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Elementary Abelian Regular Coverings of Cube

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Abstract: For a give finite connected graph Γ, a group H of automorphisms of Γ and a finite group A, a natural question can be raised as follows: Find all the connected regular coverings of Γ having A as its covering transformation group, on which each automorphism in H can be lifted. In this paper, we classify all the connected regular covering graphs of the cube satisfying the following two properties: (1) the covering transformation group is isomorphic to the elementary Abelian p-groups; (2) the group of fibre-preserving automorphisms acts edge-transitively.

Key Words: Connected graph, graph covering, cube, Smarandachely covering, regular covering.

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§1. Introduction

All graphs considered in this paper are finite, undirected and simple. For a graph Γ, we use $V(\Gamma)$, $E(\Gamma)$, $A(\Gamma)$ and $\mathrm{Aut}(\Gamma)$ to denote its vertex set, edge set, arc set and full automorphism group, respectively. For any $v \in V(\Gamma)$, by $N(v)$ we denote the neighborhood of v in Γ. For an arc $(u, v) \in A(\Gamma)$, we denote the corresponding undirected edge by uv.

A graph $\widetilde{\Gamma}$ is called a *covering* of the graph Γ with projection $p : \widetilde{\Gamma} \to \Gamma$ if there is a surjection $p : V(\widetilde{\Gamma}) \to V(\Gamma)$ such that $p|_{N(\tilde{v})} : N(\tilde{v}) \to N(v)$ is a bijection for any vertex $v \in V(\Gamma)$ and $\tilde{v} \in p^{-1}(v)$. The graph $\widetilde{\Gamma}$ is called the *covering graph* and Γ is the *base graph.* A $p : \widetilde{\Gamma} \to \Gamma$ is called to be a *Smarandachely covering* of Γ if there exist $u, v \in V(\Gamma)$ such that $|p^{-1}(u)| \neq |p^{-1}(v)|$. Conversely, if $|p^{-1}(v)| = n$ for each $v \in V(\Gamma)$, then such a covering p is said to be *n-fold*. Each $p^{-1}(v)$ is called a *fibre* of $\widetilde{\Gamma}$. An automorphism of $\widetilde{\Gamma}$ which maps a fibre to a fibre is said to be *fibre-preserving.* The group K of all automorphisms of $\widetilde{\Gamma}$ which fix each of the fibres setwise is called the *covering transformation group.* A covering $p : \widetilde{\Gamma} \to \Gamma$ is said to be *regular* (simply, *A-covering*) if there is a subgroup A of K acting regularly on each fibre. Moreover, if Γ is connected, then $A = K$.

A purely combinatorial description of a covering was introduced through a voltage graph

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by Gross and Tucker [4,5] and also a very similar idea was appeared in Biggs' monograph [1,2]. Let A be a finite group. An (*ordinary*) *voltage assignment* (or, A-*voltage assignment*) of Γ is a function $\phi : A(\Gamma) \to A$ with the property that $\phi(u,v) = \phi(v,u)^{-1}$ for each $(u,v) \in A(\Gamma)$. The values of ϕ are called *voltages*, and A is called the *voltage group*. The *graph* $\Gamma \times_\phi A$ *derived from* ϕ is defined by $V(\Gamma \times_\phi A) = V(\Gamma) \times A$ and $E(\Gamma \times_\phi A) = \{((u,g),(v,\phi(u,v)g)) \mid (u,v) \in E(\Gamma),\ g \in A\}$. Clearly, the graph $\Gamma \times_\phi A$ is a covering of the graph Γ with the first coordinate projection $p : \Gamma \times_\phi A \to \Gamma$, which is called the *natural projection*. For each $u \in V(\Gamma)$, $\{(u,g) \mid g \in A\}$ is a fibre of $\Gamma \times_\phi A$. Moreover, by defining $(u,g')^g := (u,g'g)$ for any $g \in A$ and $(u,g') \in V(\Gamma \times_\phi A)$, A can be identified with a fibre-preserving automorphism subgroup of $\mathrm{Aut}(\Gamma \times_\phi A)$ acting regularly on each fibre. Therefore, p can be viewed as a A-*covering*. Given a spanning tree T of the graph Γ, a voltage assignment ϕ is called T-*reduced* if the voltages on the tree arcs are identity. Gross and Tucker ([4]) showed that every regular covering of a graph Γ can be derived from an ordinary T-reduced voltage assignment ϕ with respect to an arbitrary fixed spanning tree T of Γ.

An automorphism α of Γ can be lifted to an automorphism $\tilde{\alpha}$ of a covering graph $\widetilde{\Gamma}$ if $p\tilde{\alpha} = \alpha p$, where p is the covering projection from $\widetilde{\Gamma}$ to Γ. We say a subgroup of H of $\mathrm{Aut}(\Gamma)$ can be lifted if each element of H can be lifted.

For a given finite connected graph Γ, a group H of automorphisms of Γ and a finite group A, a natural question can be raised as follows: Find all the connected regular coverings of Γ having A as its covering transformation group, on which each automorphism in H can be lifted. In [3], Du, Kawk and Xu investigate the regular coverings with $A = Z_p^n$, an elementary Abelian group and get some new matrix-theoretical characterizations for an automorphism of the base graph to be lifted, and as one of the applications, they gave a classification of all connected regular coverings of the Petersen graph with the covering transformation group Z_p^n, whose fibre-preserving automorphism subgroup acts arc-transitively.

In this paper, we use the same method to classify all the connected regular covering graphs of the cube satisfying the following two properties: (1) the covering transformation group is isomorphic to the elementary Abelian p-group; (2) the group of fibre-preserving automorphisms acts edge-transitively.

The cube is identified with the graph Γ as shown in Figure (a). Fix a spanning tree T in Γ as shown in Figure (b). Let $V_1 = \{2,6,4,7,3,5\}$. Then the induced subgraph $\Gamma(V_1)$ is a line as shown in Figure (c).

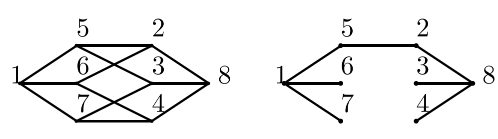

Figure (a): the graph Γ; (b) a spanning tree T of Γ.

Figure (c): the induced subgraph $\Gamma(V_1)$

First, we introduce five families of covering graphs $\Gamma \times_\phi Z_p^n$ of cube Γ by giving a T-reduced voltage assignment ϕ. Since ϕ is T-reduced, we only need to give the voltages on the cotree arcs (see Figure (c)). Let t denote the transposition of a matrix.

(1) $X(2,1) := \Gamma \times_\phi Z_2$, where $\phi_{26} = \phi_{47} = \phi_{35} = 1$ and $\phi_{46} = \phi_{37} = 0$,
$X(p,1) := \Gamma \times_\phi Z_p$, where $p = 3$ or $p \equiv 1 (\mathrm{mod}\ 6)$, $\phi_{26} = \phi_{37} = 1$, $\phi_{46} = \phi_{35} = \frac{1+\sqrt{-3}}{2}$ and $\phi_{47} = 0$.

(2) $X(p,2) := \Gamma \times_\phi Z_p^2$, where $\phi_{26} = \phi_{37} = (0,1)$, $\phi_{46} = \phi_{35} = (1,0)$ and $\phi_{47} = (0,0)$.

(3) $X(p,3) := \Gamma \times_\phi Z_p^3$, where $((\phi_{26})^t, (\phi_{47})^t, (\phi_{35})^t) = I_{3\times 3}$, $\phi_{46} = (0,1,-1)$ and $\phi_{37} = (-1,1,0)$.

(4) $X(p,4) := \Gamma \times_\phi Z_p^4$, where $p = 3$ or $p \equiv 1 (\mathrm{mod}\ 6)$, $((\phi_{26})^t,\ (\phi_{46})^t,\ (\phi_{47})^t,\ (\phi_{37})^t) = I_{4\times 4}$ and $\phi_{35} = (\dfrac{1-\sqrt{-3}}{2},\ -1,\ \dfrac{1+\sqrt{-3}}{2},\ \dfrac{1-\sqrt{-3}}{2})$.

(5) $X(p,5) := \Gamma \times_\phi Z_p^5$, where $((\phi_{26})^t, (\phi_{46})^t, (\phi_{47})^t, (\phi_{37})^t, (\phi_{35})^t) = I_{5\times 5}$.

Now we state the main theorem of this paper.

Theorem 1.1 *Let $\widetilde{\Gamma}$ be a connected regular covering of the cube Γ whose covering transformation group is isomorphic to Z_p^n and whose fibre-preserving automorphism subgroup G acts edge-transitively on $\widetilde{\Gamma}$. Then, $\widetilde{\Gamma}$ is isomorphic to one of the graphs in* (1)-(5) *listed above. Moreover, for the graphs $X(2,1)$, $X(3,1)$, $X(p,2)$, $X(p,3)$, $X(3,4)$ and $X(p,5)$,* $\mathrm{Aut}(\Gamma)$ *can be lifted, and so they are 2-arc-transitive; and for the graphs $X(p,1)$ and $X(p,4)$ for $p \equiv 1 (\mathrm{mod}\ 6)$, the subgroup isomorphic to $A_4 \times Z_2$ can be lifted but* $\mathrm{Aut}\Gamma$ *cannot, and so they are arc-transitive, in particular, all these five families of graphs are vertex transitive.*

§2. Algorithm for the Lifting

In this section, we present the algorithm given by Du, Kwak and Xu [3], which deals with the lifting problem for regular coverings of a graph Γ whose covering transformation group is elementary Abelian.

Throughout this section, let Γ be a connected graph and let $\widetilde{\Gamma} = \Gamma \times_\phi Z_p^n$ be a connected regular covering of the Γ. The voltage group Z_p^n will be identified with the additive group of the n-dimensional vector space $V(n,p)$ over the finite field $GF(p)$. Since Γ is connected, the number $\beta(\Gamma) = |E(\Gamma)| - |V(\Gamma)| + 1$ is equal to the number of independent cycles in Γ and it is referred to as the *Betti number* of Γ.

Let $V(\Gamma) = \{0, 1, \ldots, |V(\Gamma)| - 1\}$. For any arc $(i,j) \in A(\Gamma)$, by $\phi_{i,j}$ we denote the voltage on the arc, which is identified with a row vector in $V(n,p)$. An arc $(i,j) \in A(\Gamma)$ is called *positive* (resp. *negative*) if $i < j$ (resp. $i > j$). For each subset F in $E(\Gamma)$, we denote the set of its arcs, positive arcs and negative arcs by $A(F)$, $A^+(F)$ and $A^-(F)$, respectively, so that $A(F) = A^+(F) \cup A^-(F)$. In particular, if $F = E(\Gamma)$, we prefer to use $A(\Gamma)$, $A^+(\Gamma)$ and $A^-(\Gamma)$ to denote $A(F)$, $A^+(F)$ and $A^-(F)$, respectively. Fix a spanning tree T in the graph

Γ and let $E_0 = E(T)$, so that $|E(\Gamma) \setminus E_0|$ is the Betti number $\beta(\Gamma)$ of the graph Γ. From now on, the voltage assignment ϕ is assumed to be T-reduced. By the connectedness of $\widetilde{\Gamma}$, $\{\phi_{i,j} \mid (i,j) \in A^+(E(\Gamma) \setminus E_0)\}$ generates the group Z_p^n. Hence, we get $n \leqslant \beta(\Gamma)$.

Let E_1 be a set of edges such that $\phi_{A^+(E_1)} = \{\phi_{i,j} \bigm| (i,j) \in A^+(E_1)\}$ is a basis for the vector space $V(n,p)$, and let $E_2 = E(\Gamma) \setminus (E_0 \cup E_1)$. Let

$$|E_0| = k, \quad |E_1| = n \quad \text{and} \quad |E_2| = m, \tag{1}$$

so that the number of edges in Γ is $k + n + m$.

Let Φ_0 (resp. Φ_1 and Φ_2) be the $k \times n$ (resp. $n \times n$ and $m \times n$) matrix with the row vectors $\phi_{i,j}$ for the arcs (i,j) in $A^+(E_0)$ (resp. $A^+(E_1)$ and $A^+(E_2)$), according to a fixed order of the positive arcs. Since the row vectors of Φ_1 form a basis for $V(n,p)$, there exists an $m \times n$ matrix M, called a *voltage generating matrix* of ϕ, such that

$$\Phi_2 = M\Phi_1. \tag{2}$$

Let

$$\Phi = \begin{pmatrix} \Phi_0 \\ \Phi_1 \\ \Phi_2 \end{pmatrix}, \tag{3}$$

which is a $(k+n+m)\times n$ matrix over $GF(p)$, called a *voltage* (*assignment*) matrix corresponding to the voltage assignment ϕ. If we take $\phi_{A^+(E_1)}$ so that $\Phi_0 = \mathbf{0}$ and $\Phi_1 = I_{n\times n}$, the $n\times n$ identity matrix, then Φ is called a *reduced form* or a *T-reduced form* of the voltage assignment matrix Φ. From now on one may assume that Φ is in a reduced form without loss of any generality.

Let $\mathbf{V} = V(k+n+m,p)$ be the $(k+n+m)$-dimensional row vector space over the field $GF(p)$. Hereafter, we denote a vector $\mathbf{v}$ in $\mathbf{V}$ by $(\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2) \in Z_p^k \oplus Z_p^n \oplus Z_p^m$, where the coordinates of the vector $\mathbf{v}_0$ (resp. $\mathbf{v}_1$ and $\mathbf{v}_2$) are indexed by arcs in $A^+(E_0)$ (resp. $A^+(E_1)$ and $A^+(E_2)$), according to the same order of row vectors in Φ_0 (Φ_1 and Φ_2).

Given a graph Γ, its spanning tree T, and a positive cotree arc (u,v), there is a unique path from v to u in T which is denoted by $[v,\cdots,u]$. We call the closed walk $(u,[v,\cdots,u])$ the *fundamental cycle* belonging to (u,v), and denote it by $C(u,v;T)$. There are $n+m$ fundamental cycles in Γ, where $n+m$ is the Betti number of Γ.

Given a graph Γ and its spanning tree T, we keep the same order for positive arcs as the order of row vectors of the voltage matrix Φ. For each positive cotree arc (u,v), let $\mathbf{p}^{u,v}$ be the k-dimensional row vector over $GF(p)$ whose (i,j)-coordinate $\mathbf{p}_{i,j}^{u,v}$ indexed by the positive tree arc (i,j) of the given order is defined as follows:

$$\mathbf{p}_{i,j}^{u,v} = \begin{cases} \;\;1 & \text{if } (i,j) \text{ is in } C(u,v;T), \\ -1 & \text{if } (j,i) \text{ is in } C(u,v;T), \\ \;\;0 & \text{otherwise.} \end{cases} \tag{4}$$

Let P be the $(n+m)\times k$ matrix whose row vectors are $\mathbf{p}^{u,v}$, indexed by the positive cotree arcs (u,v) of the given order. We call P the *incidence matrix* for the fundamental cycles of the graph Γ with respect to the tree T.

Now we state the algorithm for solving lifting problem for the connected regular coverings of a graph Γ whose covering transformation group is elementary Abelian.

(1st) Choose a fixed spanning tree T in Γ and write down the arcs in $A^+(E_0)$, $A^+(E_1)$ and $A^+(E_2)$ in a certain order so that $\Phi_0 = \mathbf{0}$, $\Phi_1 = I_{n\times n}$ and $\Phi_2 = M$.

(2nd) Calculate the incidence matrix P for the fundamental cycles of Γ with respect to T.

(3rd) Assume that the voltage generating matrix $M = (a_{ij})_{m\times n}$, where the entries a_{ij} are unknowns. Let $\Delta = ((-M, I_{m\times m})P, -M, I_{m\times m})$, whose columns are indexed by the arcs in $A^+(E_0), A^+(E_1), A^+(E_2)$ according to the given order. We call the matrix Δ the *discriminant matrix* for a lift of ϕ. For convenience, we write $\Delta_0 = (-M, I_{m\times m})P$, $\Delta_1 = -M$ and $\Delta_2 = I_{m\times m}$, so that $\Delta = (\Delta_0, \Delta_1, \Delta_2)$, as a block matrix.

(4th) Let $\Delta = (\cdots, \mathbf{c}_{i,j}, \cdots)$, where $c_{i,j}$ is the column indexed by $(i,j) \in A^+(\Gamma)$. For a given $\sigma \in \mathrm{Aut}(\Gamma)$, let $\mathbf{c}^\sigma_{i,j} = \mathbf{c}_{i^{\sigma^{-1}}, j^{\sigma^{-1}}}$, where we assume that $\mathbf{c}_{i,j} = -\mathbf{c}_{j,i}$ for any arc (i,j). Let $\Delta^\sigma = (\cdots, \mathbf{c}^\sigma_{i,j}, \cdots)$ for any $(i,j) \in A^+(\Gamma)$, and let $(\Delta^\sigma)_0$, $(\Delta^\sigma)_1$ and $(\Delta^\sigma)_2$ denote the first, the second and the third blocks of the matrix Δ^σ respectively, as before. Then one can say that

$$\sigma \ \text{ can be lifted} \iff (\Delta^\sigma)_1 + (\Delta^\sigma)_2 M = \mathbf{0} \iff \Delta^\sigma_1 + \Delta^\sigma_2 M = \mathbf{0}. \qquad (5)$$

Proof of Theorem 1.1

Let $E_0 = E(T)$ and $E = E(\Gamma) \setminus E_0$. Give an ordering for the arcs in $A^+(E_0)$ and $A^+(E)$ as follows:

$$A^+(E_0) = \{15, 16, 17, 25, 28, 38, 48\},$$
$$A^+(E) = \{26, 46, 47, 37, 35\}.$$

Give fundamental cycles in Γ as follows:

$$C(2,6;T) = (2,6,1,5,2),$$
$$C(4,6;T) = (4,6,1,5,2,8,4),$$
$$C(4,7;T) = (4,7,1,5,2,8,4),$$
$$C(3,7;T) = (3,7,1,5,2,8,3),$$
$$C(3,5;T) = (3,5,2,8,3).$$

It is well-known that $\mathrm{Aut}(\Gamma) \cong S_4 \times Z_2$. Take four automorphisms of Γ as follows: $\alpha = (243)(567)$, $\beta = (14)(23)(58)(67)$, $\gamma = (18)(27)(36)(45)$ and $\delta = (23)(67)$.

It is easy to check that $M =< \alpha, \beta >$ is subgroup of $\mathrm{Aut}(\Gamma)$ isomorphic to A_4, $N =< M, \gamma >$ is subgroup of $\mathrm{Aut}(\Gamma)$ isomorphic to $A_4 \times Z_2$, and $< N, \delta >= \mathrm{Aut}(\Gamma)$. Thus we have:

(1) M can be lifted if and only if α and β can be lifted;

(2) N can be lifted if and only if α, β and γ can be lifted;

(3) $\mathrm{Aut}(\Gamma)$ can be lifted if and only if α, β, γ and δ can be lifted.

Since $\beta(\Gamma)=5$, we get $n \leqslant 5$. If $n=5$, then Γ is nothing but $X(p,5)$ and by [6, Theorem 2.11], $\mathrm{Aut}(\Gamma)$ can be lifted. So, in what follows, we assume $n<5$ and divide our proof into four cases for each n with $1 \leqslant n \leqslant 4$.

3.1 The Case of $n=1$

Suppose that $n=1$. Then $K=Z_p=V(1,p)$. Since the element $(18)(25)(36)(47)$ of $\mathrm{Aut}(\Gamma)$ maps 2, 6, 4, 7, 3, 5 to 5, 3, 7, 4, 6, 2, respectively, without loss of any generality, we have the following three essentially different cases for the set E_1 and E_2:

(1) $E_1=\{26\}$ and $E_2=\{46,47,37,35\}$;

(2) $E_1=\{46\}$ and $E_2=\{26,47,37,35\}$;

(3) $E_1=\{47\}$ and $E_2=\{26,46,37,35\}$.

Case (1): $E_1=\{26\}$ and $E_2=\{46,47,37,35\}$.

In this case, the incidence matrix is:

$$P=\begin{array}{c} \\ (2,6)\\(4,6)\\(4,7)\\(3,7)\\(3,5)\end{array}\begin{array}{c}\begin{array}{ccccccc}(1,5)&(1,6)&(1,7)&(2,5)&(2,8)&(3,8)&(4,8)\end{array}\\ \begin{pmatrix} 1&-1&0&-1&0&0&0\\ 1&-1&0&-1&1&0&-1\\ 1&0&-1&-1&1&0&-1\\ 1&0&-1&-1&1&-1&0\\ 0&0&0&-1&1&-1&0\end{pmatrix}\end{array}.$$

Let $D=(D_0,D_1,D_2)$ be the discriminant matrix with $D_0=(-M,I_{4\times 4})P$, $D_1=(-M)$ and $D_2=(I_{4\times 4})$ with a voltage generating matrix $M=(a,b,c,d)^t$. A direct computation gives that $D_0=(-M,I_{4\times 4})P$ is equal to

$$D_0=\begin{array}{c}\begin{array}{ccccccc}(1,5)&(1,6)&(1,7)&(2,5)&(2,8)&(3,8)&(4,8)\end{array}\\ \begin{pmatrix} -a+1&a-1&0&a-1&1&0&-1\\ -b+1&b&-1&b-1&1&0&-1\\ -c+1&c&-1&c-1&1&-1&0\\ -d&d&0&d-1&1&-1&0\end{pmatrix}\end{array}.$$

For an automorphism σ of Γ can be lifted if and only if

$$\Delta_1^{\sigma}+\Delta_2^{\sigma}M=(c_{26})^{\sigma}+(c_{46}c_{47}c_{37}c_{35})^{\sigma}M=\mathbf{0}. \tag{6}$$

Inserting $\alpha=(243)(567)$ to (6), we have

$$\begin{aligned}\mathbf{0} &= D_1^{\alpha}+D_2^{\alpha}M=(c_{26})^{\alpha}+(c_{46}c_{47}c_{37}c_{35})^{\alpha}M\\ &= (c_{35})+(c_{25}c_{26}c_{46}c_{47})M\\ &= \begin{pmatrix} a^2-a-ab+c\\ ab-a-b^2+d\\ ac-a-bc\\ ad-a-bd+1\end{pmatrix}=(x_{i,j}).\end{aligned}$$

Inserting $\beta = (14)(23)(58)(67)$ to (6), we have

$$\begin{aligned} \mathbf{0} &= D_1^\beta + D_2^\beta M = (c_{26})^\beta + (c_{46}c_{47}c_{37}c_{35})^\beta M \\ &= (c_{37}) + (c_{17}c_{16}c_{26}c_{28})M \\ &= \begin{pmatrix} ab - b - ac + d \\ -a + b^2 - bc + d \\ -a + bc - c^2 + d + 1 \\ bd - cd + d \end{pmatrix} = (y_{ij}). \end{aligned}$$

Now assume that α and β can be lifted. By $y_{41} = 0$, we distinguish two cases: (1) $d = 0$; (2) $b - c + 1 = 0$. If (1) happens, by $x_{41} = 0$ and $x_{11} = 0$, we have $a = 1$ and $b = c$. But it doesn't satisfy $y_{21} = 0$. If (2) happens, by $y_{11} = 0$, we have $-a - b + d = 0$. By $x_{21} = 0$, we distinguish two subcases: (i) $a - b + 1 = 0$; (ii) $b = 0$. If (i) happens, by $(x_{ij}) = 0$, we have that either $a = c = 0$, $b = d = 1$ and $p = 2$ or $a = d = 2$, $b = 0$, $c = 1$ and $p = 3$. If (ii) happens, we have $c = 1$ and $a = d$. By $x_{11} = 0$, we have $a^2 - a + 1 = 0$. Thus the solution of $(x_{ij}) = (y_{ij}) = 0$ are: $a = c = 0$, $b = d = 1$ and $p = 2$; or $a = d = \frac{1\pm\sqrt{-3}}{2}$, $b = 0$, $c = 1$ and $p = 3$ or $p \equiv 1 (\mathrm{mod}\ \ 6)$. Or equivalently,

$$M_1 = (0, 1, 0, 1)^t,\ p = 2;$$

$$M_{21} = (\frac{1+\sqrt{-3}}{2}, 0, 1, \frac{1+\sqrt{-3}}{2})^t,\ p = 3 \textit{ or } p \equiv 1 (\mathrm{mod}\ \ 6);$$

$$M_{22} = (\frac{1-\sqrt{-3}}{2}, 0, 1, \frac{1-\sqrt{-3}}{2})^t,\ p = 3 \textit{ or } p \equiv 1 (\mathrm{mod}\ \ 6).$$

By $X(2, 1)$, $X(p, 1)$ and $X'(p, 1)$, we denote the covering graphs determined by M_1, M_{21} and M_{22}, respectively. In particular, $X(p, 1)$ and $X'(p, 1)$ are the same one if $p = 3$.

Inserting $\gamma = (18)(27)(36)(45)$ to (6), we have

$$\begin{aligned} \mathbf{0} &= D_1^\gamma + D_2^\gamma M = (c_{26})^\gamma + (c_{46}c_{47}c_{37}c_{35})^\gamma M \\ &= (c_{73}) + (c_{53}c_{52}c_{62}c_{64})M \\ &= \begin{pmatrix} b - ab + ac - d \\ b - b^2 + bc \\ b - bc + c^2 - 1 \\ -a + b - bd + cd \end{pmatrix}. \end{aligned}$$

Clearly, the matrices M_1, M_{21} and M_{22} satisfy this equation and γ can be lifted, and so the group N can be lifted.

Inserting $\delta = (23)(67)$ to (6), we have

$$\begin{aligned}\mathbf{0} &= D_1^\delta + D_2^\delta M = (c_{26})^\delta + (c_{46}c_{47}c_{37}c_{35})^\delta M \\ &= (c_{37}) + (c_{47}c_{46}c_{26}c_{25})M \\ &= \begin{pmatrix} b - ac + ad - d \\ a - bc + bd - d \\ -c^2 + cd - d + 1 \\ -cd + d^2 - d \end{pmatrix}.\end{aligned}$$

Clearly, the matrix M_1 satisfy this equation and δ can be lifted. But the matrices M_{21} and M_{22} satisfy this equation and δ can be lifted when $p = 3$. So, for graphs $X(2,1)$ and $X(3,1)$, $\mathrm{Aut}(\Gamma)$ can be lifted.

Finally, we show that for $p \equiv 1 (\mathrm{mod}\ \ 6)$, the graphs $X(p,1)$ and $X'(p,1)$ are isomorphic as graphs. Let $V := V(X(p,1)) = V(X'(p,1)) = \{(i;x) \mid 1 \leq \mathrm{i} \leq 8,\ x \in GF(p)\}$. Let $R = (\frac{-1+\sqrt{-3}}{2})$, and let $\zeta = (18)(25)(36)(47) \in \mathrm{Aut}(\Gamma)$. Define a permutation Υ on V by $(i;x)^\Upsilon = (i^\zeta; (x)R)$. A direct checking shows that Υ is isomorphism from $X(p,1)$ to $X'(p,1)$.

Cases (2) and (3): By a computation similar to case (1), we can get the graph $X(p,1)$ in case (2) and the graph $X(2,1)$ in the case (3).

3.2 The case of $n = 2, 3,$ and 4

In this subsection, the case $n = 2, 3,$ and 4 will be described briefly.

Case $n = 2$. In the case, $K = Z_p^2 = V(2,p)$. As before, without loss we may assume one of the following happens:

(1) $E_1 = \{46, 37\}$ and $E_2 = \{26, 47, 35\}$;

(2) $E_1 = \{26, 46\}$ and $E_2 = \{47, 37, 35\}$;

(3) $E_1 = \{26, 47\}$ and $E_2 = \{46, 37, 35\}$;

(4) $E_1 = \{26, 37\}$ and $E_2 = \{46, 47, 35\}$;

(5) $E_1 = \{26, 35\}$ and $E_2 = \{46, 47, 37\}$;

(6) $E_1 = \{46, 47\}$ and $E_2 = \{26, 37, 35\}$.

For the case of (1), let

$$M = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{pmatrix}.$$

be a voltage generating matrix. Then a computation similar to case $n = 1$ gives that $\mathrm{Aut}(\Gamma)$ can be lifted if and only if $a_1 = a_2 = b_2 = b_3$ and $b_1 = a_3 = 1$. Accordingly, we can get the graph $X(p, 2)$.

In the Cases of (2), (3), (4), (5) and (6), the group M can not be lifted.

Case $n = 3$. In this case, $K = Z_p^3 = V(3, p)$. Without any loss of generality, we may assume one of the following happens:

(1) $E_1 = \{26, 47, 35\}$ and $E_2 = \{46, 37\}$;

(2) $E_1 = \{26, 46, 47\}$ and $E_2 = \{37, 35\}$;

(3) $E_1 = \{26, 46, 37\}$ and $E_2 = \{47, 35\}$;

(4) $E_1 = \{26, 46, 35\}$ and $E_2 = \{47, 37\}$;

(5) $E_1 = \{46, 47, 37\}$ and $E_2 = \{26, 35\}$;

(6) $E_1 = \{26, 47, 37\}$ and $E_2 = \{46, 35\}$.

For the case of (1), one can show that $\mathrm{Aut}(\Gamma)$ can be lifted if and only if the voltage generation matrix

$$M = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 1 & 0 \end{pmatrix}.$$

Thus, we get the graph $X(p, 3)$.

In the case (2), (3), (4), (5) and (6), the group M can not be lifted.

Case $n = 4$. In this case, $K = Z_p^4 = V(4, p)$. Without any loss of generality, we may divide our discussion into mutually exclusive three cases as followings:

Case (1): $E_1 = \{26, 46, 47, 37\}$ and $E_2 = \{35\}$.

Let $D = (D_0, D_1, D_2)$ be the discriminant matrix with $D_0 = (-M, I_{1\times 1})P$, $D_1 = (-M)$ and $D_2 = (I_{1\times 1})$ with a voltage generating matrix $M = (a, b, c, d)$.

The group $\mathrm{Aut}(\Gamma)$ can be lifted if and only if

$$M_1 = (\frac{1-\sqrt{-3}}{2}, -1, \frac{1+\sqrt{-3}}{2}, \frac{1-\sqrt{-3}}{2}),\ p = 3 \textit{ or } p \equiv 1 (\mathrm{mod}\ \ 6);$$

$$M_2 = (\frac{1+\sqrt{-3}}{2}, -1, \frac{1-\sqrt{-3}}{2}, \frac{1+\sqrt{-3}}{2}),\ p = 3 \textit{ or } p \equiv 1 (\mathrm{mod}\ \ 6).$$

By $X(p, 4)$ and $X'(p, 4)$, we denote the covering graphs determined by M_1 and M_2, respectively. In particular, $X(p, 4)$ and $X'(p, 4)$ are the same one if $p = 3$.

For $p \equiv 1(\mathrm{mod}\ \ 6)$, the graphs $X(p, 4)$ and $X'(p, 4)$ are isomorphic as graphs. Let $V :=$

$V(X(p,4)) = V(X'(p,4)) = \{(i;x,y,z,w) \mid 1 \leq \mathrm{i} \leq 8,\ x,y,z,w \in GF(p)\}$. Let

$$R = \begin{pmatrix} c_2-1 & 1 & -c_2 & c_2-1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}.$$

And let and let $\zeta = (18)(25)(36)(47) \in \mathrm{Aut}(\Gamma)$. Define a permutation Υ on V by $(i;x,y,z,w)^{\Upsilon} = (i^{\zeta};(x,y,z,w)R)$. A direct checking shows that Υ is isomorphism from $X(p,4)$ to $X'(p,4)$.

Case (2): $E_1 = \{26,46,47,35\}$ and $E_2 = \{37\}$, and $\{\phi_{26},\phi_{46},\ \phi_{47}\,\phi_{37}\}$, $\{\phi_{46},\phi_{47},\phi_{37}\,\phi_{35}\}$ are linear dependant, hence letting

$$M = (0,b,c,0).$$

Now, one can show that M cannot be lifted.

By a similar computation, in case (3) , $E_1 = \{26,46,37,35\}$ and $E_2 = \{47\}$, and the group M can not be lifted.

Combining subsection 3.1 and 3.2, we finish the proof of Theorem 1.1. □

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