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Equations for Spacelike Biharmonic General Helices with Timelike Normal According to Bishop Frame in The Lorentzian Group of Rigid Motions $\mathbb{E}(1,1)$

Talat KÖRPINAR and Essin TURHAN

(Fırat University, Department of Mathematics 23119, Elazığ, Turkey)

E-mail: talatkorpinar@gmail.com, essin.turhan@gmail.com

Abstract: In this paper, we study spacelike biharmonic general helices according to Bishop frame in the Lorentzian group of rigid motions $\mathbb{E}(1,1)$. We characterize the spacelike biharmonic general helices in terms of their curvatures in the Lorentzian group of rigid motions $\mathbb{E}(1,1)$.

Key Words: Biharmonic curve, bienergy, bitension field, bishop frame, rigid motion.

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§1. Introduction

A helix, sometimes also called a coil, is a curve for which the tangent makes a constant angle with a fixed line. The shortest path between two points on a cylinder (one not directly above the other) is a fractional turn of a helix, as can be seen by cutting the cylinder along one of its sides, flattening it out, and noting that a straight line connecting the points becomes helical upon re-wrapping. It is for this reason that squirrels chasing one another up and around tree trunks follow helical paths.

Helices can be either right-handed or left-handed. With the line of sight along the helix's axis, if a clockwise screwing motion moves the helix away from the observer, then it is called a right-handed helix; if towards the observer then it is a left-handed helix. Handedness (or chirality) is a property of the helix, not of the perspective: a right-handed helix cannot be turned or flipped to look like a left-handed one unless it is viewed in a mirror, and vice versa.

Most hardware screw threads are right-handed helices. The alpha helix in biology as well as the A and B forms of DNA are also right-handed helices. The Z form of DNA is left-handed.

The pitch of a helix is the width of one complete helix turn, measured parallel to the axis of the helix. A double helix consists of two (typically congruent) helices with the same axis, differing by a translation along the axis.

The notions of harmonic and biharmonic maps between Riemannian manifolds have been introduced by J. Eells and J.H. Sampson (see [4]).

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A smooth map $\phi : N \longrightarrow M$ is said to be biharmonic if it is a critical point of the bienergy functional:

$$E_2(\phi) = \int_N \frac{1}{2} |\mathcal{T}(\phi)|^2 \, dv_h,$$

where $\mathcal{T}(\phi) := \mathrm{tr}\nabla^\phi d\phi$ is the tension field of ϕ

The Euler–Lagrange equation of the bienergy is given by $\mathcal{T}_2(\phi) = 0$. Here the section $\mathcal{T}_2(\phi)$ is defined by

$$\mathcal{T}_2(\phi) = -\Delta_\phi \mathcal{T}(\phi) + \mathrm{tr} R(\mathcal{T}(\phi), d\phi)\, d\phi, \tag{1.1}$$

and called the bitension field of ϕ. Non-harmonic biharmonic maps are called proper biharmonic maps.

In this paper, we study spacelike biharmonic general helices according to Bishop frame in the Lorentzian group of rigid motions $\mathbb{E}(1,1)$. We characterize the spacelike biharmonic general helices in terms of their curvatures in the Lorentzian group of rigid motions $\mathbb{E}(1,1)$. Finally, we obtain parametric equations of spacelike biharmonic general helices according to Bishop frame in the Lorentzian group of rigid motions $\mathbb{E}(1,1)$.

§2. Preliminaries

Let $\mathbb{E}(1,1)$ be the group of rigid motions of Euclidean 2-space. This consists of all matrices of the form

$$\begin{pmatrix} \cosh x & \sinh x & y \\ \sinh x & \cosh x & z \\ 0 & 0 & 1 \end{pmatrix}.$$

Topologically, $\mathbb{E}(1,1)$ is diffeomorphic to $\mathbb{R}^3$ under the map

$$\mathbb{E}(1,1) \longrightarrow \mathbb{R}^3 : \begin{pmatrix} \cosh x & \sinh x & y \\ \sinh x & \cosh x & z \\ 0 & 0 & 1 \end{pmatrix} \longrightarrow (x, y, z),$$

It's Lie algebra has a basis consisting of

$$\mathbf{X}_1 = \frac{\partial}{\partial x}, \; \mathbf{X}_2 = \cosh x \frac{\partial}{\partial y} + \sinh x \frac{\partial}{\partial z}, \; \mathbf{X}_3 = \sinh x \frac{\partial}{\partial y} + \cosh x \frac{\partial}{\partial z},$$

for which

$$[\mathbf{X}_1, \mathbf{X}_2] = \mathbf{X}_3, \;\; [\mathbf{X}_2, \mathbf{X}_3] = 0, \;\; [\mathbf{X}_1, \mathbf{X}_3] = \mathbf{X}_2.$$

Put

$$x^1 = x, \; x^2 = \frac{1}{2}(y+z), \; x^3 = \frac{1}{2}(y-z).$$

Then, we get

$$\mathbf{X}_1 = \frac{\partial}{\partial x^1}, \; \mathbf{X}_2 = \frac{1}{2}\left(e^{x^1} \frac{\partial}{\partial x^2} + e^{-x^1} \frac{\partial}{\partial x^3}\right), \; \mathbf{X}_3 = \frac{1}{2}\left(e^{x^1} \frac{\partial}{\partial x^2} - e^{-x^1} \frac{\partial}{\partial x^3}\right). \tag{2.1}$$

The bracket relations are

$$[\mathbf{X}_1, \mathbf{X}_2] = \mathbf{X}_3, \ \ [\mathbf{X}_2, \mathbf{X}_3] = 0, \ \ [\mathbf{X}_1, \mathbf{X}_3] = \mathbf{X}_2. \tag{2.2}$$

We consider left-invariant Lorentzian metrics which has a pseudo-orthonormal basis $\{\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3\}$. We consider left-invariant Lorentzian metric [10], given by

$$g = -\left(dx^1\right)^2 + \left(e^{-x^1}dx^2 + e^{x^1}dx^3\right)^2 + \left(e^{-x^1}dx^2 - e^{x^1}dx^3\right)^2, \tag{2.3}$$

where

$$g(\mathbf{X}_1, \mathbf{X}_1) = -1, \ g(\mathbf{X}_2, \mathbf{X}_2) = g(\mathbf{X}_3, \mathbf{X}_3) = 1. \tag{2.4}$$

Let coframe of our frame be defined by

$$\theta^1 = dx^1, \ \theta^2 = e^{-x^1}dx^2 + e^{x^1}dx^3, \ \theta^3 = e^{-x^1}dx^2 - e^{x^1}dx^3.$$

Proposition 2.1 *For the covariant derivatives of the Levi-Civita connection of the left-invariant metric g, defined above the following is true:*

$$\nabla = \begin{pmatrix} 0 & 0 & 0 \\ -\mathbf{X}_3 & 0 & -\mathbf{X}_1 \\ -\mathbf{X}_2 & -\mathbf{X}_1 & 0 \end{pmatrix}, \tag{2.5}$$

where the (i, j)-element in the table above equals $\nabla_{\mathbf{X}_i}\mathbf{X}_j$ for our basis

$$\{\mathbf{X}_k, k = 1, 2, 3\} = \{\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3\}.$$

§3. Spacelike Biharmonic General Helices with Timelike Normal According to Bishop Frame in the Lorentzian Group of Rigid Motions $\mathbb{E}(1,1)$

Let $\gamma : I \longrightarrow \mathbb{E}(1,1)$ be a non geodesic spacelike curve on the $\mathbb{E}(1,1)$ parametrized by arc length. Let $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ be the Frenet frame fields tangent to the $\mathbb{E}(1,1)$ along γ defined as follows:

$\mathbf{T}$ is the unit vector field γ' tangent to γ, $\mathbf{N}$ is the unit vector field in the direction of $\nabla_{\mathbf{T}}\mathbf{T}$ (normal to γ), and $\mathbf{B}$ is chosen so that $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ is a positively oriented orthonormal basis. Then, we have the following Frenet formulas:

$$\begin{aligned} \nabla_{\mathbf{T}}\mathbf{T} &= \kappa\mathbf{N}, \\ \nabla_{\mathbf{T}}\mathbf{N} &= \kappa\mathbf{T} + \tau\mathbf{B}, \\ \nabla_{\mathbf{T}}\mathbf{B} &= \tau\mathbf{N}, \end{aligned}$$

where κ is the curvature of γ and τ is its torsion and

$$\begin{aligned} g(\mathbf{T}, \mathbf{T}) &= 1, \ g(\mathbf{N}, \mathbf{N}) = -1, \ g(\mathbf{B}, \mathbf{B}) = 1, \\ g(\mathbf{T}, \mathbf{N}) &= g(\mathbf{T}, \mathbf{B}) = g(\mathbf{N}, \mathbf{B}) = 0. \end{aligned}$$

The Bishop frame or parallel transport frame is an alternative approach to defining a moving frame that is well defined even when the curve has vanishing second derivative. The Bishop frame is expressed as

$$\begin{aligned} \nabla_{\mathbf{T}}\mathbf{T} &= k_1\mathbf{M}_1 - k_2\mathbf{M}_2, \\ \nabla_{\mathbf{T}}\mathbf{M}_1 &= k_1\mathbf{T}, \\ \nabla_{\mathbf{T}}\mathbf{M}_2 &= k_2\mathbf{T}, \end{aligned} \tag{3.1}$$

where

$$\begin{aligned} &g(\mathbf{T},\mathbf{T}) = 1,\ g(\mathbf{M}_1,\mathbf{M}_1) = -1,\ g(\mathbf{M}_2,\mathbf{M}_2) = 1, \\ &g(\mathbf{T},\mathbf{M}_1) = g(\mathbf{T},\mathbf{M}_2) = g(\mathbf{M}_1,\mathbf{M}_2) = 0. \end{aligned} \tag{3.2}$$

Here, we shall call the set $\{\mathbf{T},\mathbf{M}_1,\mathbf{M}_1\}$ as Bishop trihedra, k_1 and k_2 as Bishop curvatures and $\tau(s) = \psi'(s)$, $\kappa(s) = \sqrt{|k_2^2 - k_1^2|}$. Thus, Bishop curvatures are defined by

$$\begin{aligned} k_1 &= \kappa(s)\sinh\psi(s), \\ k_2 &= \kappa(s)\cosh\psi(s). \end{aligned}$$

With respect to the orthonormal basis $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$ we can write

$$\begin{aligned} \mathbf{T} &= T^1\mathbf{e}_1 + T^2\mathbf{e}_2 + T^3\mathbf{e}_3, \\ \mathbf{M}_1 &= M_1^1\mathbf{e}_1 + M_1^2\mathbf{e}_2 + M_1^3\mathbf{e}_3, \\ \mathbf{M}_2 &= M_2^1\mathbf{e}_1 + M_2^2\mathbf{e}_2 + M_2^3\mathbf{e}_3. \end{aligned} \tag{3.3}$$

Theorem 3.1 *$\gamma : I \longrightarrow \mathbb{E}(1,1)$ is a spacelike biharmonic curve with Bishop frame if and only if*

$$\begin{aligned} &k_1^2 - k_2^2 = \text{constant} = C \neq 0, \\ &k_1'' + Ck_1 = -k_1\left[1 + 2\left(M_2^1\right)^2\right] + 2k_2M_1^1M_2^1, \\ &k_2'' + Ck_2 = -2k_1M_1^1M_2^1 - k_2\left[-1 + 2\left(M_1^1\right)^2\right]. \end{aligned} \tag{3.4}$$

Definition 3.2 *A regular spacelike curve $\gamma : I \longrightarrow \mathbb{E}(1,1)$ is called a general helix provided the spacelike unit vector $\mathbf{T}$ of the curve γ has constant angle θ with some fixed timelike unit vector u, that is*

$$g(\mathbf{T}(s), u) = \cosh\wp \text{ for all } s \in I. \tag{3.5}$$

Theorem 3.4 *Let $\gamma : I \longrightarrow \mathbb{E}(1,1)$ is a non geodesic spacelike biharmonic general helix with timelike normal in the Lorentzian group of rigid motions $\mathbb{E}(1,1)$. Then, the parametric equations* *of* γ *are*

$$\begin{aligned} x^1(s) &= \cosh\wp s + C_1, \\ x^2(s) &= \frac{\sqrt{1+\cosh^2\wp}\, e^{\cosh\wp s + C_1}}{2\cosh\wp}[\cos s + \sin s] + C_2, \\ x^3(s) &= \frac{\sqrt{1+\cosh^2\wp}\, e^{-\cosh\wp s - C_1}}{2\cosh\wp}[\cos s - \sin s] + C_3, \end{aligned} \tag{3.6}$$

where C_1, C_2, C_3 *are constants of integration.*

Proof Using (3.1) and (3.5) we have above system. □

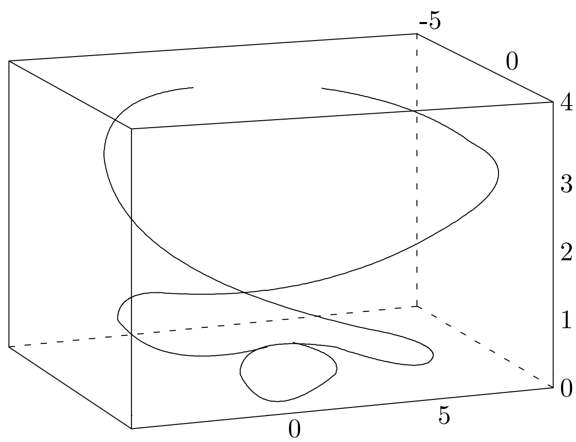

Fig.1

References

[1] K.Arslan, R.Ezentas, C.Murathan and T.Sasahara, Biharmonic submanifolds 3-dimensional (κ,μ)-manifolds, *Internat. J. Math. Sci.*, 22 (2005), 3575-3586.

[2] L.R.Bishop, There is more than one way to frame a curve, *Amer. Math. Monthly*, 82 (3) (1975) 246-251.

[3] M.P.Carmo, *Differential Geometry of Curves and Surfaces*, Pearson Education, 1976.

[4] J.Eells and J.H.Sampson, Harmonic mappings of Riemannian manifolds, *Amer. J. Math.*, 86 (1964), 109–160.

[5] G.Y.Jiang, 2-harmonic isometric immersions between Riemannian manifolds, *Chinese Ann. Math.* Ser. A 7(2) (1986), 130–144.

[6] K.İlarslan and Ö.Boyacıoğlu, Position vectors of a timelike and a null helix in Minkowski 3-space, *Chaos Solitons Fractals*, 38 (2008), 1383–1389.

[7] T.Körpınar, E.Turhan, On Horizontal Biharmonic Curves In The Heisenberg Group $Heis^3$, *Arab. J. Sci. Eng. Sect.* A Sci. 35 (1) (2010), 79-85.

[9] W.Kuhnel, *Differential Geometry, Curves-Surfaces-Manifolds,* Braunschweig, Wiesbaden, 1999.

[10] K.Onda, Lorentz Ricci Solitons on 3-dimensional Lie groups, *Geom Dedicata*, 147 (1) (2010), 313-322.

[11] E.Turhan and T.Körpınar, On characterization of timelike horizontal biharmonic curves in the Lorentzian Heisenberg Group heis^3, *Zeitschrift für Naturforschung A- A Journal of Physical Sciences*, 65a (2010), 641-648.

[12] E.Turhan, and T.Körpınar, Biharmonic slant helices according to bishop frame in E^3, *International Journal of Mathematical Combinatorics*, 3 (2010), 64-68.

[13] S.Yilmaz and M.Turgut, On the differential geometry of the curves in Minkowski spacetime I, *Int. J. Contemp. Math. Sciences*, 3 (2008),1343-1349.