International J.Math. Combin. Vol.1(2012), 58-73

Domination in Transformation Graph G^{+-+}

M.K. Angel Jebitha and J.Paulraj Joseph

(Department of Mathematics, Manonmaniam Sundaranar University, Tirunelveli- 627 012, Tamil Nadu, India)

E-mail: jebidom@gmail.com, jpaulraj_2003@yahoo.co.in

Abstract: Let $G = (V, E)$ be a simple undirected graph of order n and size m. The transformation graph of G is a simple graph with vertex set $V(G) \cup E(G)$ in which adjacency is defined as follows: (a) two elements in $V(G)$ are adjacent if and only if they are adjacent in G (b) two elements in $E(G)$ are adjacent if and only if they are non-adjacent in G and (c) one element in $V(G)$ and one element in $E(G)$ are adjacent if and only if they are incident in G. It is denoted by G^{+-+}. A set $S \subseteq V(G)$ is a *dominating set* if every vertex in $V - S$ is adjacent to at least one vertex in S. The minimum cardinality taken over all dominating sets of G is called the *domination number* of G and is denoted by $\gamma(G)$. In this paper, we investigate the domination number of transformation graph. We determine the exact values for some standard graphs and obtain several bounds. Also we prove that for any connected graph G of order $n \geq 5$, $\gamma(G^{+-+}) \leq \lceil n/2 \rceil$.

Key Words: Transformation graph, domination number, Smarandachely k-dominating set, Smarandachely k-domination number.

AMS(2010): 05C69

§1. Introduction

Let $G = (V, E)$ be a simple undirected graph of order n and size m. The *degree* of v in G is $|N(v)|$ and is denoted by $deg(v)$. The *maximum degree* of G is max $\{deg(v) : v \in V(G)\}$ and is denoted by $\Delta(G)$. The *minimum degree* of G is min$\{deg(v) : v \in V(G)\}$ and is denoted by $\delta(G)$. A vertex v is said to be *pendant vertex* if $deg(v) = 1$. A vertex u is called *support* if u is adjacent to a pendant vertex.

A path on n vertices is denoted by P_n; a cycle on n vertices is denoted by C_n and a complete graph on n vertices is denoted by K_n. A complete bipartite graph in which one partite set has r vertices and another partite set has s vertices is denoted by $K_{r,s}$. The *corona* of two graphs G_1 and G_2, is the graph $G = G_1 \circ G_2$ formed from one copy of G_1 and $|V(G_1)|$ copies of G_2 where the i^{th} vertex of G_1 is adjacent to every vertex in the i^{th} copy of G_2. If G and H are any two graphs, then $G + H$ is the graph obtained from $G \cup H$ by joining each vertex of G to every

[1]Received September 9, 2011. Accepted March 1, 2012.

vertex of H. $C_{n-1} + K_1$ is called the wheel on n vertices.

A set of vertices S in a graph G is said to be a *Smarandachely k-dominating set* if each vertex of G is dominated by at least k vertices of S and the *Smarandachely k-domination number* $\gamma_k(G)$ of G is the minimum cardinality of a Smarandachely k-dominating set of G. Particularly, if $k = 1$, such a set is called a dominating set of G and the Smarandachely 1-domination number of G is called the *domination number* of G and denoted by $\gamma(G)$ in general.

In [8], Paulraj Joseph and Arumugam studied domination parameters in subdivision graphs. Wallis [10] studied domination parameters of line graphs of designs and variations of Chess-board graph. The domination number of transformation graph G^{-+-} was studied in [1]. Terms not defined are used in the sense of [5].

The *transformation graph* of G is a simple graph with vertex set $V(G) \cup E(G)$ in which adjacency is defined as follows:(a) two elements in $V(G)$ are adjacent if and only if they are adjacent in G (b) two elements in $E(G)$ are adjacent if and only if they are non-adjacent in G and (c) one element in $V(G)$ and one element in $E(G)$ are adjacent if and only if they are incident in G. It is denoted by G^{+-+}. A graph G and it's transformation graph are given in Fig.1.1.

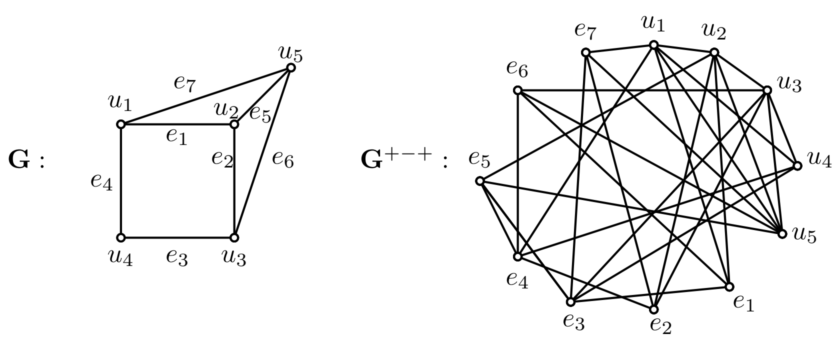


Fig.1.1 **A graph** G **and** G^{+-+}.

In this paper we study about domination number of the transformation graph G^{+-+}. We need the following theorems to obtain an upper bound for G^{+-+} in Section 5.

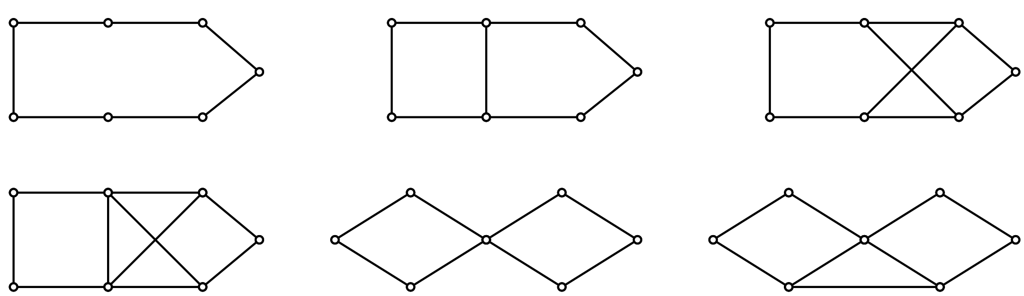


Fig.1.2 **Graphs in family** $\mathcal{A}$.

Theorem 1.1([7]) *If a graph G has no isolated vertices, then $\gamma(G) \leq n/2$.*

Theorem 1.2([3,4]) *For a graph G with even order n and no isolated vertices $\gamma(G) = n/2$ if and only if the components of G are the cycle C_4 or the corona $H \circ K_1$ for any connected graph H.*

Theorem 1.3([6]) *If G is a connected graph with $\delta(G) \geq 2$ and $G \notin \mathcal{A} \cup C_4$, then $\gamma(G) \leq 2n/5$.*

§2. Exact Values for Standard Graphs

In this section, we obtain the exact values for the domination number of the transformation graph where G is the star, complete graph, complete bipartite graph and wheel.

Theorem 2.1 $\gamma(G^{+-+}) = 1$ *if and only if* $G \cong K_{1,r},\ r \geq 1$.

Proof Assume that $G \cong K_{1,r},\ r \geq 1$. Let v be the full vertex of G. Then v is adjacent to all the vertices and all the edges of G in G^{+-+}. Therefore $D = \{v\}$ is a dominating set of G^{+-+}. Hence $\gamma(G^{+-+}) = 1$. Conversely, assume that $\gamma(G^{+-+}) = 1$. Let D be a minimum dominating set of G^{+-+}. If $D = \{e\}$, then e is incident with exactly two vertices of G and hence $G \cong K_2 \cong K_{1,1}$. If $D = \{v\}$, then v is adjacent to all the vertices and incident with all the edges of G. Hence $G \cong K_{1,r}$. □

Theorem 2.2 $\gamma(K_n^{+-+}) = 3,\ n \geq 4$.

Proof Let $V(K_n) = \{v_1, v_2, \ldots, v_n\}$. By Theorem 2.1, $\gamma(K_n^{+-+}) \geq 2$. Let S be a subset of $V(K_{r,s}^{+-+})$.

Claim 1 No 2-element subset of $V(K_n^{+-+})$ is a dominating set of K_n^{+-+}.

Suppose $S = \{v_i, v_j\}$. Since $n \geq 4$, there exist two vertices v_k and v_r such that $v_kv_r \in E(G)$ which is adjacent to neither v_i nor v_j in K_n^{+-+}. Suppose $S = \{v_i, v_jv_k\}$. If $v_i = v_j$, then an edge which is incident with v_k is adjacent to neither v_i nor v_jv_k in K_n^{+-+}. If $v_i \neq v_j$, then since $n \geq 4$, there exists a vertex v_r such that $v_jv_r, v_kv_r \in E(G)$ are adjacent to neither v_i nor v_jv_k in K_n^{+-+}. Suppose $S = \{v_iv_j, v_kv_r\}$. If v_iv_j and v_kv_r are adjacent in K_n, then there exists a vertex $v_s \in V(K_n)$ which is adjacent to neither v_iv_j nor v_kv_r in K_n^{+-+}. If v_iv_j and v_kv_r are non-adjacent in K_n, then $v_iv_k,\ v_iv_r$ are adjacent to neither v_iv_j nor v_kv_r in K_n^{+-+}. Thus in all cases, S is not a dominating set of K_n^{+-+}.

Let v_i and v_j be any two vertices of K_n. Clearly v_i dominates all the vertices of K_n in K_n^{+-+}. All the edges which are adjacent to v_iv_j are dominated by the 2-element subset $\{v_i, v_j\}$ in K_n^{+-+} and all the remaining edges of K_n which are not adjacent to v_iv_j are dominated by v_iv_j in K_n^{+-+}. Hence $\{v_i, v_j, v_iv_j\}$ is a dominating set of K_n^{+-+}. Thus $\gamma(K_n^{+-+}) = 3$. □

Theorem 2.3 $\gamma(K_{r,s}^{+-+}) = 3$, *where* $r,\ s > 2$.

Proof Let $(X,\ Y)$ be the bipartition of $K_{r,s}$ with $|X| = r$ and $|Y| = s$. Since $r,\ s > 2$, by Theorem 2.1, $\gamma(K_{r,s}^{+-+}) \geq 2$. Let S be a subset of $V(K_{r,s}^{+-+})$.

Claim 1 No 2-element subset of $V(K_{r,s}^{+-+})$ is a dominating set of $K_{r,s}^{+-+}$.

Suppose $S = \{x, y\}$. If $x,\ y \in X$, then there exists a vertex $z \in X$ which is adjacent to neither x nor y in $K_{r,s}^{+-+}$. If $x \in X$ and $y \in Y$, then there exists an edge e whose end vertices are not in S. Then e is adjacent to neither x nor y in $K_{r,s}^{+-+}$. If x and y are edges of $K_{r,s}$, then at most four vertices of $K_{r,s}$ are dominated by S. Therefore, there exists a vertex of $K_{r,s}$ which is adjacent to neither x nor y in $K_{r,s}^{+-+}$. If $x \in X$ and $y \in E(K_{r,s})$, then at most two vertices of X are dominated by S in $K_{r,s}^{+-+}$. Therefore there exists a vertex $z \in X$ which is adjacent to neither x nor y in $K_{r,s}^{+-+}$. Thus in all cases, S is not a dominating set of $K_{r,s}^{+-+}$. Hence the claim.

Let $u \in X$ and $v \in Y$. Then all the vertices of $K_{r,s}$ and all the edges which are adjacent to uv are dominated by $\{u, v\}$ in $K_{r,s}^{+-+}$. Further, the remaining edges which are not adjacent to uv are dominated by uv. Hence $\{u, v, uv\}$ is a dominating set of $K_{r,s}^{+-+}$. Thus $\gamma(K_{r,s}^{+-+}) = 3$.□

Theorem 2.4 $\gamma(W_n^{+-+}) = 3,\ n \geq 4.$

Proof Let $v_1, v_2, \ldots, v_{n-1}$ be the vertices of degree 3 and $v$be the vertex of degree $n-1$ in W_n. Let $e_i = vv_i$ where $1 \leq i \leq n-1$, $e_{i(i+1)} = v_i v_{i+1}$ where $1 \leq i \leq n-2$ and $e_{1(n-1)} = v_1 v_{n-1}$.

If $n = 4$, then $W_n \cong K_n$ and hence by Theorem 2.2, $\gamma(W_n^{+-+}) = 3$. Now, let $n \geq 5$. By Theorem 2.1, $\gamma(W_n^{+-+}) \geq 2$.

Claim 1 No 2-element subset of $V(W_n^{+-+})$ is a dominating set of W_n^{+-+}.

Now,$\{v, v_i\}$ is not a dominating set of W_n^{+-+} since there exists an edge e_{jk} which is not adjacent to either v or v_i in W_n^{+-+}; $\{v_i, v_j\}$ is not a dominating set of W_n^{+-+} since there exists an edge e_k which is not adjacent to either v_i or v_j in W_n^{+-+}. Hence no 2-element subset of $V(W_n)$ is a dominating set of W_n^{+-+}. Since any two edges of W_n are adjacent to at most 4 vertices of W_n in W_n^{+-+}, no 2-element subset of $E(W_n)$ is a dominating set of W_n^{+-+}. Now, $\{v, e_i\}$ is not a dominating set of W_n^{+-+} since adjacent edges of e_i in the cycle C_{n-1} are not adjacent to either v or e_i in W_n^{+-+}; $\{v, e_{ij}\}$ is not a dominating set of W_n^{+-+} since adjacent edges of e_{ij} in the cycle C_{n-1} are not adjacent to either v or e_{ij} in W_n^{+-+}; $\{v_i, e_j\}$ is not a dominating set of W_n^{+-+} since there exists an edge e_r which is not adjacent to either v_i or e_j; $\{v_i, e_{jk}\}$ is not a dominating set of W_n^{+-+} since at least one of $\{e_j, e_k\}$ is not adjacent to either v_i or e_{jk}. Hence no 2-element subset containing one vertex and one edge of W_n is not a dominating set of W_n^{+-+}. Thus the claim.

If $D' = \{v, e_{ij}, e_{jk}\}$, then in W_n^{+-+}, v is adjacent to all the vertices of W_n and all the spokes of W_n; e_{ij} is adjacent to all the non-adjacent edges of e_{ij}; e_{jk} is adjacent to all the adjacent edges of e_{ij} in the cycle C_{n-1}. Hence D' is a dominating set of W_n^{+-+}. Thus $\gamma(W_n^{+-+}) = 3$.□

§3. Some Bounds for $\gamma(G^{+-+})$

In this section, we obtain the upper bounds for the domination number of the transformation graph by considering the order, maximum and minimum degrees of G.

Theorem 3.1 *If G is a connected graph with $\Delta(G) = n-2$, then $\gamma(G^{+-+}) \le 3$.*

Proof Let v be a vertex of degree $\Delta(G)$ and $V - N[v] = \{u\}$. Let u be adjacent to $v_i \in N(v)$. Then v dominates all the vertices except u of G and all the edges incident with v in G^{+-+}. Also v_i dominates u and all the edges incident with v_i in G^{+-+}. Further, all the remaining edges are dominated by vv_i in G^{+-+}. Hence $\{v, v_i, vv_i\}$ is a dominating set of G^{+-+}. Thus $\gamma(G^{+-+}) \le 3$. □

Theorem 3.2 *If a graph G has $diam(G) = 2$, then $\gamma(G^{+-+}) \le \delta(G) + 1$ and the bound is sharp.*

Proof Let $v \in V(G)$ such that $deg(v) = \delta(G)$ and $N(v) = \{v_1, v_2, \ldots, v_\delta\}$. Then vv_i is adjacent to all the non-adjacent edges of vv_i in G^{+-+}. Further $N(v)$ dominates all the edges incident with v or v_i and also all the vertices of G in G^{+-+}. Hence $N(v) \cup \{vv_i\}$ is a dominating set of G^{+-+}. Thus $\gamma(G^{+-+}) \le \delta(G) + 1$. Further, for the graph G in Fig.3.1, $\gamma(G^{+-+}) = 3 = \delta(G) + 1$. □

G :

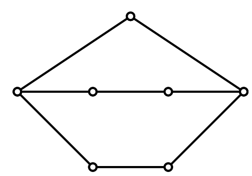


Fig.3.1 **A graph** G **with** $\gamma(G^{+-+}) = \delta(G) + 1$.

Theorem 3.3 *For any connected graph G with $\Delta(G) < n-1$, $\gamma(G^{+-+}) \le n - \Delta(G) + 1$ and the bound is sharp.*

Proof Let $deg(v) = \Delta(G)$ and $N(v) = \{v_1, v_2, \ldots, v_\Delta\}$. Since G is connected, there is a vertex $u \in V(G) - N[v]$ which is adjacent to at least one vertex $v_i \in N(v)$. Then $[(V(G) - N(v)) - \{u\}] \cup \{v_i\}$ dominates all the vertices of G. Now v dominates all the edges incident with v in G^{+-+}. The vertex v_i dominates all the edges incident with v_i and the edge vv_i dominates all the non-adjacent edges of vv_i in G^{+-+}. Therefore $[(V(G) - N(v)) - \{u\}] \cup \{v_i, vv_i\}$ is a dominating set of G^{+-+}. Hence $\gamma(G^{+-+}) \le n - \Delta(G) + 1$. Further, for the graph G in Fig.3.2, $\gamma(G^{+-+}) = 4 = n - \Delta(G) + 1$. □

G :

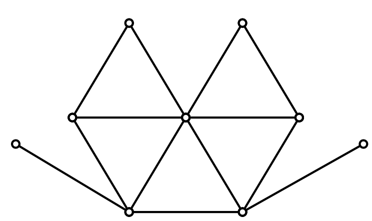


Fig.3.2 **A graph with** $\gamma(G^{+-+}) = n - \Delta(G) + 1$.

Theorem 3.4 *Let G be a connected graph of order $n > 2$ with $\Delta(G) = n-1$ and v be a vertex* *of degree $\Delta(G)$. Then $\gamma(G^{+-+}) = 2$ if and only if $\langle N(v)\rangle$ is non-empty and it contains K_1 or K_2 or is isomorphic to $K_{1,r},\ r \geq 2$.*

Proof Assume that $\gamma(G^{+-+}) = 2$. By Theorem 2.1, $\langle N(v)\rangle$ is non-empty. Suppose that $\langle N(v)\rangle$ is not isomorphic to $K_{1,r},\ r \geq 2$ and contains neither K_1 nor K_2. For $n \leq 4$, the result is easily verified.

Now, let $n \geq 5$. Then $\langle N(v)\rangle$ contains P_4 or C_3.

Since any 2-element subset of $E(G)$ is adjacent to at most 4 vertices of G in G^{+-+}, no 2-element subset of $E(G)$ is a dominating set of G^{+-+}. Let $S = \{x, y\}$. Suppose $x \in V(G)$ and $y \in E(G)$. If $x = v$, then there exists an edge in $\langle N(v)\rangle$ which is not adjacent to either x or y in G^{+-+}. If $x \in N(v)$, then there exists an edge which is incident with v which is not adjacent to either x or y in G^{+-+}. Suppose $x, y \in V(G)$. Since $n \geq 5$, there exist two vertices u and w of G such that $uw \in E(G)$ which is not adjacent to either x or y in G^{+-+}. Hence $\gamma(G^{+-+}) \geq 3$ which is a contradiction. Therefore $\langle N(v)\rangle$ contains K_1 or K_2 or is isomorphic to $K_{1,r},\ r \geq 2$.

Conversely, assume that $\langle N(v)\rangle$ is non-empty and it contains K_1 or K_2 or is isomorphic to $K_{1,r},\ r \geq 2$. By Theorem 2.1, $\gamma(G^{+-+}) \geq 2$. If $\langle N(v)\rangle$ contains $K_1 = \{u\}$, then $\{v, uv\}$ is a dominating set of G^{+-+}. If $\langle N(v)\rangle$ contains $K_2 = \{e\}$, then $\{v, e\}$ is a dominating set of G^{+-+}. If $\langle N(v)\rangle \cong K_{1,r},\ r \geq 2$ and u is the centre vertex, then $\{v, u\}$ is a dominating set of G^{+-+}. Hence $\gamma(G^{+-+}) = 2$. □

Remark 3.5 If $\Delta(G) = n - 1$, then $\gamma(G^{+-+})$ may be 3 which is greater than $n - \Delta(G) + 1$. For the graphs G_1 and G_2 in Fig.3.3, $\gamma(G_1^{+-+}) = \gamma(G_2^{+-+}) = 3$.

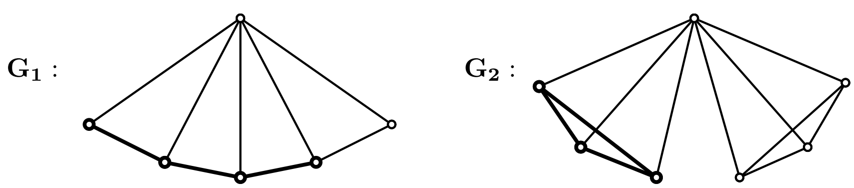


Fig.3.3 **Graphs with** $\gamma(G_1^{+-+}) = \gamma(G_2^{+-+}) > n - \Delta(G) + 1$.

Theorem 3.6 *For any graph G, $\gamma(G) \leq \gamma(G^{+-+}) \leq \gamma(G) + 2$ and the bounds are sharp.*

Proof Let D be a minimum dominating set of G and D' be a minimum dominating set of G^{+-+}.

Claim 1 $\gamma(G) \leq \gamma(G^{+-+})$.

Suppose $\gamma(G) > \gamma(G^{+-+})$. Then $|D| > |D'|$. If D' contains no edge of G, then D' is a dominating set of G with $|D'| < |D|$ which is a contradiction. If D' contains $k \geq 1$ edges of G, say $u_1v_1, u_2v_2, \ldots, u_kv_k$, then this k edges dominate $k_1 \leq 2k$ vertices of G in G^{+-+}. Therefore $[D' - \{u_1v_1, u_2v_2, \ldots, u_kv_k\}] \cup \{u_1, u_2, \ldots, u_k\}$ is a dominating set of cardinality $|D'| < |D|$ of G which is a contradiction.

Claim 2 $\gamma(G^{+-+}) \leq \gamma(G) + 2$.

All the vertices of G are dominated by D in G^{+-+}. If G is an empty graph, then $\gamma(G^{+-+}) =$

$\gamma(G) = n$. If G is non-empty, then G has an edge $e = uv$ whose one end is in D. Without loss of generality, let $u \in D$. Then every edge which is adjacent to e is adjacent to u or v in G^{+-+} and all the edges non-adjacent to e in G are adjacent to e in G^{+-+}. Hence $D \cup \{e, v\}$ is a dominating set of G^{+-+}. Thus $\gamma(G^{+-+}) \leq \gamma(G) + 2$.

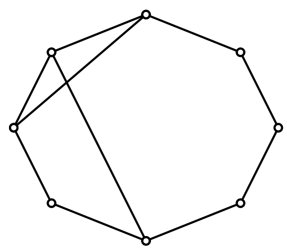
Further, $\gamma((H \circ K_1)^{+-+}) = n/2 = \gamma(H \circ K_1)$ and $\gamma((K_n)^{+-+}) = 3 = \gamma(K_n) + 2$, where $n \geq 4$. □

§4. Domination Transformation (DT) Classes

All the graphs are classified into three broad categories according to the lower and upper bounds in Theorem 3.6.

Definition 4.1 *A graph belongs to DT-class 1 if* $\gamma(G^{+-+}) = \gamma(G) + 1$.

G₁ :

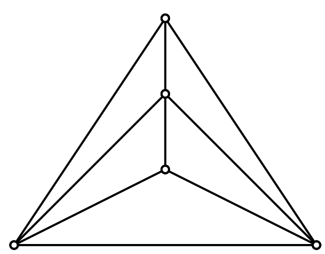


$\gamma(\mathbf{G_1}) = \mathbf{3};\ \gamma(\mathbf{G_1^{+-+}}) = \mathbf{3+1=4}$

Fig.4.1 **A graph in DT-class 1.**

Definition 4.2 *A graph belongs to DT-class 2 if* $\gamma(G^{+-+}) = \gamma(G) + 2$.

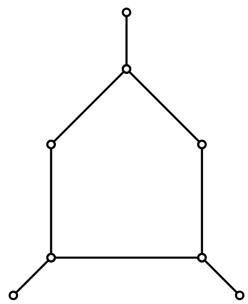
G₂ :



$\gamma(\mathbf{G_2}) = \mathbf{1};\ \gamma(\mathbf{G_2^{+-+}}) = \mathbf{1+2=3}$

Fig.4.2 **A graph in DT-class 2.**

Definition 4.3 *A graph G belongs to DT-class 3 if* $\gamma(G^{+-+}) = \gamma(G)$.



$\gamma(\mathbf{G_3}) = \gamma(\mathbf{G_3^{+-+}}) = \mathbf{3}$

Fig.4.3 **A graph in DT-class 3.**

Theorem 4.4 *For $n \neq 4$, C_n is in DT-class 1.*

Proof We know that $\gamma(C_n) = \lceil n/3 \rceil$. We have to prove that $\gamma(C_n^{+-+}) = \lceil n/3 \rceil + 1$. By Theorem 3.6, $\gamma(C_n^{+-+}) \geq \lceil n/3 \rceil$. If $n = 3$, then $\gamma(C_n^{+-+}) = 2 = \lceil n/3 \rceil + 1$. Now let $n \geq 5$.

Claim 1 There is no dominating set with cardinality $\lceil n/3 \rceil$ in C_n^{+-+}.
Let S be any subset of $V(C_n^{+-+})$ with cardinality $\lceil n/3 \rceil$.

Case 1: $S \subseteq V(C_n)$.

If S is not a dominating set of C_n, then S is not a dominating set of C_n^{+-+}. Suppose S is a minimum dominating set of C_n. Since $n \geq 5$, there exist two adjacent vertices u and v which are not in S. Therefore, the edge $e = uv$ is not dominated by S. Hence S is not a dominating set of C_n^{+-+}.

Case 2 $S \subseteq E(C_n)$.

Since each edge of C_n is adjacent to exactly two vertices of C_n in C_n^{+-+}, at most $2|S| = 2\lceil n/3 \rceil$ vertices of C_n are dominated by S. Since $n \geq 5$, $2\lceil n/3 \rceil < n$. Therefore S is not a dominating set of C_n^{+-+}.

Case 3 $S \subseteq V(C_n) \cup E(C_n)$.

Let S contain $k \geq 1$ edges. Then at most $2k$ vertices of C_n are dominated by k edges of C_n in C_n^{+-+}. Further, at most $3(|S| - k)$ vertices of C_n are dominated by $|S| - k$ vertices of C_n. Now, $3|S| - 3k + 2k = 3\lceil n/3 \rceil - k = n - k < n$. Therefore, at most $n - k$ vertices of C_n are dominated by S. Hence S is not a dominating set of C_n^{+-+}. Thus in all cases, S is not a dominating set of C_n^{+-+}.

Let D be a minimum dominating set of C_n. Since $n \geq 5$, there exist two adjacent vertices x and y of C_n which are not in D and two vertices x' and y' which are adjacent to x and y respectively are in D. Then the edge $e = xy \in E(C_n)$is adjacent to all the edges whose end vertices are not in D. Further, two adjacent edges of e are adjacent to either x' or y' and all the edges whose one end in D are dominated by D in C_n^{+-+}. Hence $D \cup \{e\}$ is a dominating set of C_n^{+-+}. Thus $\gamma(C_n^{+-+}) = \lceil n/3 \rceil + 1$. $\square$

Theorem 4.5 *If G is a graph with $\delta(G) = 1$, then G is not in DT-class 2.*

Proof Let D be a minimum dominating set of G with all the supports. Let v be a pendant vertex and u be the support of v. Hence $u \in D$. Then all the vertices of G are dominated by D in G^{+-+}. Further, all the edges which are non-adjacent to the edge uv are adjacent to uv in G^{+-+} and all the edges which are adjacent to uv are adjacent to u in G^{+-+}. Hence $D \cup \{uv\}$ is a dominating set of G^{+-+}. Therefore, $\gamma(G^{+-+}) \leq \gamma(G) + 1$. Hence by Theorem 3.6, G is in DT-class 1 or DT-class 3. □

Theorem 4.6 *If $n \equiv 0, 2$ (mod 3), $n \geq 6$, then P_n is in DT-class 1; otherwise DT-class 3.*

Proof The result can be easily verified for $n \leq 5$. Let $P_n = v_1 v_2 \ldots v_n$ and D be a minimum dominating set of P_n. By Theorem 3.6, $\gamma(P_n^{+-+}) \geq |D| = \lceil n/3 \rceil$.

Case 1 $n = 3k + 1, \ k \geq 2$.

Consider $D_1 = \{v_2, v_5, \ldots, v_{3k-1}\}$. Now, all the vertices of P_n except v_{3k+1} are dominated by D_1 in G^{+-+}. Also all edges whose one end vertex is in D_1 are dominated by D_1 in P_n^{+-+}. Further, $v_{3k}v_{3k+1}$ dominates v_{3k+1} and all the edges whose end vertices are not in D_1 at P_n^{+-+}. Hence $D_1 \cup \{v_{3k}v_{3k+1}\}$ is a dominating set of P_n^{+-+}. Therefore $\gamma(P_n^{+-+}) \leq |D_1| + 1 = \lceil n/3 \rceil$. Hence $\gamma(P_n^{+-+}) \leq \lceil n/3 \rceil$.

Case 2 $n = 3k, \ k \geq 2$.

Let S be any subset of $V(P_n^{+-+})$ with k elements.

Subcase 1 $S \subseteq V(P_n)$.

If S is not a dominating set of P_n, then S is not a dominating set of P_n^{+-+}. If S is a dominating set of P_n, then since $n = 3k$ and $n > 5$ there is an edge whose end vertices are not in S. Hence S is not a dominating set of P_n^{+-+}.

Subcase 2 $S \subseteq E(P_n)$.

Since each edge of P_n is adjacent to exactly two vertices of P_n in P_n^{+-+}, at most $2|S| = 2\lceil n/3 \rceil$ vertices of P_n are dominated by S. Since $n > 5, \ 2\lceil n/3 \rceil < n$. Therefore, S is not a dominating set of G^{+-+}.

Subcase 3 $S \subseteq V(P_n) \cup E(P_n)$.

If S contains $r \geq 1$ edges, then at most $2r$ vertices are dominated by r edges. Further, at most $3(k-r)$ vertices are dominated by $|S| - r$ vertices in P_n^{+-+}. Now, $2r + 3(k-r) = 3k - r = n - r < n$.

Hence S is not a dominating set of P_n^{+-+}. By Theorem 4.5, $\gamma(P_n^{+-+}) \leq \lceil n/3 \rceil + 1$ and hence $\gamma(P_n^{+-+}) = \lceil n/3 \rceil + 1$.

Case 3 $n = 3k + 2, \ k \geq 2$.

Let $Q = (v_1 v_2 v_3 \ldots v_{3k} v_{3k+1})$ be a path on $3k+1$ vertices and D' be a minimum dominating set of Q^{+-+}. Then $|D'| = k + 1$.

Claim 1 D' contains no pendant vertex of Q.

Suppose $v_1 \in D'$. Then v_1 dominates v_2 and v_1v_2 in Q^{+-+}. By Case(ii), the remaining vertices and edges of Q are not dominated by any k-element subset of $V(Q^{+-+})$. Therefore, $|D'| > k+1$ which is a contradiction. Hence $v_1 \notin D'$. Similarly, $v_{3k+1} \notin D'$.

Therefore $v_{3k}v_{3k+1}$ or $v_{3k} \in D'$. Then v_{3k+2} is not dominated by D' in P_n^{+-+}. Hence $\gamma(P_n^{+-+}) > k+1$. But by Theorem 4.5, $\gamma(P_n^{+-+}) \leq k+2$. Thus $\gamma(P_n^{+-+}) = k+2 = \lceil n/3 \rceil + 1$. □

Definition 4.7 *Two supports u and v are said to be successive supports if no internal vertex of any $u-v$ path is a support.*

Theorem 4.8 *Let T be a tree. If any two successive supports are of distance 1, 2 or 4 in T, then T is in DT-class 3.*

Proof Let D be a minimum dominating set of T containing all the supports of T. If any two successive supports are at distance 1, or 2, then D contains supports only. Then all the vertices and edges of T are dominated by D in T^{+-+}. Therefore $\gamma(T^{+-+}) \leq |D|$. By Theorem 3.6, $\gamma(T^{+-+}) = |D| = \gamma(T)$. If there exist two successive supports x and y at distance 4, then $x,\ y \in D$ and $w \in D$ where w is the vertex of distance 2 from x and y. Therefore, in T^{+-+} all the edges of $x-y$ path are dominated by $\{x, y, w\}$ which is also subset of D. Hence $\gamma(T^{+-+}) = \gamma(T)$. □

Theorem 4.9 *Let T be a tree. If there exists a support v such that $d(v,x) \equiv 1(mod\ 3)$ for every successive support x of v, then T is in DT-class 3.*

Proof Let v be a support of T such that $d(v,x) \equiv 1(mod\ 3)$ for every successive support x of v. Let v' be the pendant vertex adjacent to v and x' be the pendant vertex adjacent to x. Let D be a minimum dominating set of G. Then v or $v' \in D$. Since $d(v,x) \equiv 1(mod\ 3)$, we can choose a minimum dominating set D_1 of T containing the neighbors of v such that $|D| = |D_1|$. Then $(D_1 - \{v'\}) \cup \{vv'\}$ dominates all the vertices of T in T^{+-+}. Further, vv' dominates all the edges which are non-adjacent to vv' and the adjacent edges of vv' are dominated by the neighbor of v in D_1. Hence $(D_1 - \{v'\}) \cup \{vv'\}$ is a dominating set of T^{+-+} of cardinality $|D| = \gamma(T)$. □

Theorem 4.10 *If G is a disconnected graph with K_2 as one of the components of G, then G is in DT-class 3.*

Proof Let $G_1,\ G_2, \ldots, G_k$ be the components of G and $G_i \cong K_2 = uv$. Let $D_1,\ D_2, \ldots, D_i, \ldots, D_k$ be minimum dominating sets of $G_1, G_2, \ldots, G_i, \ldots, G_k$ respectively. Therefore, $|D_i| = 1$. All the vertices except u and v of G are dominated by $D_1 \cup D_2 \cup \ldots \cup D_{i-1} \cup D_{i+1} \cup \ldots \cup D_k$ in G^{+-+}. Further, $u,\ v$ and all the edges of G are dominated by uv in G^{+-+}. Hence $D_1 \cup D_2 \cup \ldots \cup D_{i-1} \cup D_{i+1} \cup \ldots \cup D_k \cup \{uv\}$ is a dominating set of cardinality $\gamma(G)$ of G^{+-+}. Hence by Theorem 3.6, $\gamma(G^{+-+}) = \gamma(G)$. □

§5. The Upper Bound $\lceil n/2 \rceil$

In this section, we consider the connected graphs of order 6, 8, 10 and prove that $\lceil n/2 \rceil$ is an upper bound for the domination number of the transformation graph where G is any connected graph of order n.

Lemma 5.1 *Let G be a connected graph of order 6. Then $\gamma(G^{+-+}) \leq 3$ and the bound is sharp.*

Proof If $\gamma(G) = 1$, then by Theorem 3.6, $\gamma(G^{+-+}) \leq 3$. If $\gamma(G) = 3$, then $G \cong H \circ K_1$ where $H \cong P_3$ or C_3. Then $V(H)$ is a minimum dominating set of G^{+-+}. Hence $\gamma(G^{+-+}) = 3$.

Now, let $\gamma(G) = 2$. If $\delta(G) = 1$, then by Theorem 4.5, $\gamma(G^{+-+}) \leq 3$. Assume that $\delta(G) \geq 2$. If $\Delta(G) = 5$ or 4, then by Theorem 3.1 and Theorem 3.6, $\gamma(G^{+-+}) \leq 3$. If $\Delta(G) = 2$, then $G \cong C_6$ and hence $\gamma(G^{+-+}) = 3$. Now, assume that $\Delta(G) = 3$. Let v be a vertex of degree 3. Then let $N(v) = \{v_1, v_2, v_3\}$ and $V - N[v] = \{u_1, u_2\}$. If $N(u_1) \cap N(u_2) \neq \phi$ in $N(v)$, then let $v_i \in N(u_1) \cap N(u_2)$. Now, v dominates all the vertices of $N[v]$ and all the edges which are incident with v in G^{+-+}. The vertex v_i dominates all the vertices of $V - N[v]$ and all the edges which are incident with v_i in G^{+-+}. Further vv_i dominates all the remaining edges which are not adjacent to vv_i. Hence $\{v, v_i, vv_i\}$ is a dominating set of G^{+-+}. If $N(u_1) \cap N(u_2) = \phi$ in $N(v)$, then let u_1 be adjacent to v_1 and u_2 be adjacent to v_2. Then u_1v_1 dominates u_1, v_1 and all the edges which are not adjacent to u_1v_1 in G^{+-+}; u_2v_2 dominates u_2, v_2 and all the edges which are adjacent to u_1v_1 except u_1u_2, v_1v_2 in G^{+-+}; vv_3 dominates v, v_3 and the edges u_1u_2, v_1v_2 in G^{+-+}. Hence $\{u_1v_1, u_2v_2, vv_3\}$ is a dominating set of G^{+-+}. Thus $\gamma(G^{+-+}) \leq 3$. Further, $\gamma(C_6^{+-+}) = 3$ and hence the bound is sharp. □

Lemma 5.2 *Let G be a connected graph of order 8. Then $\gamma(G^{+-+}) \leq 4$ and the bound is sharp.*

Proof If $\gamma(G) \leq 2$, then by Theorem 3.6, $\gamma(G^{+-+}) \leq 4$. If $\gamma(G) = 4$, then $G \cong H \circ K_1$ for some connected graph H of order 4. Then $V(H)$ is a minimum dominating set of G^{+-+}. Hence $\gamma(G^{+-+}) = 4$.

Now, let $\gamma(G) = 3$. If $\delta(G) = 1$, then by Theorem 4.5, $\gamma(G^{+-+}) \leq 4$. Assume that $\delta(G) \geq 2$. If $\Delta(G) = 6$ or 7, then by Theorem 3.1 and Theorem 3.6, $\gamma(G^{+-+}) \leq 3$. If $\Delta(G) = 2$, then $G \cong C_8$ and hence $\gamma(G^{+-+}) = 4$. Let v be a vertex of degree $\Delta(G)$. We consider the following three cases.

Case 1 $\Delta(G) = 5$.

Let $N(v) = \{v_1, v_2, v_3, v_4, v_5\}$ and $V - N[v] = \{u_1, u_2\}$. If $N(u_1) \cap N(u_2) \neq \phi$ in $N(v)$, then let $v_i \in N(u_1) \cap N(u_2)$. Now, v dominates all the vertices of $N[v]$ and all the edges which are incident with v in G^{+-+}. The vertex v_i dominates all the vertices of $V - N[v]$ and all the edges which are incident with v_i in G^{+-+}. Further vv_i dominates all the remaining edges which are not adjacent to vv_i. Hence $\{v, v_i, vv_i\}$ is a dominating set of G^{+-+}.

If $N(u_1) \cap N(u_2) = \phi$ in $N(v)$, then let u_1 be adjacent to v_1 and u_2 be adjacent to v_2. Let $D = \{v, v_1, vv_1, u_2\}$. Now, v dominates all the vertices of $N[v]$ and all the edges which are incident with v in G^{+-+}; v_1 dominates all the edges which are incident with v_1 and the vertex u_1 in G^{+-+}; vv_1 dominates all the remaining edges which are not adjacent to vv_1 in G^{+-+}. Also u_2 dominates itself. Hence D is a dominating set of G^{+-+}. Thus $\gamma(G^{+-+}) \leq 4$.

Case 2 $\Delta(G) = 4$.

Let $N(v) = \{v_1, v_2, v_3, v_4\}$ and $V - N[v] = \{u_1, u_2, u_3\}$. If $N(u_i) \cap N(u_j) \neq \phi$ in $N(v)$, then let $v_k \in N(u_i) \cap N(u_j)$. Let $D = \{v, v_k, vv_k, u_r\}$ where $r \notin \{i, j\}$. All the vertices of G are dominated by D in G^{+-+}. Also all the adjacent edges of vv_k are dominated by $\{v, v_k\}$ and all the non-adjacent edges of vv_k are dominated by the edge vv_k in G^{+-+}. Hence D is a dominating set of G^{+-+}.

Now, let $N(u_i) \cap N(u_j) = \phi$ for all i and j in $N(v)$. Then the induced subgraph $\langle V - N[v]\rangle$ is not isomorphic to $\overline{K_3}$ and hence it is isomorphic to $K_2 \cup K_1$ or P_3 or C_3.

If $\langle V - N[v]\rangle \cong K_2 \cup K_1$, then let $u_1u_2 \in E(G)$ and $u_3v_i, u_3v_j \in E(G)$. Now, v dominates all the vertices of $N[v]$; u_1u_2 dominates u_1, u_2, all the edges in $N[v]$ and all the edges incident with u_3; u_3v_i dominates u_3 and all the edges adjacent to u_1u_2. Therefore $\{v, u_1u_2, u_3v_i\}$ is a dominating set of G^{+-+}. If $\langle V - N[v]\rangle \cong P_3$, then let u_1 and u_3 be pendant vertices in $\langle V - N[v]\rangle$ and $u_1v_i, u_3v_j \in E(G)$. v dominates all the vertices of $N[v]$ and all the edges which are incident with v in G^{+-+}. A vertex v_i dominates u_1 and all the edges incident with v_i in G^{+-+}. The edge vv_i dominates all the remaining edges which are non-adjacent to vv_i in G^{+-+}. Also, u_2 dominates u_3 and itself in G^{+-+}. Hence $\{v, v_i, vv_i, u_2\}$ is a dominating set of G^{+-+}. If $\langle V - N[v]\rangle \cong C_3$, then let u_1 be adjacent to v_i. Then $\{v, v_i, vv_i, u_2\}$ is a dominating set of G^{+-+}. Hence $\gamma(G^{+-+}) \leq 4$.

Case 3 $\Delta(G) = 3$.

Let $N(v) = \{v_1, v_2, v_3\}$ and $V - N[v] = \{u_1, u_2, u_3, u_4\}$. If $N(u_i) \cap N(u_j) \neq \phi$ for some $i,\ j$ in $N(v)$, then let $v_k \in N(u_1) \cap N(u_2)$. If $u_3u_4 \in E(G)$, then $\{v, v_k, vv_k, u_3u_4\}$ is a dominating set of G^{+-+}. If u_3 and u_4 are adjacent to a common vertex $x \in V(G)$, then $\{v, v_k, vv_k, x\}$ is a dominating set of G^{+-+}. If u_3 and u_4 are not adjacent to a common vertex and $u_3u_4 \notin E(G)$, then let u_3 be adjacent to v_i and u_1; and u_4 is adjacent to v_j and u_2. Hence $\{v, u_1, u_2, vv_k\}$ is a dominating set of G^{+-+}.

Now, let $N(u_i) \cap N(u_j) = \phi$ for all $i,\ j$ in $N(v)$. Then the induced subgraph $\langle V - N[v]\rangle$ is not isomorphic to $\overline{K_4}$ or $P_3 \cup K_1$ and hence it is isomorphic to $C_3 \cup K_1$ or P_4 or a graph with at least one vertex u_i of $V - N[v]$ is of degree three.

If $\langle V - N[v]\rangle \cong C_3 \cup K_1$ where $C_3 = u_1u_2u_3u_1$, then u_4 is adjacent to a vertex v_i. Therefore, $\{v, v_i, vv_i, u_1\}$ is a dominating set of G^{+-+}.

If $\langle V - N[v]\rangle \cong P_4 = u_1u_2u_3u_4$, then u_1 is adjacent to at least one vertex of v_i and u_4 is adjacent to at least one vertex of v_j. Let $D = \{v, u_2u_3, u_1v_i, u_4v_j\}$. The vertex v dominates all the vertices of $N[v]$ and all the edges incident with v in G^{+-+}. The edge u_2u_3 dominates $u_2,\ u_3$, all the edges in $N[v]$ and all the edges incident with u_1 or u_4 except u_1u_2 and u_3u_4 in G^{+-+}. Also the edge u_1v_i dominates $u_1,\ u_3u_4$ and all the edges incident with u_2 or u_3 except $u_1u_2,\ v_iu_2,\ v_iu_3$ in G^{+-+}. Further, u_4v_j dominates $u_4,\ u_1u_2,\ v_iu_2,\ v_iu_3$ in G^{+-+}. Hence D is a dominating set of G^{+-+}.

If at least one vertex u_i of $V - N[v]$ is of degree three in $\langle V - N[v]\rangle$, then at least one vertex

$u_j(\neq u_i)$ in $V - N[v]$ is adjacent to a vertex v_k of $N(v)$. Then $\{v, v_k, vv_k, u_i\}$ is a dominating set of G^{+-+}. Further, $\gamma(C_8^{+-+}) = 4$ and hence the bound is sharp. □

Lemma 5.3 *Let G be a connected graph of order 10. Then $\gamma(G^{+-+}) \leq 5$. and the bound is sharp.*

Proof If $\gamma(G) \leq 3$, then by Theorem 3.6, $\gamma(G^{+-+}) \leq 5$. If $\gamma(G) = 5$, then $G = H \circ K_1$ for some connected graph H of order 5. Then $V(H)$ is a minimum dominating set of G^{+-+}. Hence $\gamma(G^{+-+}) = 5$. Now, let $\gamma(G) = 4$. If $\delta(G) = 1$, then by Theorem 4.5 $\gamma(G^{+-+}) \leq 5$. Now, assume that $\delta(G) \geq 2$. If $\Delta(G) = 8$ or 9, then by Theorem 3.1 and Theorem 3.6, $\gamma(G^{+-+}) \leq 3$. If $\Delta(G) = 2$, then $G \cong C_{10}$. Therefore $\gamma(G^{+-+}) = 5$. Let v be a vertex of degree $\Delta(G)$. Then we consider the following five cases.

Case 1 $\Delta(G) = 7$.

Let $N(v) = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$ and $V - N[v] = \{u_1, u_2\}$. Therefore $\{v, u_1, u_2\}$ is a dominating set of G and hence $\gamma(G^{+-+}) \leq 3 + 2 = 5$.

Case 2 $\Delta(G) = 6$.

Let $N(v) = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ and $V - N[v] = \{u_1, u_2, u_3\}$. If $N(u_i) \cap N(u_j) \neq \phi$ in $N(v)$, then let $v_k \in N(u_i) \cap N(u_j)$. Let $D' = \{v, v_k, vv_k, u_r\}$ where $r \notin \{i, j\}$. Clearly, all the vertices are dominated by D' in G^{+-+}. Also, all the adjacent edges of vv_k are dominated by $\{v, v_k\}$ and all the non-adjacent edges of vv_k are dominated by vv_k in G^{+-+}. Hence D' is a dominating set of G^{+-+}.

Assume that $N(u_i) \cap N(u_j) = \phi$ for all i, j in $N(v)$. Then the induced subgraph $\langle V - N[v] \rangle$ is isomorphic to $\overline{K_3}$ or $K_2 \cup K_1$ or P_3 or C_3.If $\langle V - N[v] \rangle \cong \overline{K_3}$, let u_i be adjacent to v_i. Then $\{v, v_i, vv_i, u_j, u_k\}$ is a dominating set of G^{+-+}. If $\langle V - N[v] \rangle \cong K_2 \cup K_1$, let $u_1u_2 \in E(G)$ and $u_3v_i, u_3v_j \in E(G)$. Then $\{v, u_1u_2, u_3v_i\}$ is a dominating set of G^{+-+}. If $\langle V - N[v] \rangle \cong P_3$ where $P_3 = u_1u_2u_3$, then $u_1v_i, u_3v_j \in E(G)$ and hence $\{v, v_i, vv_i, u_2\}$ is a dominating set of G^{+-+}. If $\langle V - N[v] \rangle \cong C_3$, let u_i be adjacent to v_i. Then $\{v, v_i, vv_i, u_j\}$ is a dominating set of G^{+-+}. Hence $\gamma(G^{+-+}) \leq 5$.

Case 3 $\Delta(G) = 5$.

Let $N(v) = \{v_1, v_2, v_3, v_4, v_5\}$ and $V - N[v] = \{u_1, u_2, u_3, u_4\}$. If $N(u_i) \cap N(u_j) \neq \phi$ in $N(v)$, let $v_k \in N(u_1) \cap N(u_2)$. Then $\{v, v_k, vv_k, u_3, u_4\}$ is a dominating set of G^{+-+}.

Now, let $N(u_i) \cap N(u_j) = \phi$ for all i and j in $N(v)$. Then the induced subgraph $\langle V - N[v] \rangle$ is not isomorphic to $\overline{K_4}$, or $K_2 \cup 2K_1$ and hence it is isomorphic to $P_3 \cup K_1$ or $C_3 \cup K_1$ or P_4 or C_4 or a graph with at least one vertex of $V - N[v]$ of degree 3. If $\langle V - N[v] \rangle \cong P_3 \cup K_1$ where $P_3 = u_1u_2u_3$, then u_1 is adjacent to at least one vertex v_k in $N(v)$ and hence $\{v, v_k, vv_k, u_2u_3, u_4\}$ is a dominating set of G^{+-+}. If $\langle V - N[v] \rangle \cong C_3 \cup K_1$ where $C_3 = u_1u_2u_3u_1$, then u_4 is adjacent to a vertex v_i. Therefore $\{v, v_i, vv_i, u_1\}$ is a dominating set of G^{+-+}. If $\langle V - N[v] \rangle \cong P_4 = u_1u_2u_3u_4$, then u_1 is adjacent to $v_i \in N(v)$ and u_4 is adjacent to v_j. Therefore $\{v, v_2v_3, u_1v_i, u_4v_j\}$ is a dominating set of G^{+-+}. If $\langle V - N[v] \rangle \cong C_4 = u_1u_2u_3u_4u_1$, let u_1 be adjacent to a vertex $v_j \in N(v)$. Then $\{v, v_j, vv_j, u_3\}$ is a dominating set of G^{+-+}. If at least one vertex u_i of $V - N[v]$ is of degree 3 in $\langle V - N[v] \rangle$, then at least one vertex u_j(may be u_i) in $V - N[v]$

is adjacent to a vertex $v_k \in N(v)$. Then $\{v, v_k, vv_k, u_i\}$ is a dominating set of G^{+-+}. Hence $\gamma(G^{+-+}) \le 5$.

Case 4 $\Delta(G) = 4$.

Let $N(v) = \{v_1, v_2, v_3, v_4\}$ and $V - N[v] = \{u_1, u_2, u_3, u_4, u_5\}$. If $v_k \in N(u_i) \cap N(u_j) \cap N(u_k)$ for some i, j and k in $N(v)$, then $\{v, v_k, vv_k, u_r u_s\}$ is a dominating set of G^{+-+}. Now, let $N(u_i) \cap N(u_j) \cap N(u_k) = \phi$ for all i, j and k. Then the induced subgraph $\langle V - N[v]\rangle$ is non-empty and hence we consider two subcases.

Subcase 1 $\langle V - N[v]\rangle$ is disconnected.

If $\langle V - N[v]\rangle \cong K_2 \cup 3K_1$, let $u_1u_2 \in E(G)$. Then each of u_3, u_4 and u_5 are adjacent to at least two vertices of $N(v)$ and each of u_1 and u_2 are adjacent to at least one vertex of $N(v)$. Hence $\{v_1, v_2, v_3, v_4, u_1u_2\}$ is a dominating set of G^{+-+}. If $\langle V - N[v]\rangle \cong 2K_2 \cup K_1$, let $u_1u_2, u_3u_4 \in E(G)$ and u_5 be adjacent to v_i and v_j of $N(v)$. Then, $\{v, v_i, vv_i, u_1u_2, u_3u_4\}$ is a dominating set of G^{+-+}. If $\langle V - N[v]\rangle \cong P_3 \cup 2K_1$ where $P_3 = u_1u_2u_3$, let u_4 be adjacent to v_i and u_5 be adjacent to v_j. Then $\{v, v_i, vv_i, u_2, u_5\}$ is a dominating set of G^{+-+}. If $\langle V - N[v]\rangle \cong C_3 \cup 2K_1$ where $C_3 = u_1u_2u_3u_1$, let u_4 be adjacent to v_i. Then $\{v, v_i, vv_i, u_2, u_5\}$ is a dominating set of G^{+-+}. If $\langle V - N[v]\rangle \cong P_4 \cup K_1$ or $C_4 \cup K_1$ where $P_4 = u_1u_2u_3u_4$ and $C_4 = u_1u_2u_3u_4u_1$, let u_5 be adjacent to v_i. Then, $\{v, v_i, vv_i, u_1u_2, u_3u_4\}$ is a dominating set of G^{+-+}. If $\langle V - N[v]\rangle$ has exactly one isolated vertex u_i and a vertex u_j of degree 3, let u_k (or u_j) be adjacent to v_k and u_i be adjacent to v_r. Then $\{v, v_r, vv_r, u_j\}$ is a dominating set of G^{+-+}.

Subcase 2 $\langle V - N[v]\rangle$ is connected.

If $\langle V - N[v]\rangle \cong P_5$ where $P_5 = u_1u_2u_3u_4u_5$, then u_1 and u_5 must be adjacent to v_i and v_j respectively. Therefore $\{v, v_i, vv_i, u_2, u_4\}$ is a dominating set of G^{+-+}. If $\langle V - N[v]\rangle \cong C_5$, where $C_5 = u_1u_2u_3u_4u_5u_1$, then $u_iv_j \in E(G)$. Then $\{v, v_j, vv_j, u_2, u_4\}$ is a dominating set of G^{+-+}. If $\langle V - N[v]\rangle$ has a vertex u_i of degree 3 and no isolated vertex, then there exists $u_j \in V - N[v]$ such that $u_iu_j \notin E(G)$ and there is u_k (may be u_i or u_j) which is adjacent to v_k. Therefore $\{v, v_k, vv_k, u_i, u_j\}$ is a dominating set of G^{+-+}. If $\langle V - N[v]\rangle$ has a vertex u_i of degree 4, then a vertex $u_j (\ne u_i)$ is adjacent to v_j and hence $\{v, v_j, vv_j, u_j\}$ is a dominating set of G^{+-+}. Thus $\gamma(G^{+-+}) \le 5$.

Case 5 $\Delta(G) = 3$.

Let $N(v) = \{v_1, v_2, v_3\}$ and $V - N[v] = \{u_1, u_2, u_3, u_4, u_5, u_6\}$. Then $\langle V - N[v]\rangle$ has at most two isolated vertices and $\langle V - N[v]\rangle$ is not isomorphic to $2K_1 \cup 2K_2$. Hence it is isomorphic to $P_4 \cup 2K_1$ or $C_4 \cup 2K_1$ or $P_5 \cup K_1$ or $C_5 \cup K_1$ or $3K_2$ or P_6 or C_6 or a graph with a vertex of degree 3.

If $\langle V - N[v]\rangle \cong P_4 \cup 2K_1$ where $P_4 = u_1u_2u_3u_4$, let $u_1v_1 \in E(G)$. Then $\{v_1, v_2, v_3, u_3, vv_1\}$ is a dominating set of G^{+-+}. If $\langle V - N[v]\rangle \cong C_4 \cup 2K_1$ where $C_4 = u_1u_2u_3u_4u_1$, let $u_iv_j \in E(G)$. Then $\{v_1, v_2, v_3, vv_j, u_3\}$ is a dominating set of G^{+-+}. If $\langle V - N[v]\rangle$ has two isolated vertices and a vertex u_i of degree 3, then $\{v_1, v_2, v_3, vv_1, u_i\}$ is a dominating set of G^{+-+}. If $\langle V - N[v]\rangle \cong P_5 \cup K_1$, or $C_5 \cup K_1$ where $P_5 = u_1u_2u_3u_4u_5$ and $C_5 = u_1u_2u_3u_4u_5u_1$, let u_5 be adjacent to v_i. Then $\{v, v_i, vv_i, u_1, u_4\}$ is a dominating set of G^{+-+}. If $\langle V - N[v]\rangle$

has exactly one isolated vertex u_i and a vertex u_j of degree 3, let $u_ju_k \notin E(G)$ and u_i be adjacent to v_i. Then $\{v, v_i, vv_i, u_j, u_k\}$ is a dominating set of G^{+-+}. If $\langle V - N[v]\rangle \cong 3K_2$ and $u_1u_2, u_3u_4, u_5u_6 \in E(G)$, let $u_1v_i \in E(G)$. Then $\{v, v_i, vv_i, u_3, u_5\}$ is a dominating set of G^{+-+}. If $\langle V - N[v]\rangle \cong P_6$ or C_6 where $P_6 = u_1u_2u_3u_4u_5u_6$ and $C_6 = u_1u_2u_3u_4u_5u_6u_1$, then $\{u_2, u_5, v, v_i, vv_i\}$ is a dominating set of G^{+-+}. If $\langle V - N[v]\rangle$ has a vertex u_i of degree 3 and $u_iu_j, u_iu_k \notin E(G)$, then u_j is adjacent to a vertex v_j of $N(v)$. Therefore $\{v, v_j, vv_j, u_i, u_k\}$ is a dominating set of G^{+-+}. Hence $\gamma(G^{+-+}) \le 5$. Further, $\gamma(C_{10}^{+-+}) = 5$ and hence the bound is sharp. □

Theorem 5.4 *Let G be a connected graph of order $n \ge 5$. Then $\gamma(G^{+-+}) \le \lceil n/2\rceil$ and the bound is sharp.*

Proof If for $k \ge 2$, $\gamma(G) \le \lceil n/2\rceil - k$, then by Theorem 3.6, $\gamma(G^{+-+}) \le \lceil n/2\rceil$. Hence it is enough if we consider the following two cases.

Case 1 $\gamma(G) = \lceil n/2\rceil$.

By Theorem 1.1, $\gamma(G) \le n/2$ and hence n is even. Therefore, by Theorem 1.2, $G \cong C_4$ or $H \circ K_1$ for some connected graph H. If $G \cong C_4$, then $\gamma(G^{+-+}) = \gamma(G) = n/2$. If $G \cong H \circ K_1$, then all the supports of G from a minimum dominating set D of G. Since each edge of G is incident with at least one vertex of D, D is also a dominating set of G^{+-+}. Therefore, $\gamma(G^{+-+}) \le n/2$.

Case 2 $\gamma(G) = \lceil n/2\rceil - 1$.

If $\delta(G) = 1$, then by Theorem 4.5, $\gamma(G^{+-+}) \le \gamma(G) + 1 = \lceil n/2\rceil$.

Now, let $\delta(G) \ge 2$. If $G \in \mathcal{A}$, then $\gamma(G^{+-+}) = 3 < \lceil n/2\rceil$. If $G \notin \mathcal{A}$, then by Theorem 1.3, $\gamma(G) \le 2n/5$.

Subcase 1 n is odd.

Then $\gamma(G) = \lceil n/2\rceil - 1 \le 2n/5$. Therefore, $n \le 5$. Hence $n = 5$. Clearly for all connected graphs on 5 vertices, $\gamma(G^{+-+}) \le 3 = \lceil n/2\rceil$.

Subcase 2 n is even.

Then $n/2 - 1 \le 2n/5$. Thus $n \le 10$. If $n = 6$, then by Lemma 5.1, $\gamma(G^{+-+}) \le 3$. If $n = 8$, then by Lemma 5.2, $\gamma(G^{+-+}) \le 4$. If $n = 10$, then by Lemma 5.3, $\gamma(G^{+-+}) \le 5$. Further, for the graph G given in Fig.5.1, $\gamma(G^{+-+}) = 3 = \lceil n/2\rceil$ and hence the bound is sharp. □

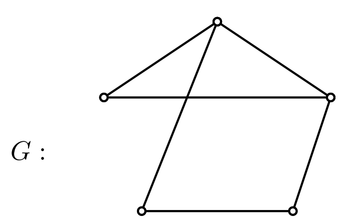


Fig.5.1 **A graph G with $G^{+-+} = \lceil n/2\rceil$.**

Open Problems

We present open problems following:

1. Characterize the graphs which attain the bound given in Theorem 3.3.
2. Characterize the extremal graphs in Theorem 5.4.

Acknowledgment

The research of the first author is supported by the University Grants Commission, New Delhi through the Basic Science Research Fellowship(Grant No. F.4-1/2006(BSR)/7-201-2007).

References

[1] M.K.Angel Jebitha and J.Paulraj Joseph, Domination in transformation graph G^{-+-}, *J. Discrete Mathematical Sciences and Cryptography*, 14 (3)(2011), 279-303.

[2] T.Y.Chang, *Domination Numbers of Grid Graphs*, Ph.D. Thesis, Univ. South Florida, 1992.

[3] Charles Payan and Nguyen Huy Xuong, Domination-balanced graphs, *J. Graph Theory*, 6 (1982), 23-32.

[4] J.F.Fink, M.S.Jacobson, L.F.Kinch and J. Roberts, On graphs having domination number half their order, *Period. Math. Hungar.*, 16 (1985), 287 - 293.

[5] Gary Chartrand and Ping Zhang, *Introduction to Graph Theory* , Tata McGraw-Hill Edition, 2006.

[6] W.McCuaig and B.Shepherd, Domination in graphs with minimum degree two, *J. Graph Theory*, 13(1989), 749-762.

[7] O.Ore, *Theory of Graphs*, Amer. Math. Soc. Colloq. Publ., 38 (Amer. Math. soc., Providence, RI), 1962.

[8] J.Paulraj Joseph and S.Arumugam, Domination in subdivision graphs, *J. Indian Math. Soc.*, 62 (1996), 274-282.

[9] Teresa W.Haynes, Stephen T. Hedetniemi and Peter J. Slater, *Fundamentals of Domination in Graphs*, Marcel Dekker, 1998.

[10] C.K.Wallis, *Domination parameters of line graphs of design and variations of chessboard graph*, Ph.D. Thesis, Clemson Univ., 1994.

[11] L.Xu and B.Wu, Transformation graph G^{-+-}, *Discrete Mathematics*, 308 (2008) 5144-5148.