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The Arc Energy of Digraph

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Abstract: We study the energy of the arc-adjacency matrix of a directed graph D, which is simply called the arc energy of D. In particular, we give upper and lower bounds for the arc energy of D. We show that arc energy of a directed tree is independent of its orientation. We also compute arc energies of directed cycles and some unitary cayley digraphs.

Keywords: Smarandache arc k-energy, digraph, arc adjacency matrix, arc energy.

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§1. Introduction

Let D be a simple digraph with vertex set $V(D) = \{v_1, v_2, \cdots, v_n\}$ and arc set $\Gamma(D) \subset V(D) \times V(D)$. Let $|\Gamma(D)| = m$. The arc adjacency matrix of D is the $n \times n$ matrix $A = [a_{ij}]$, where

$$a_{ij} = \begin{cases} 1 & \text{if} \quad i < j \quad \text{and} \quad (v_i, v_j) \in \Gamma(D) \\ -1 & \text{if} \quad i < j \quad \text{and} \quad (v_j, v_i) \in \Gamma(D) \\ 0 & \text{if} \quad v_i \text{ and } v_j \text{ are not adjacent.} \end{cases}$$

For $i > j$ we define $a_{ij} = a_{ji}$. A is a symmetric matrix of order n and all its eigenvalues are real. We denote the eigenvalues of A by $\lambda_1, \lambda_2, \cdots, \lambda_n$ with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. The set $\{\lambda_1, \lambda_2, \cdots, \lambda_n\}$ is called the arc spectrum of D. The characteristic polynomial $|xI - A|$ of the arc adjacency matrix A is called the arc characteristic polynomial of D and it is denoted by $\Phi(D; x)$. The arc energy of D is defined by

$$E_a(D) = \sum_{i=1}^{n} |\lambda_i|.$$

For the majority of conjugated hydrocarbons, The total $\pi-$electron energy, E_π satisfies the relation

$$E_\pi(D) = \sum_{i=1}^{n} |\lambda_i|$$

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where $\lambda_1, \lambda_2, \cdots, \lambda_n$ are the eigenvalues of the molecular graph of the conjugated hydrocarbons. In view of this, Gutman [3] introduced the concept of graph energy $E(G)$ of a simple undirected graph G and he defined it as

$$E(G) = \sum_{i=1}^{n} |\lambda_i|$$

where $\lambda_1, \lambda_2, \cdots, \lambda_n$ are the eigenvalues of the adjacency matrix of G. Survey of development of this topic before 2001 can be found in [4]. For recent development, one can consult [2]. The energy of a graph has close links to chemistry [5]. In many situations chemists use digraph rather than graphs. In this paper we are interested in studying mathematical aspects of arc energy of digraphs. The skew energy of a digraph is recently studied in [1].

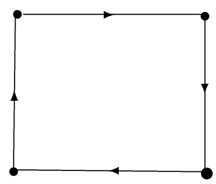
In Section 2 of this paper we study some basic properties of the arc energy and also derive an upper bound for $E_a(D)$. In Section 3 we study arc energy of directed trees. We compute arc energies of directed cycles and some unitary Cayley digraphs in Section 4 and 5 respectively.

§2. Basic Properties of Arc Energy

We begin with the definition of arc energy.

Definition 2.1 *Let A be the arc adjacency matrix of a digraph D. Then its Smarandache arc k-energy $E_a^K(D)$ is defined as $\sum_{i=1}^{n} |\lambda_i|^k$, where n is the order of D and $\lambda_i, 1 \le i \le n$ are the eigenvalues of A. Particularly, if $k = 1$, the Smarandache arc k-energy $E_a^1(D)$ is called the arc energy of D and denoted by $E_a(D)$ for abbreviatation.*

Example 2.2 Let D be a directed cycle on four vertices.



Then $A = \begin{bmatrix} 0 & 1 & 0 & -1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ -1 & 0 & 1 & 0 \end{bmatrix}$ and the characteristic polynomial of A is $\lambda^4 - 4\lambda^2 + 4$, and hence the eigenvalues of A are $-\sqrt{2}, \sqrt{2}, -\sqrt{2}, \sqrt{2}$, and the arc energy of D is $4\sqrt{2}$.

Theorem 2.3 *Let D be a digraph with the arc adjacency characteristic polynomial*

$$\Phi(D; x) = b_0 x^n + b_1 x^{n-1} + \cdots + b_n.$$

Then

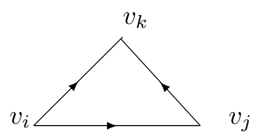
(i) $b_0 = 1$;

(ii) $b_1 = 0$;

(iii) $b_2 = -m$, *the number of arcs of* D;

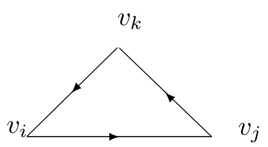
(iv) For $i < j < k$, *we define*

$(i, j) =$ *number of triangles of the form*



and

$(i, j, k) =$ *number of triangles of the form*



$$b_3 = -2[(i, j) + (j, k) + (k, i) + (k, j, i) - (j, i) - (k, j) - (i, k) - (i, j, k)].$$

Proof

(i) It follows from the definition, $\Phi(D; x) = \det(xI - A)$, that $b_0 = 1$.

(ii) Since the diagonal elements of A are all zero, the sum of determinants of all 1×1 principal submatrices of $A =$ trace of A $= 0$. So $b_1 = 0$.

(iii) The sum of determinants of all 2×2 principal submatrices of

$$A = \sum_{j<k} \det \begin{bmatrix} 0 & a_{jk} \\ a_{kj} & 0 \end{bmatrix} = \sum_{j<k} -a_{jk}a_{kj} = -\sum_{j<k} a_{jk}^2 = -m.$$

Thus $b_2 = -m$.

(iv) We have

$$b_3 = (-1)^3 \sum_{i<j<k} \begin{vmatrix} 0 & a_{ij} & a_{ik} \\ a_{ji} & 0 & a_{jk} \\ a_{ki} & a_{kj} & 0 \end{vmatrix}$$

$$= (-1)^3 \sum_{i<j<k} \begin{vmatrix} 0 & a_{ij} & a_{ik} \\ a_{ij} & 0 & a_{jk} \\ a_{ik} & a_{jk} & 0 \end{vmatrix}$$

$$= -2 \sum_{i<j<k} s_{ij} s_{ik} s_{jk}$$

$$= -2[(i,j)+(j,k)+(k,i)+(k,j,i)-(j,i)-(k,j)-(i,k)-(i,j,k)].$$

□

Theorem 2.4 *If $\lambda_1, \lambda_2, \cdots, \lambda_n$ are the arc eigenvalues of a digraph D, then*

(i) $\sum_{i=1}^n \lambda_i^2 = 2m$;

(ii) For $1 \le i \le n$, $|\lambda_i| \le \Delta$, the maximum degree of the underlying graph G_D.

Proof (i) We have $\sum_{i=1}^n \lambda_i^2 = \text{trace of } A^2 = \sum_{i=1}^n \sum_{j=1}^n a_{ij} a_{ji}$

$$= \sum_{i=1}^n \sum_{j=1}^n (a_{ij})^2 = 2m.$$

(ii) Let $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n$ be the eigenvalues of A. The Cauchy-Schwartz inequality state that if $(a_1, a_2, \cdots, a_n)$ and $(b_1, b_2, \cdots, b_n)$ are real n-vectors then

$$\left(\sum_{i=1}^n a_i b_i\right)^2 \le \left(\sum_{i=1}^n a_i^2\right)\left(\sum_{i=1}^n b_i^2\right).$$

Let $a_i = 1$ and $b_i = |\lambda_i|$ for $1 \le i \le n$, and $i \ne j$. Then

$$\left(\sum_{\substack{i=1 \\ i\ne j}}^n |\lambda_i|\right)^2 \le (n-1)\left(\sum_{\substack{i=1 \\ i\ne j}}^n |\lambda_i|^2\right). \tag{2.1}$$

Since $\sum_{i=1}^n \lambda_i = 0$ we have $\sum_{\substack{i=1, \\ i\ne j}}^n \lambda_i = -\lambda_j$. Thus

$$|\sum_{\substack{i=1, \\ j\ne i}}^n \lambda_i|^2 = |-\lambda_j|^2.$$

Hence

$$|-\lambda_j|^2 \le \left(\sum_{\substack{i=1 \\ i\ne j}}^n |\lambda_i|\right)^2.$$

Using (2.1) in the above inequality we get

$$|-\lambda_j|^2 \le (n-1)\sum_{i=1}^n \left(|\lambda_i|^2 - |\lambda_j|^2\right).$$

i.e.,

$$n|\lambda_j|^2 \leq 2m(n-1),$$

$$|\lambda_j|^2 \leq (n-1)^2.$$

Hence

$$|\lambda_j| \leq \Delta.$$

□

Corollary 2.5 $E_a(D) \leq n\Delta$.

Theorem 2.6 $\sqrt{2m + n(n-1)p^{2/n}} \leq E_a(D) \leq \sqrt{2mn} \leq n\sqrt{\Delta}$ *where* $p = |det\ A| = \prod_{i=1}^{n} |\lambda_i|$.

Proof We have

$$(E_a(D))^2 = \left(\sum_{i=1}^{n} |\lambda_i|\right)^2 = \sum_{i=1}^{n} \lambda_i^2 + \sum_{i\neq j} |\lambda_i|\ |\lambda_j|$$

and by the inequality between the arithmetic and geometric means,

$$\frac{1}{n}E_a(D) \geq \left(\prod_{i=1}^{n} |\lambda_i|\right)^{\frac{1}{n}} = |\det\ A|^{\frac{1}{n}}$$

$$\therefore \frac{1}{n(n-1)} \sum_{i\neq j} |\lambda_i|\ |\lambda_j| \geq \left(\prod_{i\neq j} |\lambda_i|\ |\lambda_j|\right)^{\frac{1}{n(n-1)}}$$

$$= \left(\prod_{i=1}^{n} |\lambda_i|^{2(n-1)}\right)^{\frac{1}{n(n-1)}}$$

$$= \left(\prod_{i=1}^{n} |\lambda_i|\right)^{\frac{2}{n}} = |\prod_{i=1}^{n} \lambda_i|^{\frac{2}{n}} = p^{\frac{2}{n}}.$$

Therefore

$$(E_a(D))^2 \geq 2m + n(n-1)p^{\frac{2}{n}}.$$

To prove the right hand side inequality , we apply Schwartz's inequality to the Euclidean vectors $u = (|\lambda_1|, |\lambda_2|, \cdots, |\lambda_n|)$ and $v = (1, 1, \cdots, 1)$ to get

$$E_a(D) = \sum_{i=1}^{n} |\lambda_i| \leq \sqrt{\sum_{i=1}^{n} |\lambda_i|^2}\ \sqrt{n} = \sqrt{2mn} \leq \sqrt{n\Delta n} = n\sqrt{\Delta}. \quad (2.2)$$

□

Corollary 2.7 $E_a(D) = n\sqrt{\Delta}$ *if and only if* $A^2 = \Delta I_n$ *where* I_n *is the identity matrix of order* n.

Proof Equality holds in (2.2) if and only if the Schwartz's inequality becomes equality and trace $A^2 = \sum_{i=1}^{n} \lambda_i^2 = 2m = n\Delta$, if and only if, there exists a constant α such that $|\lambda_i|^2 = \alpha$ for all i and G_D is a Δ-regular graph, if and only if, $A^2 = \alpha I_n$ and $\alpha = \Delta$. □

Theorem 2.8 *Each even positive integer* $2p$ *is the arc energy of a directed star.*

Proof Let $V(K_{1,n}) = \{v_1, \ldots, v_{n+1}\}$. If v_{n+1} is the center of $K_{1,n}$, orient all the edges toward v_{n+1}. Then

$$A = \begin{bmatrix} 0 & 0 & \ldots & 0 & 1 \\ 0 & 0 & \ldots & 0 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & \ldots & 1 & 0 \end{bmatrix},$$

and its eigenvalues are $\{\sqrt{n}, -\sqrt{n}, 0, 0, \ldots, 0\}$, and so $E_a(K_{1,n}) = 2\sqrt{n}$. Now take $n = p^2$. □

§3. Arc Energies of Trees

We begin with a basic lemma.

Lemma 3.1 *Let* D *be a simple digraph. and let* D' *be the digraph obtained from* D *by reversing the orientations of all the arcs incident with a particular vertex of* D*. Then* $E_a(D) = E_a(D')$*.*

Proof Let $A(D)$ be the arc adjacency matrix of D with respect to a labeling of its vertex set. Suppose the orientations of all the arcs incident at vertex v_i of D are reversed. Let the resulting digraph be D'. Then $A(D') = P_i A(D) P_i$ where P_i is the diagonal matrix obtained from the identity matrix by changing the i-th diagonal entry to -1. Hence $A(D)$ and $A(D')$ are orthogonally similar, and so have the same eigenvalues, and hence D and D' have the same arc energy. □

Lemma 3.2 *Let* T *be a labeled directed tree rooted at vertex* v*. It is possible, through reversing the orientations of all arcs incident at some vertices other than* v*, to transform* T *to a directed tree* T' *in which the orientations of all the arcs go from low labels to high labels.*

Proof The proof is by induction on n, the order of the tree. For $n = 2$, there is only one arc and the result is true. Assume that any labeled directed tree of order less than n can be transformed in the manner described to a directed tree T' such that the orientations of all the arcs go from low labels to high labels. Consider a labeled directed tree T of order n rooted at v. Let $N(v)$ be the neighbor set of v. For each $w \in N(v)$, reverse the orientations of all the arcs incident at w, if necessary, so that the orientation of the arc between v and w is from low to high labels. Now, by induction assumption, the old-labeled new-orientation subtree T_w rooted at $w \in N(v)$ can be transformed to a directed subtree T'_w such that the orientations of all the arcs go from low labels to high labels. Now combine all the subtrees T'_w and the root v to obtain the required tree T'. □

Theorem 3.3 *The arc energy of a directed tree is independent of its orientation.*

Proof Let T be a labeled directed tree. Since the underlying graph is a tree, it is a bipartite graph, and hence we can label T such that $A(T) = \begin{bmatrix} 0 & Y \\ Y^T & 0 \end{bmatrix}$. By Lemma 3.2, we can transform T to T' such that $A(T') = \begin{bmatrix} 0 & X \\ X^T & 0 \end{bmatrix}$, where X is nonnegative. By applying Lemma 3.1 repeatedly, we conclude that $A(T)$ and $A(T')$ are orthogonally similar, and hence have the same eigenvalues and so the same arc energy. Consequently, T has the same arc energy as the special directed tree T' in which the orientations of all the arcs go from low labels to high labels. □

Corollary 3.4 *The arc energy of a directed tree is the same as the energy of its underlying tree.*

Proof From the proof of Theorem 3.3, the arc energy of a directed tree is equal to the sum of the singular values of $\begin{bmatrix} 0 & X \\ X^T & 0 \end{bmatrix}$, which is nothing but the adjacency matrix of underlying undirected tree and so the arc energy of a directed tree is the same as the energy of its underlying undirected tree. □

Corollary 3.5 *Energy of a special tournament of order n with vertex set $\{1, 2, \ldots, n\}$ in which all its arcs point from low labels to high labels is same as its underlying tournament.*

§4. Computation of Arc Energies of Cycles

In this section, we compute the arc energies of cycles under different orientations. Given a directed cycle, fix a vertex and label the vertices consecutively. Reversing the arcs incident at a vertex if necessary, we obtain a new directed cycle with arcs going from low labels to high labels with a possible exception of one arc. Hence the arc adjacency matrix of a directed cycle is orthogonally similar to either A^+ or A^- where,

$$A^+ = \begin{bmatrix} 0 & 1 & 0 & \ldots & 1 \\ 1 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & 0 & 0 & \ldots & 0 \end{bmatrix} \text{ and } A^- = \begin{bmatrix} 0 & 1 & 0 & \ldots & -1 \\ 1 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ -1 & 0 & 0 & \ldots & 0 \end{bmatrix}.$$

Case (i): Let C_n^+ be the directed cycle with arc adjacency matrix A^+. We have $A^+ = Z + Z^{n-1}$

where

$$Z = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & 0 & 0 & \dots & 0 \end{bmatrix}$$

which is a circulant matrix. Since $Z^n = I$, the characteristic polynomial of Z is $x^n - 1$. Hence we have $Sp(Z) = \{1, w, w^2, \cdots, w^{n-1}\}$ where $w = e^{\frac{2\pi i}{n}}$ and so

$$\begin{aligned} Sp(C_n^+) &= \{w^j + w^{j(n-1)} \ : \ j = 0, 1, 2, \cdots, n-1\} \\ &= \{w^j + w^{-j} \ : \ j = 0, 1, 2, \cdots, n-1\} \\ &= \{2\cos(\frac{2j\pi}{n}) \ : \ j = 0, 1, 2, \cdots, n-1\}. \end{aligned}$$

For $n = 2k+1$, we have

$$\begin{aligned} E_a(C_n^+) &= \sum_{j=0}^{n-1} 2|\cos(\frac{2j\pi}{n})| = 2 + 4\sum_{j=1}^{k} |\cos(\frac{2j\pi}{(2k+1)})| \\ &= 2 + 4\sum_{j=1}^{k} \cos(\frac{j\pi}{(2k+1)}) = 2 + 4\left(\frac{\sin\frac{(2k+1)\pi}{2(2k+1)}}{2\sin\frac{\pi}{2(2k+1)}} - \frac{1}{2}\right) \\ &= 2\csc(\frac{\pi}{2(2k+1)}) = 2\csc(\frac{\pi}{2n}). \end{aligned}$$

For $n = 4k$,

$$\begin{aligned} E_a(C_n^+) &= \sum_{j=0}^{n-1} 2|\cos(\frac{2j\pi}{n})| = 4 + 8\sum_{j=1}^{k-1} \cos(\frac{j\pi}{2k}) \\ &= 4 + 8\left(\frac{\sin\frac{(2k-1)\pi}{4k}}{2\sin\frac{\pi}{4k}} - \frac{1}{2}\right) = 4\cot(\frac{\pi}{4k}) = 4\cot(\frac{\pi}{n}). \end{aligned}$$

Similarly for $n = 4k+2$

$$E_a(C_n^+) = 4\csc(\frac{\pi}{n}).$$

Putting together the results above, we obtain the following formulas for arc energy of C_n^+ :

$$E_a(C_n^+) = \begin{cases} 2\csc\frac{\pi}{2n} & \text{if} \quad n \equiv 1(\text{mod}2), \\ 4\cot\frac{\pi}{n} & \text{if} \quad n \equiv 0(\text{mod}4), \\ 4\csc\frac{\pi}{n} & \text{if} \quad n \equiv 2(\text{mod}4). \end{cases}$$

Case (ii): Let C_n^- be the directed cycle with arc adjacency matrix A^-. We have $A^- = Z - Z^{n-1}$ where

$$Z = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ -1 & 0 & 0 & \dots & 0 \end{bmatrix}.$$

Since $Z^n = -I$, the characteristic polynomial of Z is $x^n + 1$. Hence we have $Sp(Z) = \{e^{\frac{(2j+1)\pi i}{n}} \mid j = 0, 1, \cdots, (n-1)\}$. So $Sp(A^-) = \{z - z^{n-1} \mid z \in Sp(Z)\}$.

For $n = 2k+1$, we have

$$\begin{aligned} E_a(C_n^-) &= \sum_{j=0}^{n-1} 2|\cos(\frac{(2j+1)\pi}{2k+1})| = 2\left(\sum_{m=0}^{k} \cos(\frac{m\pi}{2k+1}) - \sum_{m=k+1}^{2k} \cos(\frac{m\pi}{2k+1})\right) \\ &= 2\left(1 + \sum_{m=1}^{k} \cos(\frac{m\pi}{2k+1}) - \sum_{m=k+1}^{2k} \cos\left(\pi - \frac{2k+1-m}{2k+1}\right)\right) \\ &= 2\left(1 + \sum_{m=1}^{k} \cos(\frac{m\pi}{2k+1}) + \sum_{m=k+1}^{2k} \cos\left(\frac{2k+1-m}{2k+1}\right)\pi\right) \\ &= 2\left(1 + 2\sum_{m=1}^{k} \cos(\frac{m\pi}{2k+1})\right) = 2 + 4\sum_{m=1}^{k} \cos(\frac{m\pi}{2k+1}) \\ &= 2\csc(\frac{\pi}{2n}). \end{aligned}$$

For $n = 4k$, we have

$$\begin{aligned} E_a(C_n^-) &= \sum_{j=0}^{n-1} |\cos(\frac{(2j+1)\pi}{4k})| = 8\sum_{j=0}^{k-1} \cos(\frac{(2j+1)\pi}{4k}) \\ &= 8\sum_{j=1}^{k} \cos(\frac{(2j-1)\pi}{4k}) = 8\left(\frac{\sin\frac{(k+1)\pi}{4k}\cos(\frac{\pi}{4} - \frac{\pi}{4k})}{\sin\frac{\pi}{4k}}\right). \end{aligned}$$

Similarly for $n = 4k+2$, we get

$$E_a(C_n^-) = \frac{\sin(\frac{(k+1)\pi}{2(2k+1)})\cos(\frac{k\pi}{2(2k+1)} - \frac{\pi}{2(2k+1)})}{\sin\frac{\pi}{2(2k+1)}}.$$

Putting together the results above, we obtain the following formulas for arc energy of C_n^- :

$$E_a(C_n^-) = \begin{cases} 2\csc(\frac{\pi}{2n}) & \text{if} \quad n \equiv 1(\text{mod}2), \\ 8\left(\frac{\sin\frac{(k+1)\pi}{4k}\cos(\frac{\pi}{4} - \frac{\pi}{4k})}{\sin\frac{\pi}{4k}}\right) & \text{if} \quad n \equiv 0(\text{mod}4), \\ \frac{\sin(\frac{(k+1)\pi}{2(2k+1)})\cos(\frac{k\pi}{2(2k+1)} - \frac{\pi}{2(2k+1)})}{\sin\frac{\pi}{2(2k+1)}} & \text{if} \quad n \equiv 2(\text{mod}4). \end{cases}$$

§4. On the Arc Energies of Some Unitary Cayley Digraphs

We now define the unit Cayley digraph D_n, $n > 1$. The vertex set of D_n is $V(D_n) = \{0, 1, 2, \cdots, (n-1)\}$ and the arc set of D_n is $\Gamma(D_n)$ and is defined as follows:

For $i, j \in \{0, 1, 2, \cdots, (n-1)\}$ with $i < j$ and $(j-i, n) = 1$, $(i, j) \in \Gamma(D_n)$ or $(j, i) \in \Gamma(D_n)$ according as $j - i$ is a quadratic residue or a quadratic non-residue modulo n. In this section we compute arc energies of unitary Cayley digraphs D_n for $n = 2^{\alpha_0} p_1^{\alpha_1} \cdots p_r^{\alpha_r}$, $\alpha_0 = 0$ or 1,

$p_i \equiv 1 \pmod 4$, $i = 1, 2, 3, \cdots, r$. We make use of the following well-known result to establish a formula for arc energy of D_n for certain values of n.

Theorem 5.1 *Let $n = 2^{\alpha_0} p_1^{\alpha_1} \cdots p_r^{\alpha_r}$, $n > 1$ and $(a, n) = 1$. Then $x^2 \equiv a$ (mod n) is solvable if and only if*

(i) $\left(\frac{a}{p_i}\right) = 1$ *for* $i = 1, 2, \cdots, r$

and

(ii) $a \equiv 1$ *(mod 4) if* $4 \mid n$ *but* $8 \nmid n$ *;* $a \equiv 1$ *(mod 8) if* $8 \mid n$.

Here $\left(\frac{a}{p_i}\right)$ *is the Legendre symbol.*

Theorem 5.2 *For $n = 2^{\alpha_0} p_1^{\alpha_1} \cdots p_r^{\alpha_r}$, $\alpha_0 = 0$ or 1, $p_i \equiv 1$ (mod 4), $i = 1, 2, 3, \cdots, r$, the arc adjacency eigenvalues of the unitary Cayley digraph D_n are the Gauss sums $G(r, \chi_f)$, $r = 0, 1, 2, \cdots, n-1$, associated with quadratic character f.*

Proof The arc adjacency matrix of D_n with respect to the natural order of the vertices $0, 1, \cdots, n-1$ is

$$A_n = \begin{pmatrix} (\frac{0}{n}) & (\frac{1}{n}) & (\frac{2}{n}) & \cdots & (\frac{i-1}{n}) & \cdots & (\frac{n-1}{n}) \\ \\ (\frac{1}{n}) & (\frac{0}{n}) & (\frac{1}{n}) & \cdots & (\frac{i-2}{n}) & \cdots & (\frac{n-2}{n}) \\ \vdots \\ (\frac{i-1}{n}) & (\frac{i-2}{n}) & (\frac{i-3}{n}) & \cdots & (\frac{0}{n}) & \cdots & (\frac{n-i}{n}) \\ \vdots \\ (\frac{n-1}{n}) & (\frac{n-2}{n}) & (\frac{n-3}{n}) & \cdots & (\frac{n-i}{n}) & \cdots & (\frac{0}{n}) \end{pmatrix}$$

where

$$\left(\frac{a}{n}\right) = \begin{cases} 1 \text{ if } \ (a,n) = 1 \text{ and } x^2 \equiv a (\text{mod } n) \text{ is solvable,} \\ -1 \text{ if } \ (a,n) = 1 \text{ and } x^2 \equiv a (\text{mod } n) \text{ is not solvable,} \\ 0 \ \text{ otherwise.} \end{cases}$$

Since $n = 2^{\alpha_0} p_1^{\alpha_1} \cdots p_r^{\alpha_r}$, $n > 1$, where $\alpha_0 = 0$ or 1 and $p_i \equiv 1 \pmod 4$, $i = 1, 2, 3, \cdots, r$, it follows from Theorem 5.1 that $x^2 \equiv -1$ (mod n) is solvable. Thus

$$\left(-\frac{1}{n}\right) = 1. \tag{5.1}$$

Moreover, if $(a, n) = 1$ then

$$\left(\frac{n-a}{n}\right) = \left(\frac{-a}{n}\right) = \left(\frac{-1}{n}\right)\left(\frac{a}{n}\right) = \left(\frac{a}{n}\right) \qquad (\text{ using } (5.1)).$$

Hence the arc adjacency matrix A_n of D_n is circulant. Consequently the eigenvalues of A_n are given by

$$\begin{aligned}\lambda_r &= \sum_{m=0}^{n-1}\left(\frac{m}{n}\right)w^{rm}, \quad r=0,1,\cdots,n-1, \quad w=e^{\frac{2\pi i}{n}}\\ &= \sum_{m=1}^{n-1}\left(\frac{m}{n}\right)w^{rm}=G(r,\chi_f)\end{aligned}$$

where χ_f is the Dirichlet quadratic character mod n. □

Theorem 5.3 *If $n=2^{\alpha_0}p_1^{\alpha_1}\cdots p_r^{\alpha_r}$, $n>1$, where $\alpha_0=0$ or 1 and $p_i\equiv 1$ (mod 4), $i=1,2,\cdots,r$ then the arc energy of D_n is*

$$E_a(D_n)=\sqrt{n}\ \phi(n).$$

Proof By Theorem 5.2, the eigenvalues of D_n are

$$\lambda_r=G(r,\chi_f),\ \ 0\le r\le n-1.$$

Hence the arc energy of D_n is given by

$$\begin{aligned}E_a(D_n) &= \sum_{r=0}^{n-1}|\lambda_r|=\sum_{r=0}^{n-1}|G(r,\chi_f)|\\ &= \sum_{r=1}^{n-1}|\overline{\chi}_f(r)||G(1,\chi_f)|=|G(1,\chi_f)|\ \phi(n).\end{aligned}$$

Therefore, to complete the proof, we need to compute $|G(1,\chi_f)|$. We have

$$\begin{aligned}|G(1,\chi_f)|^2 &= G(1,\chi_f)\overline{G(1,\chi_f)}=G(1,\chi_f)\sum_{m=1}^{n}\overline{\chi}_f(m)\ e^{\frac{-2\pi i m}{n}}\\ &= \sum_{m=1}^{n}G(m,\chi_f)\ e^{\frac{-2\pi i m}{n}}=\sum_{m=1}^{n}\sum_{j=1}^{n}\left(\frac{j}{n}\right)e^{\frac{2\pi i j m}{n}}e^{\frac{-2\pi i m}{n}}\\ &= \sum_{j=1}^{n}\left(\frac{j}{n}\right)\sum_{m=1}^{n}w^{m(j-1)}, \text{ where } w=e^{\frac{2\pi i}{n}}\\ &= \left(\frac{1}{n}\right)\sum_{m=1}^{n}1, \text{ since} \sum_{m=1}^{n}w^{m(j-1)}=0, \text{ if } j>1\\ &= n.\end{aligned}$$

Hence $|G(1,\chi_f)|=\sqrt{n}$ and $E(D_n)=\sqrt{n}\ \phi(n)$.

Conclusion The arc spectrum and arc energy of D_n when $n\equiv 1$ or 2(mod 4) was computed (Theorems 5.2 and 5.3.) using fact that the associated arc adjacency matrix A_n was circulant. Since in general A_n is not circulant, we leave open the problem of computing the arc spectrum and arc energy of D_n for any natural number n.

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