

Singed Total Domatic Number of a Graph

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Abstract: Let G be a finite and simple graph with vertex set $V(G)$, $k \geq 1$ an integer and let $f : V(G) \rightarrow \{-k, k-1, \dots, -1, 1, \dots, k-1, k\}$ be $2k$ valued function. If $\sum_{x \in N(v)} f(x) \geq k$ for each $v \in V(G)$, where $N(v)$ is the open neighborhood of v , then f is a Smarandachely k -Signed total dominating function on G . A set $\{f_1, f_2, \dots, f_d\}$ of Smarandachely k -Signed total dominating function on G with the property that $\sum_{i=1}^d f_i(x) \leq k$ for each $x \in V(G)$ is called a Smarandachely k -Signed total dominating family (function) on G . Particularly, a Smarandachely 1-Signed total dominating function or family is called signed total dominating function or family on G . The maximum number of functions in a signed total dominating family on G is the signed total domatic number of G . In this paper, some properties related signed total domatic number and signed total domination number of a graph are studied and found the sign total domatic number of certain class of graphs such as fans, wheels and generalized Petersen graph.

Key Words: Smarandachely k -signed total dominating function, signed total domination number, signed total domatic number.

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§1. Terminology and Introduction

Various numerical invariants of graphs concerning domination were introduced by means of dominating functions and their variants [1] and [4]. We considered finite, undirected, simple graphs $G = (V, E)$ with vertex set $V(G)$ and edge set $E(G)$. The order of G is given by $n = |V(G)|$. If $v \in V(G)$, then the open neighborhood of v is $N(v) = \{u \in V(G) | uv \in E(G)\}$ and the closed neighborhood of v is $N[v] = \{v\} \cup N(v)$. The number $d_G(v) = d(v) = |N(v)|$ is the *degree* of the vertex $v \in V(G)$, and $\delta(G)$ is the *minimum degree* of G . The *complete graph* and the *cycle* of order n are denoted by K_n and C_n respectively. A fan and a wheel is a graph obtained from a path and a cycle by adding a new vertex and edges joining it to all the vertices of the path and cycle respectively. The generalized Petersen graph $P(n, k)$ is defined to be a graph on $2n$ vertices with $V(P(n, k)) = \{v_i u_i : 1 \leq i \leq n\}$ and $E(P(n, k)) =$

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$\{v_i v_{i+1}, v_i u_i, u_i u_{i+k} : 1 \leq i \leq n, \text{subscripts modulo } n\}$. If $A \subseteq V(G)$ and f is a mapping from $V(G)$ into some set of numbers, then $f(A) = \sum_{x \in A} f(x)$.

Let $k \geq 1$ be an integer and let $f : V(G) \rightarrow \{-k, k-1, \dots, -1, 1, \dots, k-1, k\}$ be a $2k$ valued function. If $\sum_{x \in N(v)} f(x) \geq k$ for each $v \in V(G)$, where $N(v)$ is the open neighborhood of v , then f is a Smarandachely k -Signed total dominating function on G . A set $\{f_1, f_2, \dots, f_d\}$ of Smarandachely k -Signed total dominating function on G with the property that $\sum_{i=1}^d f_i(x) \leq k$ for each $x \in V(G)$ is called a Smarandachely k -Signed total dominating family (function) on G . Particularly, a Smarandachely 1-Signed total dominating function or family is called signed total dominating function or family on G . The signed total dominating function is defined in [6] as a two valued function $f : V(G) \rightarrow \{-1, 1\}$ such that $\sum_{x \in N(v)} f(x) \geq 1$ for each $v \in V(G)$. The minimum of weights $w(f)$, taken over all signed total dominating functions f on G , is called the signed total domination number $\gamma_t^s(G)$. Signed total domination has been studied in [3].

A set $\{f_1, f_2, \dots, f_d\}$ of signed total dominating functions on G with the property that $\sum_{i=1}^d f_i(x) \leq 1$ for each $x \in V(G)$, is called a signed total dominating family on G . The maximum number of functions in a signed total dominating family is the signed total domatic number of G , denoted by $d_t^s(G)$. Signed total domatic number was introduced by Guan Mei and Shan Er-fang [2]. Guan Mei and Shan Er-fang [2] have determined the basic properties of $d_t^s(G)$. Some of them are analogous to those of the signed domatic number in [5] and studied sharp bounds of the signed total domatic number of regular graphs, complete bipartite graphs and complete graphs. Guan Mei and Shan Er-fang [2] presented the following results which are useful in our investigations.

Proposition 1.1([6]) *For Circuit C_n of length n we have $\gamma_t^s(C_n) = n$.*

Proof Here no other signed total dominating exists than the constants equal to 1. $\square$

Theorem 1.2([3]) *Let T be a tree of order $n \geq 2$. then, $\gamma_t^s(T) = n$ if and only if every vertex of T is a support vertex or is adjacent to a vertex of degree 2.*

Proposition 1.3([2]) *The signed total domatic number $d_t^s(G)$ is well defined for each graph G .*

Proposition 1.4([2]) *For any graph G of order n , $\gamma_t^s(G) \cdot d_t^s(G) \leq n$.*

Proposition 1.5([2]) *If G is a graph with the minimum degree $\delta(G)$, then $1 \leq d_t^s(G) \leq \delta(G)$.*

Proposition 1.6([2]) *The signed total domatic number is an odd integer.*

Corollary 1.7([2]) *If G is a graph with the minimum degree $\delta(G) = 1$ or 2, then $d_t^s(G) = 1$. In particular, $d_t^s(C_n) = d_t^s(P_n) = d_t^s(K_{1,n-1}) = d_t^s(T) = 1$, where T is a tree.*

§2. Properties of the Signed Total Domatic Number

Proposition 2.1 *If G is a graph of order n and $\gamma_t^s(G) \geq 0$ then, $\gamma_t^s(G) + d_t^s(G) \leq n + 1$ equality*

holds if and only if G is isomorphic to C_n or tree T of order $n \geq 2$.

Proof Let G be a graph of order n . The inequality follows from the fact that for any two non-negative integers a and b , $a + b \leq ab + 1$. By Proposition 1.4 we have,

$$\gamma_t^s(G) + d_t^s(G) \leq \gamma_t^s(G) \cdot d_t^s(G) + 1 \leq n + 1$$

Suppose that $\gamma_t^s(G) + d_t^s(G) = n + 1$ then, $n + 1 = \gamma_t^s(G) + d_t^s(G) \leq \gamma_t^s(G) \cdot d_t^s(G) + 1 \leq n + 1$.

This implies that $\gamma_t^s(G) + d_t^s(G) = \gamma_t^s(G) \cdot d_t^s(G) + 1$. This shows that $\gamma_t^s(G) \cdot d_t^s(G) = n$. Solving equations 1 and 2 simultaneously, we have either $\gamma_t^s(G) = 1$ and $d_t^s(G) = n$ or $\gamma_t^s(G) = n$ and $d_t^s(G) = 1$. If $\gamma_t^s(G) = 1$ and $d_t^s(G) = n$ then $n = d_t^s(G) \leq \delta(G)$. There fore, $\delta(G) \geq n$ a contradiction.

If $\gamma_t^s(G) = n$ and $d_t^s(G) = 1$ then by Proposition 1.1 and Proposition 1.2, we have $\gamma_t^s(C_n) = n$ and $d_t^s(C_n) = 1$ and By Theorem 1.2, If T is a tree of order $n \geq 2$ then, $\gamma_t^s(T) = n$ if and only if every vertex of T is a support vertex or is adjacent to a vertex of degree 2 and $d_t^s(T) = 1$. $\square$

Theorem 2.2 Let G be a graph of order n then $d_t^s(G) + d_t^s(\bar{G}) \leq n - 1$.

Proof Let G be a regular graph order n , By Proposition 1.5 we have $d_t^s(G) \leq \delta(G)$ and $d_t^s(\bar{G}) \leq \delta(\bar{G})$. Thus we have,

$$d_t^s(G) + d_t^s(\bar{G}) \leq \delta(G) + \delta(\bar{G}) = \delta(G) + (n - 1 - \Delta(G)) \leq n - 1.$$

Thus the inequality holds. $\square$

§3. Signed Total Domatic Number of Fans, Wheels and Generalized

Petersen Graph

Proposition 3.1 Let G be a fan of order n then $d_t^s(G) = 1$.

Proof Let $n \geq 2$ and let $x_1, x_2, \dots, x_n$ be the vertex set of the fan G such that $x_1, x_2, \dots, x_n, x_1$ is a cycle of length n and x_n is adjacent to x_i for each $i = 2, 3, \dots, n - 2$. By Proposition 1.5 and Proposition 1.6, $1 \leq d_t^s(G) \leq \delta(G) = 2$, which implies $d_t^s(G) = 1$ which proves the result. $\square$

Proposition 3.2 If G is a wheel of order n then $d_t^s(G) = 1$.

Proof Let $x_1, x_2, \dots, x_n$ be the vertex set of the wheel G such that $x_1, x_2, \dots, x_{n-1}, x_1$ is a cycle of length $n - 1$ and x_n is adjacent to x_i for each $i = 1, 2, 3, \dots, n - 1$. According to the Proposition 1.5 and Proposition 1.6, we observe that either $d_t^s(G) = 1$ or $d_t^s(G) = 3$. Suppose to the contrary that $d_t^s(G) = 3$. Let $\{f_1, f_2, f_3\}$ be a corresponding signed total dominating family. Because of $f_1(x_n) + f_2(x_n) + f_3(x_n) \leq 1$, there exists at least one function say f_1 with $f_1(x_n) = -1$. The condition $\sum_{x \in N(v)} f_1(x) \geq 1$ for each $v \in (V(G) - \{x_n\})$ yields $f_1(x) = 1$ for each some $i \in \{1, 2, \dots, n - 1\}$ and $t = 2, 3$ then it follows that $f_t(x_{i+1}) = f_t(x_{i+2}) = 1$, where the indices are taken modulo $n - 1$ and $f_t(x_n) = 1$. Consequently, the function f_t has at most $\lfloor \frac{n}{2} \rfloor - 1$ for n is odd and $\frac{n}{2} - 1$ for n is even number of vertices $x \in V(G)$ such that

$f_t(x) = -1$. Thus there exist at most $\lfloor \frac{n}{2} \rfloor - 1$ for n is odd and $\frac{n}{2} - 1$ for n is even number of vertices $x \in V(G)$ such that $f_t(x) = -1$ for at least one $i = 1, 2, 3$. Since $n \geq 4$, we observe that $2(\lfloor \frac{n}{2} \rfloor + 1) = 2(\frac{n}{2} - 1) + 1 < n$ for n is odd and $2(\frac{n}{2} - 1) + 1 < n$, a contradiction to $f_1(x_n) + f_2(x_n) + f_3(x_n) \leq 1$ for each $x \in V(G)$. $\square$

Proposition 3.3 *Let $G = P(n, k)$ be a generalized Petersen graph then for $k = 1, 2$, $d_t^s(G) = 1$.*

Proof The generalized Petersen graph $P(n, 1)$ is a graph on $2n$ vertices with

$$V(P(n, k)) = \{v_i u_i : 1 \leq i \leq n\}$$

and $E(P(n, k)) = \{v_i v_{i+1}, v_i u_i, u_i u_{i+1} : 1 \leq i \leq n, \text{subscripts modulo } n\}$. According to the Proposition 1.5, Proposition 1.6, we observe that $d_t^s(G) = 1$ or $d_t^s(G) = 3$.

Case 1: $k = 1$

Let $\{f_1, f_2, f_3\}$ be a corresponding signed total dominating functions. Because of $f_1(v_n) + f_2(v_n) + f_3(v_n) \leq 1$ for each $i \in \{1, 2, \dots, 2n\}$, there exist at least one number $j \in \{1, 2, 3\}$ such that $f_j(v_i) = -1$. Let, for example, $f_1(v_k) = -1$ for for any $t \in \{1, 2, \dots, 2n\}$ then $\sum_{x \in N(v_t)} f_1(v) \geq 1$ implies that $f_1(v_k) = f_1(v_{k+1}) = -1$ for $k \cong 0, 1 \pmod{4}$ and $f_1(v_k) = -1$ for $k \cong 0 \pmod{3}$. This implies, there exist at most $8r, 8r + 2, 8r + 4, 8r + 6, r \geq 1$ vertices such that $f_t(v) = -1$ for each $t = 2, 3$ when $P(n, 1)$ is of order $2(6r + l)$ for $0 \leq l \leq 2, 2(6r + 3), 2(6r + 4), 2(6r + 5)$ respectively. Thus there exist $3(8r) = 3(8(\frac{n}{12} - \frac{l}{6})) < n$ (similarly $< n$ for all values of vertex set) a contradiction to $f_1(v_n) + f_2(v_n) + f_3(v_n) \leq 1$ for each $v \in V(G)$.

Case 2: $k = 2$

Similar to the proof of Case 1, we can prove the claim in this case. $\square$

References

- [1] Cockayne E.J, Hedetniemi S.T. *Towards a theory of domination in graphs*, J. Networks, 1977, 7: 247-261.
- [2] Guan Mei, Shan Er-fang. *Signed total domatic number of a graph*, J.Shanghai Univ (Engl Ed) 2008 12 (1): 31-34.
- [3] Henning M.A. *Signed total domination in graphs*, J. Discrete Mathematics, 2001, 278: 109-125.
- [4] Haynes T.W, Hedetniemi S.T, Slater P.J. *Fundamentals of domination in graphs*, M. New York: Marcel Dekker, 1998.
- [5] Volkmann L, Zelinka B. *Signed domatic number of a graph*, J. Discrete Applied Mathematics, 2005, 150: 261-267.
- [6] Zelinka B, Liberec. *Signed total domination number of a graph*, J. Czechoslovak Mathematical Journal, 2001, 51: 225-229.