left(x^1, x^2, x^3\right) \in \mathbb{R}^3 : \left(x^1, x^2, x^3\right) \neq (0,0,0) \right\},$$

where $\left(x^1, x^2, x^3\right)$ are the standard coordinates in $\mathbb{R}^3$. We choose the vector fields

$$\mathbf{e}_1 = e^{x^1} \frac{\partial}{\partial x^2}, \quad \mathbf{e}_2 = e^{x^1} \left(\frac{\partial}{\partial x^2} - \frac{\partial}{\partial x^3} \right), \quad \mathbf{e}_3 = -\frac{\partial}{\partial x^1} \tag{2.6}$$

are linearly independent at each point of $\mathbb{P}$.

Let η be the 1-form defined by

$$\eta(Z) = g(Z, \mathbf{e}_3) \text{ for any } Z \in \chi(\mathbb{P}).$$

Let be the (1,1) tensor field defined by

$$\phi(\mathbf{e}_1) = \mathbf{e}_2, \ \phi(\mathbf{e}_2) = \mathbf{e}_1, \ \phi(\mathbf{e}_3) = 0.$$

Then using the linearity of and g we have

$$\eta(\mathbf{e}_3) = 1,$$

$$\phi^2(Z) = Z - \ \eta(Z)\mathbf{e}_3,$$

$$g(\phi Z, \phi W) = g(Z, W) - \eta(Z)\eta(W),$$

for any $Z, W \in \chi(\mathbb{P})$. Thus for $\mathbf{e}_3 = \xi$, (ϕ, ξ, η, g) defines an almost para-contact metric structure on $\mathbb{P}$.

Let ∇ be the Levi-Civita connection with respect to g. Then, we have

$$[\mathbf{e}_1, \mathbf{e}_2] = 0, \quad [\mathbf{e}_1, \mathbf{e}_3] = \mathbf{e}_1, \quad [\mathbf{e}_2, \mathbf{e}_3] = \mathbf{e}_2.$$

Taking $\mathbf{e}_3 = \xi$ and using the Koszul's formula, we obtain

$$\begin{aligned} \nabla_{\mathbf{e}_1}\mathbf{e}_1 &= -\mathbf{e}_3, \ \nabla_{\mathbf{e}_1}\mathbf{e}_2 = 0, \qquad \nabla_{\mathbf{e}_1}\mathbf{e}_3 = \mathbf{e}_1, \\ \nabla_{\mathbf{e}_2}\mathbf{e}_1 &= 0, \qquad \nabla_{\mathbf{e}_2}\mathbf{e}_2 = -\mathbf{e}_3, \ \nabla_{\mathbf{e}_2}\mathbf{e}_3 = \mathbf{e}_2, \\ \nabla_{\mathbf{e}_3}\mathbf{e}_1 &= 0, \qquad \nabla_{\mathbf{e}_3}\mathbf{e}_2 = 0, \qquad \nabla_{\mathbf{e}_3}\mathbf{e}_3 = 0. \end{aligned} \tag{2.7}$$

Moreover we put

$$R_{ijk} = R(\mathbf{e}_i, \mathbf{e}_j)\mathbf{e}_k, \quad R_{ijkl} = R(\mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_k, \mathbf{e}_l),$$

where the indices i, j, k and l take the values $1, 2$ and 3.

$$R_{122} = -\mathbf{e}_{1,}, \quad R_{133} = -\mathbf{e}_{1,}, \ R_{233} = -\mathbf{e}_2,$$

and

$$R_{1212} = \ R_{1313} = R_{2323} = 1. \tag{2.8}$$

§3. Biharmonic Curves in the Special Three-Dimensional $\phi-$Ricci Symmetric Para-Sasakian Manifold $\mathbb{P}$

Let us consider biharmonicity of curves in the special three-dimensional $\phi-$Ricci symmetric para-Sasakian manifold $\mathbb{P}$. Let $\{\mathbf{T},\mathbf{N},\mathbf{B}\}$ be the Frenet frame field along γ. Then, the Frenet frame satisfies the following Frenet–Serret equations:

$$\begin{aligned}\nabla_{\mathbf{T}}\mathbf{T} &= \kappa\mathbf{N},\\ \nabla_{\mathbf{T}}\mathbf{N} &= -\kappa\mathbf{T}+\tau\mathbf{B},\\ \nabla_{\mathbf{T}}\mathbf{B} &= -\tau\mathbf{N},\end{aligned} \tag{3.1}$$

where κ is the curvature of γ and τ its torsion.

With respect to the orthonormal basis $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$, we can write

$$\begin{aligned}\mathbf{T} &= T_1\mathbf{e}_1+T_2\mathbf{e}_2+T_3\mathbf{e}_3,\\ \mathbf{N} &= N_1\mathbf{e}_1+N_2\mathbf{e}_2+N_3\mathbf{e}_3,\\ \mathbf{B} &= \mathbf{T}\times\mathbf{N}=B_1\mathbf{e}_1+B_2\mathbf{e}_2+B_3\mathbf{e}_3.\end{aligned} \tag{3.2}$$

Theorem 3.1([12]) *$\gamma : I \longrightarrow \mathbb{P}$ is a biharmonic curve if and only if*

$$\begin{aligned}\kappa &= \text{constant}\neq 0,\\ \kappa^2+\tau^2 &= 1,\\ \tau' &= 0.\end{aligned} \tag{3.3}$$

Theorem 3.2([12]) *All of biharmonic curves in the special three-dimensional $\phi-$Ricci symmetric para-Sasakian manifold $\mathbb{P}$ are helices.*

§4. New Approach for Biharmonic Curves in $\mathbb{P}$

A map

$$\exp : R\times\mathbb{P}_3^3\to GL(3,\mathbb{R})\subset\mathbb{P}_3^3,\ \ (t,\mathcal{A})\to\exp(t,\mathcal{A})=\sum_{k=0}^{\infty}\frac{t^k}{k!}\mathcal{A}^k$$

is called exponential map in para-Sasakian Manifold $\mathbb{P}$.

Definition 4.1 *$\langle\mathcal{A},\mathcal{B}\rangle_{\mathbb{P}}=\mathfrak{trace}\left(\mathcal{A}\mathcal{B}^T\right)$ is called an inner product for $\mathcal{A},\mathcal{B}\in\mathbb{P}_3^3$.*

Firstly, let us calculate the arbitrary parameter t according to the arclength parameter s. It is well known that

$$s=\int_0^t\|\gamma'(t)\|_{\mathbb{P}}\,dt, \tag{4.1}$$

where

$$\gamma'(t)=\mathcal{A}\gamma. \tag{4.2}$$

The norm of Equation (4.1), we obtain

$$\|\mathcal{A}\gamma\|_{\mathbb{P}} = \sqrt{-\mathfrak{trace}\left(\mathcal{A}^2\right)},$$

where $\gamma\gamma^T = I$.

Substituting above equation in (4.1), we have

$$s = \sqrt{-\mathfrak{trace}\left(\mathcal{A}^2\right)}t$$

Lemma 4.2 *Let $\mathcal{A}$ be a be an anti-symmetric matrix and $n \in \mathbb{N}$. Then,*

i) If n is odd, $\mathcal{A}^n$ is an anti-symmetric matrix.
ii) If n is even, $\mathcal{A}^n$ is a symmetric matrix.
iii) The trace of an anti-symmetric matrix is zero.

The first, second and third derivatives of γ are given as follows:

$$\begin{aligned} \gamma'(s) &= \frac{\mathcal{A}\gamma}{\sqrt{-\mathfrak{trace}\left(\mathcal{A}^2\right)}}, \\ \gamma''(s) &= \frac{\mathcal{A}^2\gamma}{\left(\sqrt{-\mathfrak{trace}\left(\mathcal{A}^2\right)}\right)^2}, \\ \gamma'''(s) &= \frac{\mathcal{A}^3\gamma}{\left(\sqrt{-\mathfrak{trace}\left(\mathcal{A}^2\right)}\right)^3}. \end{aligned} \tag{4.3}$$

§5. Matrix Representation of Biharmonic Curves in Terms of Exponential Maps in the Special Three-Dimensional $\phi-$Ricci Symmetric Para-Sasakian Manifold

Using above sections we obtain following results.

Theorem 5.1 *Let $\gamma : I \longrightarrow \mathbb{P}$ be a unit speed non-geodesic biharmonic curve in the special three-dimensional $\phi-$Ricci symmetric para-Sasakian manifold $\mathbb{P}$. Then,*

$$\begin{aligned} \mathcal{A}\gamma &= \sqrt{-\mathfrak{trace}\left(\mathcal{A}^2\right)}(-\cos\varphi, \sin\varphi e^{-s\cos\varphi + C_1}\left(\sin\left[\Bbbk s + C\right] + \cos\left[\Bbbk s + C\right]\right), \\ &\quad \sin\varphi e^{-s\cos\varphi + C_1}\sin\left[\Bbbk s + C\right]), \end{aligned}$$

$$\begin{aligned} \mathcal{A}^2\gamma &= \frac{\left(\sqrt{\mathfrak{trace}\left(\mathcal{A}^4\right)}\right)}{\kappa}(-\frac{\sin^2\varphi}{2}s^2 + \overline{C}_1 s + \overline{C}_2, \\ &\quad e^{-\frac{\sin^2\varphi}{2}s^2 + \overline{C}_1 s + \overline{C}_2}\left(\Bbbk\sin\varphi\sin\left[\Bbbk s + C\right] + \cos\varphi\sin\varphi\cos\left[\Bbbk s + C\right]\right) \\ &\quad + e^{-\frac{\sin^2\varphi}{2}s^2 + \overline{C}_1 s + \overline{C}_2}\left(-\Bbbk\sin\varphi\cos\left[\Bbbk s + C\right] + \cos\varphi\sin\varphi\sin\left[\Bbbk s + C\right]\right), \\ &\quad - e^{-\frac{\sin^2\varphi}{2}s^2 + \overline{C}_1 s + \overline{C}_2}\left(-\Bbbk\sin\varphi\cos\left[\Bbbk s + C\right] + \cos\varphi\sin\varphi\sin\left[\Bbbk s + C\right]\right)), \end{aligned} \tag{5.1}$$

$$\begin{aligned}\mathcal{A}^3\gamma &= \frac{\left(\sqrt{-\mathfrak{trace}\left(\mathcal{A}^2\right)}\right)^3}{\kappa}\left[\frac{\mathfrak{trace}\left(\mathcal{A}^6\right)}{\left(\sqrt{-\mathfrak{trace}\left(\mathcal{A}^2\right)}\right)^6}-\frac{\left(\mathfrak{trace}\left(\mathcal{A}^4\right)\right)^2}{\left(\sqrt{-\mathfrak{trace}\left(\mathcal{A}^2\right)}\right)^5}\right]^{\frac{1}{2}}(-\sin\varphi e^{-s\cos\varphi+C_1}(\sin[\Bbbk s+C]\\
&+\cos[\Bbbk s+C])e^{-\frac{\sin^2\varphi}{2}s^2+\overline{C}_1 s+\overline{C}_2}.(-\Bbbk\sin\varphi\cos[\Bbbk s+C]+\cos\varphi\sin\varphi\sin[\Bbbk s+C])\\
&-\sin\varphi e^{-s\cos\varphi+C_1}\sin[\Bbbk s+C]\,e^{-\frac{\sin^2\varphi}{2}s^2+\overline{C}_1 s+\overline{C}_2}((\Bbbk\sin\varphi\sin[\Bbbk s+C]+\cos\varphi\sin\varphi\cos[\Bbbk s+C])\\
&+(-\Bbbk\sin\varphi\cos[\Bbbk s+C]+\cos\varphi\sin\varphi\sin[\Bbbk s+C])),\\
&(-\frac{\sin^2\varphi}{2}s^2+\overline{C}_1 s+\overline{C}_2)\sin\varphi e^{-s\cos\varphi+C_1}\sin[\Bbbk s+C]\\
&-\cos\varphi e^{-\frac{\sin^2\varphi}{2}s^2+\overline{C}_1 s+\overline{C}_2}(-\Bbbk\sin\varphi\cos[\Bbbk s+C]+\cos\varphi\sin\varphi\sin[\Bbbk s+C]),\\
&-\cos\varphi e^{-\frac{\sin^2\varphi}{2}s^2+\overline{C}_1 s+\overline{C}_2}((\Bbbk\sin\varphi\sin[\Bbbk s+C]+\cos\varphi\sin\varphi\cos[\Bbbk s+C])\\
&+(-\Bbbk\sin\varphi\cos[\Bbbk s+C]+\cos\varphi\sin\varphi\sin[\Bbbk s+C]))\\
&-\sin\varphi e^{-s\cos\varphi+C_1}(\sin[\Bbbk s+C]+\cos[\Bbbk s+C])(-\frac{\sin^2\varphi}{2}s^2+\overline{C}_1 s+\overline{C}_2)\\
&-\frac{\mathfrak{trace}\left(\mathcal{A}^4\right)}{\left(\sqrt{-\mathfrak{trace}\left(\mathcal{A}^2\right)}\right)}(-\cos\varphi,\sin\varphi e^{x^1}(\sin[\Bbbk s+C]+\cos[\Bbbk s+C]),\sin\varphi e^{x^1}\sin[\Bbbk s+C]).\end{aligned}$$

where C, $\overline{C}_1,\overline{C}_2$ are constants of integration and $\Bbbk=\frac{\sqrt{\kappa^2-\sin^2\varphi}}{\sin\varphi}$.

Proof From (3.1) and Theorem 3.2, imply

$$\mathbf{T}=\sin\varphi\cos[\Bbbk s+C]\,\mathbf{e}_1+\sin\varphi\sin[\Bbbk s+C]\,\mathbf{e}_2+\cos\varphi\mathbf{e}_3,$$

where $\Bbbk=\frac{\sqrt{\kappa^2-\sin^2\varphi}}{\sin\varphi}$.

Using (2.4) in above equation, we obtain

$$\mathbf{T}=(-\cos\varphi,\sin\varphi e^{x^1}(\sin[\Bbbk s+C]+\cos[\Bbbk s+C]),\sin\varphi e^{x^1}\sin[\Bbbk s+C]). \tag{5.2}$$

On the other hand, first equation of (3.3) we have

$$\begin{aligned}\mathcal{A}\gamma &= \sqrt{-\mathfrak{trace}\left(\mathcal{A}^2\right)}(-\cos\varphi,\sin\varphi e^{-s\cos\varphi+C_1}(\sin[\Bbbk s+C]+\cos[\Bbbk s+C]),\\
&\sin\varphi e^{-s\cos\varphi+C_1}\sin[\Bbbk s+C]).\end{aligned}$$

Using Gram-Schmidt method

$$\bar{\mathbf{N}}=\gamma''(s)-\frac{\langle\gamma''(s),\mathbf{T}\rangle_{\mathbb{P}}}{\|\mathbf{T}\|_{\mathbb{P}}^2}\mathbf{T}.$$

Therefore

$$\langle\gamma''(s),\mathbf{T}\rangle_{\mathbb{P}}=<\frac{\mathcal{A}^2\gamma}{\left(\sqrt{-\mathfrak{trace}\left(\mathcal{A}^2\right)}\right)^2},\frac{\mathcal{A}\gamma}{\sqrt{-\mathfrak{trace}\left(\mathcal{A}^2\right)}}>_{\mathbb{P}}$$

or

$$\langle \gamma''(s), \mathbf{T} \rangle_{\mathbb{P}} = \left(\sqrt{-\mathfrak{trace}\left(\mathcal{A}^2\right)}\right)^{-3} < \mathcal{A}^2\gamma, \mathcal{A}\gamma > .$$

Also from Definition 3.1 and Lemma 3.2, we obtain

$$< \mathcal{A}^2\gamma, \mathcal{A}\gamma >_{\mathbb{P}} = \mathfrak{trace}\left(-\mathcal{A}^3\right) = 0.$$

Since

$$\bar{\mathbf{N}} = \left(\sqrt{-\mathfrak{trace}\left(\mathcal{A}^2\right)}\right)^{-2} \mathcal{A}^2\gamma.$$

So we immediately arrive at

$$\mathbf{N} = \frac{\bar{\mathbf{N}}}{\left\|\bar{\mathbf{N}}\right\|_{\mathbb{P}}} = \frac{\left(\sqrt{-\mathfrak{trace}\left(\mathcal{A}^2\right)}\right)^{-2} \mathcal{A}^2\gamma}{\left\|\left(\sqrt{-\mathfrak{trace}\left(\mathcal{A}^2\right)}\right)^{-2} \mathcal{A}^2\gamma\right\|_{\mathbb{P}}} = \frac{\mathcal{A}^2\gamma}{\sqrt{< \mathcal{A}^2\gamma, \mathcal{A}^2\gamma >_{\mathbb{P}}}}. \tag{5.3}$$

Also from Definition 4.1 and Lemma 4.2 we obtain

$$< \mathcal{A}^2\gamma, \mathcal{A}^2\gamma >_{\mathbb{P}} = \mathfrak{trace}\left(\mathcal{A}^4\right). \tag{5.4}$$

Substituting (5.4) in (5.3), we have

$$\mathbf{N} = \left(\sqrt{\mathfrak{trace}\left(\mathcal{A}^4\right)}\right)^{-1} \mathcal{A}^2\gamma.$$

On the other hand, using (2.7), we have

$$\nabla_{\mathbf{T}}\mathbf{T} = \left(T_1' + T_1T_3\right)\mathbf{e}_1 + \left(T_2' + T_2T_3\right)\mathbf{e}_2 + \left(T_3' - \left(T_1^2 - T_2^2\right)\right)\mathbf{e}_3. \tag{5.5}$$

From (4.1) and (5.5), we get

$$\begin{aligned} \nabla_{\mathbf{T}}\mathbf{T} &= \sin\varphi\left(-\mathbb{k}\sin\left[\mathbb{k}s + C\right] + \cos\varphi\cos\left[\mathbb{k}s + C\right]\right)\mathbf{e}_1 \\ &\quad + \sin\varphi\left(\mathbb{k}\cos\left[\mathbb{k}s + C\right] + \cos\varphi\sin\left[\mathbb{k}s + C\right]\right)\mathbf{e}_2 \\ &\quad - \sin^2\varphi\mathbf{e}_3, \end{aligned} \tag{5.6}$$

where $\mathbb{k} = \frac{\sqrt{\kappa^2 - \sin^2\varphi}}{\sin\varphi}$. By the use of above equation, we get

$$\begin{aligned} \mathcal{A}^2\gamma &= \left(\sqrt{\mathfrak{trace}\left(\mathcal{A}^4\right)}\right)\mathbf{N} \\ &= \frac{\left(\sqrt{\mathfrak{trace}\left(\mathcal{A}^4\right)}\right)}{\kappa}[\left(\mathbb{k}\sin\varphi\sin\left[\mathbb{k}s + C\right] + \cos\varphi\sin\varphi\cos\left[\mathbb{k}s + C\right]\right)\mathbf{e}_1 \\ &\quad + \left(-\mathbb{k}\sin\varphi\cos\left[\mathbb{k}s + C\right] + \cos\varphi\sin\varphi\sin\left[\mathbb{k}s + C\right]\right)\mathbf{e}_2 \\ &\quad - \sin^2\varphi\mathbf{e}_3]. \end{aligned} \tag{5.7}$$

Substituting (2.4) in (5.7), we have

$$\begin{aligned} \mathcal{A}^2\gamma &= \frac{\left(\sqrt{\mathfrak{trace}\left(\mathcal{A}^4\right)}\right)}{\kappa}(-\frac{\sin^2\varphi}{2}s^2 + \overline{C}_1 s + \overline{C}_2, \\ &\quad e^{-\frac{\sin^2\varphi}{2}s^2 + \overline{C}_1 s + \overline{C}_2}\left(\mathbb{k}\sin\varphi\sin\left[\mathbb{k}s + C\right] + \cos\varphi\sin\varphi\cos\left[\mathbb{k}s + C\right]\right) \\ &\quad + e^{-\frac{\sin^2\varphi}{2}s^2 + \overline{C}_1 s + \overline{C}_2}\left(-\mathbb{k}\sin\varphi\cos\left[\mathbb{k}s + C\right] + \cos\varphi\sin\varphi\sin\left[\mathbb{k}s + C\right]\right), \\ &\quad - e^{-\frac{\sin^2\varphi}{2}s^2 + \overline{C}_1 s + \overline{C}_2}\left(-\mathbb{k}\sin\varphi\cos\left[\mathbb{k}s + C\right] + \cos\varphi\sin\varphi\sin\left[\mathbb{k}s + C\right]\right)), \end{aligned} \tag{5.8}$$

where $\overline{C}_1, \overline{C}_2$ are constants of integration.

Using same calculations we get

$$\mathbf{B} = [\frac{\mathfrak{trace}\left(\mathcal{A}^6\right)}{\left(\sqrt{-\mathfrak{trace}\left(\mathcal{A}^2\right)}\right)^6} - \frac{\left(\mathfrak{trace}\left(\mathcal{A}^4\right)\right)^2}{\left(\sqrt{-\mathfrak{trace}\left(\mathcal{A}^2\right)}\right)^5}]^{-\frac{1}{2}}[\frac{\mathcal{A}^3\gamma}{\left(\sqrt{-\mathfrak{trace}\left(\mathcal{A}^2\right)}\right)^3} + \frac{\mathfrak{trace}\left(\mathcal{A}^4\right)}{\left(\sqrt{-\mathfrak{trace}\left(\mathcal{A}^2\right)}\right)^5}\mathcal{A}\gamma].$$

From above equation we have

$$\mathcal{A}^3\gamma = \left(\sqrt{-\mathfrak{trace}\left(\mathcal{A}^2\right)}\right)^3[\frac{\mathfrak{trace}\left(\mathcal{A}^6\right)}{\left(\sqrt{-\mathfrak{trace}\left(\mathcal{A}^2\right)}\right)^6} - \frac{\left(\mathfrak{trace}\left(\mathcal{A}^4\right)\right)^2}{\left(\sqrt{-\mathfrak{trace}\left(\mathcal{A}^2\right)}\right)^5}]^{\frac{1}{2}}\mathbf{B} - \frac{\mathfrak{trace}\left(\mathcal{A}^4\right)}{\left(\sqrt{-\mathfrak{trace}\left(\mathcal{A}^2\right)}\right)^2}\mathcal{A}\gamma.$$

Cross product of $\mathbf{T} \times \mathbf{N} = \mathbf{B}$ gives us

$$\begin{aligned}
\mathcal{A}^3\gamma \quad = \quad & \frac{\left(\sqrt{-\mathfrak{trace}\left(\mathcal{A}^2\right)}\right)^3}{\kappa}[\frac{\mathfrak{trace}\left(\mathcal{A}^6\right)}{\left(\sqrt{-\mathfrak{trace}\left(\mathcal{A}^2\right)}\right)^6} - \frac{\left(\mathfrak{trace}\left(\mathcal{A}^4\right)\right)^2}{\left(\sqrt{-\mathfrak{trace}\left(\mathcal{A}^2\right)}\right)^5}]^{\frac{1}{2}}(-\sin\varphi e^{-s\cos\varphi + C_1}(\sin[\Bbbk s + C] \\
& + \cos[\Bbbk s + C])e^{-\frac{\sin^2\varphi}{2}s^2 + \overline{C}_1 s + \overline{C}_2}.(-\Bbbk\sin\varphi\cos[\Bbbk s + C] + \cos\varphi\sin\varphi\sin[\Bbbk s + C]) \\
& - \sin\varphi e^{-s\cos\varphi + C_1}\sin[\Bbbk s + C]\, e^{-\frac{\sin^2\varphi}{2}s^2 + \overline{C}_1 s + \overline{C}_2}((\Bbbk\sin\varphi\sin[\Bbbk s + C] + \cos\varphi\sin\varphi\cos[\Bbbk s + C]) \\
& + (-\Bbbk\sin\varphi\cos[\Bbbk s + C] + \cos\varphi\sin\varphi\sin[\Bbbk s + C])), \\
& (-\frac{\sin^2\varphi}{2}s^2 + \overline{C}_1 s + \overline{C}_2)\sin\varphi e^{-s\cos\varphi + C_1}\sin[\Bbbk s + C] \qquad (5.9) \\
& - \cos\varphi e^{-\frac{\sin^2\varphi}{2}s^2 + \overline{C}_1 s + \overline{C}_2}\left(-\Bbbk\sin\varphi\cos[\Bbbk s + C] + \cos\varphi\sin\varphi\sin[\Bbbk s + C]\right), \\
& - \cos\varphi e^{-\frac{\sin^2\varphi}{2}s^2 + \overline{C}_1 s + \overline{C}_2}((\Bbbk\sin\varphi\sin[\Bbbk s + C] + \cos\varphi\sin\varphi\cos[\Bbbk s + C]) \\
& + (-\Bbbk\sin\varphi\cos[\Bbbk s + C] + \cos\varphi\sin\varphi\sin[\Bbbk s + C])) \\
& - \sin\varphi e^{-s\cos\varphi + C_1}\left(\sin[\Bbbk s + C] + \cos[\Bbbk s + C]\right)(-\frac{\sin^2\varphi}{2}s^2 + \overline{C}_1 s + \overline{C}_2) \\
& - \frac{\mathfrak{trace}\left(\mathcal{A}^4\right)}{\left(\sqrt{-\mathfrak{trace}\left(\mathcal{A}^2\right)}\right)}(-\cos\varphi, \sin\varphi e^{x^1}\left(\sin[\Bbbk s + C] + \cos[\Bbbk s + C]\right), \sin\varphi e^{x^1}\sin[\Bbbk s + C]).
\end{aligned}$$

So the proof is completed. □

In the light of Theorem 5.1, we express the following corollary without proof.

Corollary 5.2

$$\mathcal{A}^3\gamma = [\frac{\mathfrak{trace}\left(\mathcal{A}^6\right)}{\left(\sqrt{-\mathfrak{trace}\left(\mathcal{A}^2\right)}\right)^3} - \frac{\left(\mathfrak{trace}\left(\mathcal{A}^4\right)\right)^2}{\left(\sqrt{-\mathfrak{trace}\left(\mathcal{A}^2\right)}\right)^2}]^{\frac{1}{2}}\mathbf{B} - \frac{\mathfrak{trace}\left(\mathcal{A}^4\right)}{\sqrt{-\mathfrak{trace}\left(\mathcal{A}^2\right)}}\mathbf{T}.$$

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