International J.Math. Combin. Vol.1 (2011), 88-93

A Note on Admissible Mannheim Curves in Galilean Space G_3

S.Ersoy, M.Akyigit AND M.Tosun

Department of Mathematics, Faculty of Arts and Sciences,

Sakarya University, Sakarya, 54187 Turkey

E-mail: sersoy@sakarya.edu.tr, makyigit@sakarya.edu.tr, tosun@sakarya.edu.tr

Abstract: In this paper, the definition of the Mannheim partner curves in Galilean space G_3 is given. The relationship between the curvatures and the torsions of the Mannheim partner curves with respect to each other are obtained.

Key Words: Galilean space, Mannheim curve, admissible curve.

AMS(2010): 53A35, 51M30

§1. Introduction

The theory of curves is a fundamental structure of differential geometry. An increasing interest of the theory curves makes a development of special curves to be examined. A way for classification and characterization of curves is the relationship between the Frenet vectors of the curves. The well known of such curve is Bertrand curve which is characterized as a kind of corresponding relation between the two curves. A Bertrand curve is defined as a spatial curve which shares its principal normals with another spatial curve (called Bertrand mate). Another kind of associated curve has been called Mannheim curve and Mannheim partner curve. A space curve whose principal normal is the binormal of another curve is called Mannheim curve. The notion of Mannheim curve was discovered by A. Mannheim in 1878. The articles concerning Mannheim curves are rather few. In [1], R. Blum studied a remarkable class of Mannheim curves. O. Tigano obtained general Mannheim curves in Euclidean 3-space in [2]. Recently, H.Liu and F. Wang studied the Mannheim partner curves in Euclidean 3-space and Minkowski 3-space and obtained the necessary and sufficient conditions for the Mannheim partner curves in [3]. On the other hand, a range of new types of geometries have been invented and developed in the last two centuries. They can be introduced in a variety of manners. One possible way is through projective manner, where one can express metric properties through projective relations. For this purpose a fixed conic in infinity is taken and all metric relations with respect to the absolute. This approach is due to A. Cayley and F. Klein who noticed that due to the nature of the absolute, various geometries are possible. Among this geometries, there are also Galilean and pseudo-Galilean geometries. They are very important in physics. Galilean space-time plays the same role in non relativistic physics. The Geometry of the Galilean space

[1]Received October 12, 2010. Accepted February 26, 2011.

G_3 has treated in detail in Röschel's habilitiation, [4]. Furthermore, Kamenarovic and Sipus studied about Galilean space, [5]-[6]. The properties of the curves in the Galilean space are studied [7]-[9]. The aim of this paper is to study the Mannheim partner curves in Galilean space G_3 and give some characterizations for this curves.

§2. Preliminaries

The Galilean space G_3 is a Cayley-Klein space equipped with the projective metric of signature (0,0,+,+), as in [10]. The absolute figure of the Galilean Geometry consist of an ordered triple $\{w, f, I\}$, where w is the ideal (absolute) plane, f is the line (absolute line) in w and I is the fixed elliptic involution of points of f, [5]. In the non-homogeneous coordinates the similarity group H_8 has the form

$$\begin{aligned}\overline{x} &= a_{11} + a_{12}x \\ \overline{y} &= a_{21} + a_{22}x + a_{23}y\cos\varphi + a_{23}z\sin\varphi \\ \overline{z} &= a_{31} + a_{32}x - a_{23}y\sin\varphi + a_{23}z\cos\varphi\end{aligned} \tag{2.1}$$

where a_{ij} and φ are real numbers, [7].

In what follows the coefficients a_{12} and a_{23} will play the special role. In particular, for $a_{12} = a_{23} = 1$, (2.1) defines the group $B_6 \subset H_8$ of isometries of Galilean space G_3.

In G_3 there are four classes of lines:

i) (proper) non-isotropic lines- they don't meet the absolute line f.

ii) (proper) isotropic lines- lines that don't belong to the plane w but meet the absolute line f.

iii) un-proper non-isotropic lines-all lines of w but f.

iv) the absolute line f.

Planes x =constant are Euclidean and so is the plane w. Other planes are isotropic, [6].

The Galilean scalar product can be written as

$$< u_1, u_2 > = \begin{cases} x_1x_2 & , \ if \ \ x_1 \neq 0 \ \vee \ x_2 \neq 0 \\ y_1y_2 + z_1z_2 & , \ if \ \ x_1 = 0 \ \wedge \ x_2 = 0 \end{cases}$$

where $u_1 = (x_1, y_1, z_1)$ and $u_2 = (x_2, y_2, z_2)$. It leaves invariant the Galilean norm of the vector $u = (x, y, z)$ defined by

$$\|u\| = \begin{cases} x & , \ x \neq 0 \\ \sqrt{y^2 + z^2} & , \ x = 0. \end{cases}$$

Let α be a curve given in the coordinate form

$$\left(\begin{array}{l} \alpha : I \to G_3, \qquad I \subset \mathbb{R} \\ t \to \alpha(t) = (x(t), y(t), z(t) \end{array} \right) \tag{2.2}$$

where $x(t), y(t), z(t) \in C^3$ and t is a real interval. If $x'(t) \neq 0$, then the curve α is called admissible curve.

Let α be an admissible curve in G_3 parameterized by arc length s and given by

$$\alpha(s) = (s, y(s), z(s))$$

where the curvature $\kappa(s)$ and the torsion $\tau(s)$ are

$$\kappa(s) = \sqrt{y''^2(s) + z''^2(s)} \quad \text{and} \quad \tau(s) = \frac{\det\left[\alpha'(s), \alpha''(s), \alpha'''(s)\right]}{\kappa^2(s)}, \tag{2.3}$$

respectively. The associated moving Frenet frame is

$$\begin{aligned} T(s) &= \alpha'(s) = (1, y'(s), z'(s)) \\ N(s) &= \tfrac{1}{\kappa(s)}\alpha'(s) = \tfrac{1}{\kappa(s)}(0, y''(s), z''(s)) \\ B(s) &= \tfrac{1}{\kappa(s)}(0, -z''(s), y''(s)). \end{aligned} \tag{2.4}$$

Here T, N and B are called the tangent vector, principal normal vector and binormal vector fields of the curve α, respectively. Then for the curve α, the following Frenet equations are given by

$$\begin{aligned} T'(s) &= \kappa(s)N(s) \\ N'(s) &= \tau(s)B(s) \\ B'(s) &= -\tau(s)N(s) \end{aligned} \tag{2.5}$$

where T, N, B are mutually orthogonal vectors, [6].

§3. Admissible Mannheim Curves in Galilean Space G_3

In this section, we defined the admissible Mannheim curve and gave some theorems related to these curves in G_3.

Definition 3.2 *Let α and α^* be an admissible curves with the Frenet frames along $\{T, N, B\}$ and $\{T^*, N^*, B^*\}$, respectively. The curvature and torsion of α and α^*, respectively, $\kappa(s), \tau(s)$ and $\kappa^*(s), \tau^*(s)$ never vanish for all $s \in I$ in G_3. If the principal normal vector field N of α coincidence with the binormal vector field B^* of α^* at the corresponding points of the admissible curves α and α^*. Then α is called an admissible Mannheim curve and α^* is an admissible Mannheim mate of α. Thus, for all $s \in I$*

$$\alpha^*(s) = \alpha(s) + \lambda(s)N(s). \tag{3.1}$$

The mate of an admissible Mannheim curve is denoted by (α, α^*), (see Figure 1).

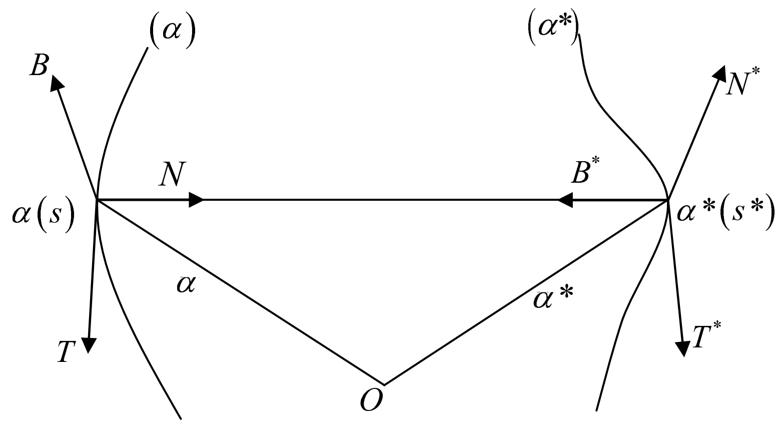

Figure 1. The admissible Mannheim partner curves

Theorem 3.1 *Let (α, α^*) be a mate of admissible Mannheim pair in G_3. Then function λ is constant.*

Proof Let α be an admissible Mannheim curve in G_3 and α^* be an admissible Mannheim mate of α. Let the pair of $\alpha(s)$ and $\alpha^*(s)$ be corresponding points of α and α^*. Then the curve $\alpha^*(s)$ is given by (3.1). Differentiating (3.1) with respect to s and using Frenet equations,

$$T^* \frac{ds^*}{ds} = T + \lambda' N + \lambda\tau B \tag{3.2}$$

is obtained.

Here and here after prime denotes the derivative with respect to s. Since N is coincident with B^* in the same direction, we have

$$\lambda'(s) = 0, \tag{3.3}$$

that is, λ is constant. This theorem proves that the distance between the curve α and its Mannheim mate α^* is constant at the corresponding points of them. It is notable that if α^* is an admissible Mannheim mate of α. Then α is also Mannheim mate of α^* because the relationship obtained in theorem between a curve and its Mannheim mate is reciprocal one. □

Theorem 3.2 *Let α be an admissible curve with arc length parameter s. The curve α is an admissible Mannheim curve if and only if the torsion τ of α is constant.*

Proof Let (α, α^*) be a mate of an admissible Mannheim curves, then there exists the relation

$$\begin{aligned} T(s) &= \cos\theta T^*(s) + \sin\theta N^*(s) \\ B(s) &= -\sin\theta T^*(s) + \cos\theta N^*(s) \end{aligned} \tag{3.4}$$

and

$$\begin{aligned} T^*(s) &= \cos\theta T(s) - \sin\theta B(s) \\ N^*(s) &= \sin\theta T(s) + \cos\theta B(s) \end{aligned} \tag{3.5}$$

where θ is the angle between T and T^* at the corresponding points of $\alpha(s)$ and $\alpha^*(s)$, (see Figure 1).

By differentiating (3.5) with respect to s, we get

$$\tau^* B^* \tfrac{ds^*}{ds} = \tfrac{d(\sin\theta)}{ds} T + \sin\theta\kappa N + \cos\theta\tau N + \tfrac{d(\cos\theta)}{ds} B. \tag{3.6}$$

Since the principal normal vector field N of the curve α and the binormal vector field B^* of its Mannheim mate curve, then it can be seen that θ is a constant angle.

If the equations (3.2) and (3.5) is considered, then

$$\lambda\tau\cot\theta = 1 \tag{3.7}$$

is obtained. According to Theorem 3.1 and constant angle θ, $u = \lambda\cot\theta$ is constant. Then from equation (3.7), $\tau = \frac{1}{u}$ is constant, too. Hence the proof is completed. □

Theorem 3.3(Schell's Theorem) *Let (α, α^*) be a mate of an admissible Mannheim curves with torsions τ and τ^*, respectively. The product of torsions τ and τ^* is constant at the corresponding points $\alpha(s)$ and $\alpha^*(s)$.*

Proof Since α is an admissible Mannheim mate of α^*, then equation (3.1) also can be given by

$$\alpha = \alpha^* - \lambda B^*. \tag{3.8}$$

By taking differentiation of last equation and using equation (3.4),

$$\tau^* = \tfrac{1}{\lambda}\tan\theta \tag{3.9}$$

can be given. By the helps of (3.7), the equation below is easily obtained;

$$\tau\tau^* = \tfrac{\tan^2\theta}{\lambda^2} = constant \tag{3.10}$$

This completes the proof. □

Theorem 3.4 *Let (α, α^*) be an admissible Mannheim mate with curvatures κ, κ^* and torsions τ, τ^* of α and α^*, respectively. Then their curvatures and torsions satisfy the following relations*

$$\begin{aligned} &i)\ \kappa^* = -\tfrac{d\theta}{ds^*} \\ &ii)\ \kappa = \tau^* \tfrac{ds^*}{ds}\sin\theta \\ &iii)\ \tau = -\tau^* \tfrac{ds^*}{ds}\cos\theta. \end{aligned}$$

Proof i) Let us consideration equation (3.4), then we have

$$< T, T^* >= \cos\theta. \tag{3.11}$$

By differentiating last equation with respect to s^* and using the Frenet equations of α and α^*, we reach

$$< \kappa(s)N(s)\tfrac{ds}{ds^*}, T^*(s) > + < T(s), \kappa^*(s)N^*(s) >= -\sin\theta\tfrac{d\theta}{ds^*}. \tag{3.12}$$

Since the principal normal N of α and binormal B^* of α^* are linearly dependent. By considering equations (3.4) and (3.12), we reach

$$\kappa^*(s) = -\tfrac{d\theta}{ds^*}. \tag{3.13}$$

If we take into consideration $< T, B^* >$, $< B, B^* >$ scalar products and (2.5), (3.4), (3.5) equations, then we can easily prove *ii)* and *iii)* items of the theorem, respectively.
The relations given in *ii)* and *iii)* of the last theorem, we obtain

$$\frac{\kappa}{\tau} = -\tan\theta = constant.$$

$\square$

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