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Edge Maximal C_3 and C_5-Edge Disjoint Free Graphs

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Abstract: For a positive integer k, let $\mathcal{G}(n; E_{2k+1})$ be the class of graphs on n vertices containing no two $2k+1$-edge disjoint cycles. Let $f(n; E_{2k+1}) = \max\{\mathcal{E}(G) : G \in \mathcal{G}(n; E_{2k+1})\}$. In this paper we determine $f(n; E_{2k+1})$ and characterize the edge maximal members in $\mathcal{G}(n; E_{2k+1})$ for $k = 1$ and 2.

Key Words: Extremal graphs; Edge disjoint: Cycles, Smarandache-Turán graph.

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§1. Introduction

For our purposes a graph G is finite, undirected and has no loops or multiple edges. We denote the vertex set of G by $V(G)$ and edge set of G by $E(G)$. The cardinalities of these sets are denoted by $\nu(G)$ and $\mathcal{E}(G)$, respectively. The cycle on n vertices is denoted by C_n. Let G be a graph and $u \in V(G)$. The degree of u in G, denoted by $d_G(u)$, is the number of edges of G incident to u. The neighbor set of u in G is a subgraph H of G, denoted by $N_H(u)$, consists of the vertices of H adjacent to u; observe that $d_G(u) = |N_H(u)|$. For a proper subgraph H of G we write $G[V(H)]$ and G-$V(H)$ simply as $G[H]$ and $G - H$ respectively.

Let G_1 and G_2 be graphs. The union $G_1 \cup G_2$ of G_1 and G_2 is a graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$. G_1 and G_2 are vertex disjoint if and only if $V(G_1) \cap V(G_2) = \varnothing$; G_1 and G_2 are edge disjoint if $E(G_1) \cap E(G_2) = \varnothing$. If G_1 and G_2 are vertex disjoint, we denote their union by $G_1 + G_2$. The intersection $G_1 \cap G_2$ of graphs G_1 and G_2 is defined similarly, but in this case we need to assume that $V(G_1) \cap V(G_2) \neq \varnothing$. The join $G \vee H$ of two disjoint graphs G and H is the graph obtained from $G_1 + G_2$ by joining each vertex of G to each vertex of H. For a vertex disjoint subgraphs H_1 and H_2 of G we let $E(H_1, H_2) = \{xy \in E(G) : x \in V(H_1), y \in V(H_2)\}$ and $\mathcal{E}(H_1, H_2) = |E(H_1, H_2)|$.

Let $\mathcal{F}_1, \mathcal{F}_2$ be two graph families and n be a positive integer. Let $\mathcal{G}(n; \mathcal{F}_1, \mathcal{F}_2)$ be a Smarandache-Turán graph family consisting of graphs being $\mathcal{F}_1$-free but containing a subgraph isomorphic to a graph in $\mathcal{F}_2$ on n vertices. Define

$$f(n; \mathcal{F}_1, \mathcal{F}_2) = \max\{|E(G)| : G \in \mathcal{G}(n; \mathcal{F}_1, \mathcal{F}_2)\}.$$

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The problem of determining $f(n; \mathcal{F}_1, \mathcal{F}_2)$ is called the Smarandache-Turán-type extremal problem. It is well known that in case $\mathcal{F}_2 = \emptyset$ or $\mathcal{F}_2 = \{\text{edge}\}$ the problem is called the Turán-type extremal problem and abbreviated by $f(n; \mathcal{F}_1)$ and the class by $\mathcal{G}(n; \mathcal{F}_1)$. In this paper we consider the Turán-type extremal problem with the odd edge disjoint cycles being the forbidden subgraph. Since a bipartite graph contains no odd cycles, we only consider non-bipartite graphs. For convenience, in the case when $\mathcal{F}_1$ consists of only one member C_r, where r is an odd integer, we write

$$\mathcal{G}(n; r) = \mathcal{G}(n; \mathcal{F}_1), \qquad f(n; r) \quad = \quad f(n; \mathcal{F}_1).$$

An important problem in extremal graph theory is that of determining the values of the function $f(n; \mathcal{F}_1)$. Further, characterize the extremal graphs $\mathcal{G}(n; \mathcal{F}_1)$ where $f(n; \mathcal{F}_1)$ is attained. For a given r, the edge maximal graphs of $\mathcal{G}(n; r)$ have been studied by a number of authors [1, 2, 3, 6, 7, 8, 9, 11].

Let $\mathcal{G}(n; E_{2k+1})$ denote the class of graphs on n vertices containing no two $(2k+1)$- edge disjoint cycles. Let

$$f(n; E_{2k+1}) = \max\{\mathcal{E}(G) : G \in \mathcal{G}(n; E_{2k+1})\}.$$

In this paper we determine $f(n; E_{2k+1})$ and characterize the edge maximal members in for $k = 1$ and 2. Now, we state a number of results, which we use to prove our main results.

Lemma 1.1 (Bondy and Murty, [4]) *Let G be a graph on n vertices. If $\mathcal{E}(G) > n^2/4$, then G contains a cycle of length r for each $3 \leq r \leq \lfloor (n+3)/2 \rfloor$.*

Theorem 1.2 (Brandt, [5]) *Let G be a non-bipartite graph with n vertices and more than $\lfloor (n-1)^2/4 + 1 \rfloor$ edges. Then G contains all cycles of length between 3 and the length of the longest cycle.*

Let $\mathcal{G}^*(n)$ denote the class of graphs obtained by adding a triangle, two vertices of which are new, to the complete bipartite graph $K_{\lfloor (n-2)/2 \rfloor, \lceil (n-2)/2 \rceil}$. For an example of $\mathcal{G}^*(n)$, see Figure 1.

Theorem 1.3 (Jia, [10]) *Let $G \in \mathcal{G}(n; 5)$, $n \geq 10$. Then*

$$\mathcal{E}(G) \leq \lfloor (n-2)^2/4 \rfloor + 3.$$

Furthermore, equality holds if and only if $G \in \mathcal{G}^*(n)$.

In this paper we determine $f(n, E_{2k+1})$ and characterize the edge maximal members in $\mathcal{G}(n, E_{2k+1})$ for $k = 1, 2$, which is the first step toward solving the problem for each positive integer k.

§2. Edge-Maximal C_3-Disjoint Free Graphs

In this section we determine $f(n, E_3)$ and characterize the edge maximal members in $\mathcal{G}(n, E_3)$. We begin with some constructions. Let $\Omega(G)$ denote to the class of graphs obtained by adding

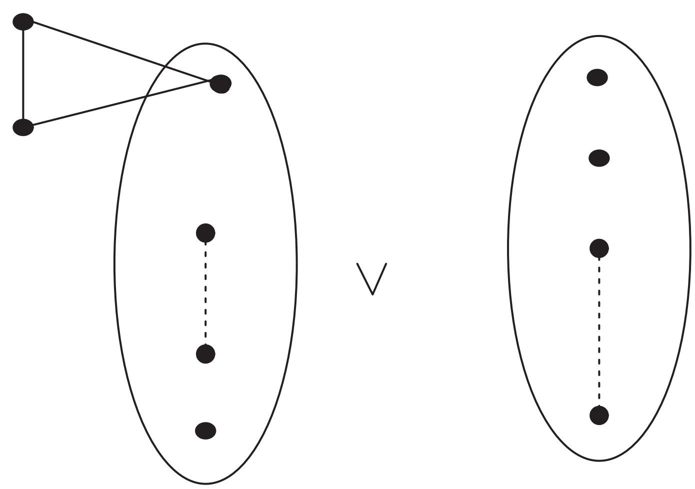

Figure 1: The figure represent a member of $\mathcal{G}^*(n)$.

an edge to the complete bipartite graph $K_{\lfloor n/2\rfloor,\lceil n/2\rceil}$. Figure 2 displays a member of $\Omega(G)$. Observe that $\Omega(G) \subseteq \mathcal{G}(n, E_{2k+1})$ and every graph in $\Omega(G)$ contains $\lfloor n^2/4\rfloor + 1$ edges. Thus, we have established that

$$f(n; E_{2k+1}) \geq \lfloor n^2/4\rfloor + 1 \quad (1)$$

We now establish that equality (1) holds for $k = 1$. Further we characterize the edge maximal members in $\mathcal{G}(n; E_3)$.

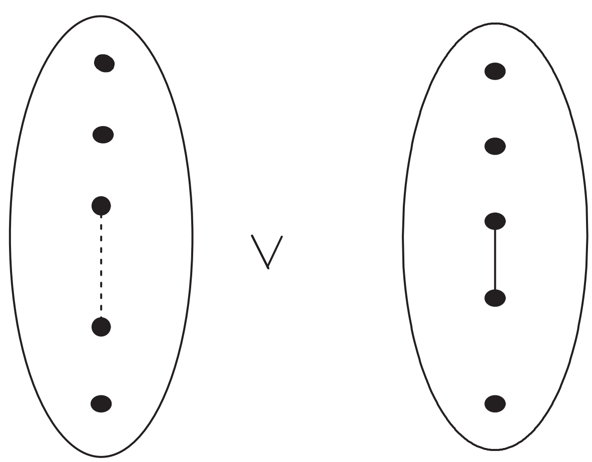

Figure 2: The figure represents a member of $\Omega(G)$.

In the following theorem we determine edge maximum members in $\mathcal{G}(n; E_3)$.

Theorem 2.1 *Let $G \in \mathcal{G}(n; E_3)$. For $n \geq 8$,*

$$f(n; E_3) \leq \lfloor n^2/4 \rfloor + 1.$$

Furthermore, equality holds if and only if $G \in \Omega(G)$.

Proof Let $G \in \mathcal{G}(n, E_3)$. If G contains no cycle of length three, then by Lemma 1.1, $\mathcal{E}(G) \leq \lfloor n^2/4 \rfloor$. Thus, $\mathcal{E}(G) < \lfloor n^2/4 \rfloor + 1$. So, we need to consider the case when G has cycles of length 3. Let $xyzx$ be a cycle of length 3 in G. Let $H = G - \{xy, xz, yz\}$. Observe that H cannot have cycles of length 3 as otherwise G would have two edge disjoint cycles of length 3. To this end we consider the following two cases.

Case1: H is not a bipartite graph. Since H contains no cycles of length 3, by Theorem 1.2,

$$\mathcal{E}(G) \leq \lfloor (n-1)^2/4 \rfloor + 1.$$

Now,

$$\begin{aligned} \mathcal{E}(H) &= \mathcal{E}(H) + 3 \\ &\leq \left\lfloor \frac{(n-1)^2}{4} \right\rfloor + 4 < \left\lfloor \frac{n^2}{4} \right\rfloor + 1, \end{aligned}$$

for $n \geq 8$.

Case 2: H is a bipartite graph. Let X and Y be the bipartition of $V(H)$. Thus, $\mathcal{E}(H) \leq |X||Y|$. Observe $|X| + |Y| = n$. The maximum of the above is when $|X| = \lfloor n/2 \rfloor$ and $|Y| = \lceil n/2 \rceil$. Thus, $\mathcal{E}(G) \leq \lfloor n^2/4 \rfloor$. Now we divide our work into two subcases.

Subcase 2.1. All the edges xy, xz, and yz are edges in on of X and Y, say in X. Observe that for any two vertices of Y, say u and v, we have that $\mathcal{E}(\{x,y,z\},\{u,v\}) \leq 5$ as otherwise G would have two edge disjoint cycles of length 3. Thus, $\mathcal{E}(\{x,y,z\}, Y) \leq 2|Y| + 1$. Now,

$$\begin{aligned} \mathcal{E}(G) &= \mathcal{E}(X - \{x,y,z\}, Y) + \mathcal{E}(\{x,y,z\}, Y) + \mathcal{E}(\{x,y,z\}) \\ &\leq (|X| - 3)|Y| + 2|Y| + 1 + 3 \\ &\leq |X||Y| - |Y| + 4 \leq (|X| - 1)|Y| + 4. \end{aligned}$$

Observe $|X| + |Y| = n$. The maximum of the above equation is when $|X| = \lfloor (n+1)/2 \rfloor$ and $|Y| = \lceil (n-1)/2 \rceil$, Thus,

$$\mathcal{E}(G) \leq \left\lfloor \frac{(n-1)^2}{4} \right\rfloor + 4 < \left\lfloor \frac{n^2}{4} \right\rfloor + 1.$$

Subcase 2.2. At most one of xy, xz and yz is an edge in X and Y. So,

$$\mathcal{E}(G) = \mathcal{E}(H) + 1 \leq \left\lfloor \frac{n^2}{4} \right\rfloor + 1.$$

This completes the proof of the theorem. □

We now characterize the extremal graphs. Through the proof, we notice that the only time we have equality is in case when G obtained by adding an edge to the complete bipartite graph $K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$. This gives rise to the class $\Omega(G)$.

§3. Edge-Maximal C_5-Disjoint Free Graphs

In this section we determine $f(n; E_{2k+1})$ and characterize the edge maximal members in $\mathcal{G}(n; E_{2k+1})$ for $k = 2$. We now establish that equality (1) holds for $k = 2$. Further we characterize the edge maximal members in $\mathcal{G}(n; E_5)$. To do that we employ the same method as in the above theorem

Theorem 3.1 *Let $G \in \mathcal{G}(n, E_5)$. For $n \geq 9$,*

$$f(n; E_5) \leq \lfloor n^2/4 \rfloor + 1.$$

Furthermore, equality holds if and only if $G \in \Omega(G)$.

Proof Let $G \in \mathcal{G}(n; E_5)$. If G does not have a cycle of length 5, then by Lemma 1.1, $\mathcal{E}(G) \leq \lfloor n^2/4 \rfloor$. Thus, $\mathcal{E}(G) < \lfloor n^2/4 \rfloor + 1$. Hence, we consider the case when G has cycles of length 5. Assume $x_1x_2 \ldots x_5x_1$ be a cycle of length 5 in G. As in the proof of the above theorem, we consider $H = G - \{e_1 = x_1x_2, e_2 = x_2x_3, \ldots, e_5 = x_5x_1\}$. Observe that H cannot have 5-cycles as otherwise G would have two 5 - edges disjoint cycles. Now, we consider two cases.

Case 1: H is not a bipartite graph. Then, by Theorem 1.3, we have

$$\mathcal{E}(H) \leq \lfloor (n-2)^2/4 \rfloor + 1.$$

Now,

$$\begin{aligned} \mathcal{E}(G) &= \mathcal{E}(H) + 5 \\ &\leq \left\lfloor \frac{(n-2)^2}{4} \right\rfloor + 8 < \left\lfloor \frac{n^2}{4} \right\rfloor - n + 9, \end{aligned}$$

for $n \geq 9$, we have

$$\mathcal{E}(G) < \left\lfloor \frac{n^2}{4} \right\rfloor + 1.$$

Case 2: H is a bipartite graph. Let X and Y be the bipartition of $V(H)$. Thus, $\mathcal{E}(G) \leq |X||Y|$. Observe $|X| + |Y| = n$. The maximum of the above is when $|X| = \lfloor n/2 \rfloor$ and $|Y| = \lceil n/2 \rceil$. Thus, $\mathcal{E}(G) < \left\lfloor \frac{n^2}{4} \right\rfloor$. Now, we consider the following subcases.

Subcase 2.1. One of X and Y contains two edges. Observe that those two edges must be consecutive, say e_1 and e_2. Let z be a vertex in X. If $|N_Y(x_1) \cap N_Y(x_2) \cap N_Y(x_3) \cap N_Y(z)| \geq 4$, then G contains two 5 edges disjoint cycles. Thus,

$$\mathcal{E}(\{x_1, x_2, x_3, z\}, Y) \leq 3|Y| + 3.$$

So,

$$\begin{aligned} \mathcal{E}(G) &= \mathcal{E}(X - x_1, x_2, x_3, z\}, Y) + \mathcal{E}(x_1, x_2, x_3, z\}, Y) + \mathcal{E}(X) + \mathcal{E}(Y) \\ &\leq (|X| - 4)|Y| + 3|Y| + 3 + 2 + 3 \\ &\leq |X||Y| - |Y| + 8 \leq (|X| - 1)|Y| + 8 \end{aligned}$$

Observe $|X| + |Y| = n$. The maximum of the above equation is when $|Y| = \lceil \frac{n-1}{2} \rceil$ and $|X| - 1 = \lfloor \frac{n-1}{2} \rfloor$. Thus,

$$\mathcal{E}(G) \leq \left\lfloor \frac{(n-1)^2}{4} \right\rfloor + 8$$

For $n \geq 9$, we have

$$\mathcal{E}(G) < \left\lfloor \frac{n^2}{4} \right\rfloor + 1.$$

Subcase 2.2. $\mathcal{E}(X) = 1$ and $\mathcal{E}(Y) = 0$ or $\mathcal{E}(X) = 0$ and $\mathcal{E}(Y) = 1$. Then

$$\mathcal{E}(G) \leq \mathcal{E}(H) + 1 \leq \left\lfloor \frac{n^2}{4} \right\rfloor + 1$$

This completes the proof of the theorem. □

We now characterize the extremal graphs. Through the proof, we notice that the only time we have equality is in case when G obtained by adding an edge to the complete bipartite graph. This gives rise to the class $\Omega(G)$.

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