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Counting Rooted Eulerian Planar Maps

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Abstract: In this paper a new method for establishing generating equations of rooted Eulerian planar maps will be provided. It is an algebraic method instead of the constructional one used as before and plays an important role in finding the kind of equations. Some equations of rooted loopless Eulerian planar maps will be obtained by using the method and some results will be corrected and simplified here.

Keywords: Eulerian map, generating function, enumerating equation, Smarandache multi-embedding, multi-surface.

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§1. Introduction

A *Smarandache multi-embedding of a graph G on a multi-surface* $\widetilde{S}$ is a continuous mapping $\varsigma : G \to \widetilde{S}$ such that there are no intersections between any two edges unless its endpoints, where $\widetilde{S}$ is an unions of surfaces underlying a graph H. Particularly, if $|V(H)| = 1$, i.e., $\widetilde{S}$ is just a surface, such multi-embedding is the common embedding of G.

With respect to the enumeration of rooted Eulerian planar maps the first result for enumerating rooted general Eulerian planar maps with vertex partition was achieved by Tutte [10] in the early 1960's. In 1986 the enumeration of rooted non-separable Eulerian planar maps with vertex partition was studied by Liu [4]. In 1992 the enumeration of rooted loopless Eulerian planar maps with vertex partition and other variables as parameters were investigated by Liu [5,6,7] too. From then on some new results were obtained [1,2,8,9], but the method used there was so difficult that one can not understand them easily. In 2004 the enumeration of unrooted Eulerian and unicursl planar maps with the number of edges was resulted by Liskovets [3] based on the rooted results. In present article we will provide an algebraic method instead of that used in the past for counting this kind of planar maps. It will paly an important role in establishing the equations of all kinds of rooted Eulerian planar maps. As examples, some equations of rooted loopless Eulerian planar maps can be derived by using the method. The procedure

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and some results in [5,6,8] will be reduced greatly and updated properly.

In general, *rooting* a map means distinguishing one edge on the boundary of *the outer face* as the *root-edge*, and one end of that edge as the *root-vertex*. In diagrams we usually represent the root-edge as an edge with an arrow in the outer face, the arrow being drawn from the root-vertex to the other end. So the outer face is also called the *root-face*. A planar map with a rooting is said to be a *rooted planar map*. We say that two rooted planar maps are *combinatorially equivalent* or up to *root-preserving isomorphism* if they are related by one to one correspondence of their elements, which maps vertices onto vertices, edges onto edges and faces onto faces,and which preserves incidence relations and the rooted elements. Otherwise, *combinatorially inequivalent* or *nonisomorphic* here.

Let $\mathcal{M}$ be any set of maps. For a map $M \in \mathcal{M}$ let $M - R$ and $M \bullet R$ be the resultant maps of deleting the root-edge $R(M)$ from M and contracting $R(M)$ into a vertex as the new root-vertex, respectively. For a vertex v of M let $val(v)$ be the valency of the vertex v. Moreover, the valency of the root-vertex of M is denoted by $val(M)$.

Terminologies and notations not explained here refer to [9].

§2. Relations on Maps

In order to set up the enumerating equation satisfied by some generating functions we have to introduce the operations on maps in $\mathcal{M}$.

Let

$$\mathcal{M}\langle R\rangle = \{M - R \mid M \in \mathcal{M}\}; \quad \mathcal{M}(R) = \{M \bullet R \mid M \in \mathcal{M}\}, \tag{31}$$

and let

$$\begin{cases} \widetilde{\bigtriangledown}\mathcal{M} = \sum\limits_{M\in\mathcal{M}} \{\bigtriangledown_i M \mid i = 1, 2, \cdots, l(M) - 1\}; \\ \bigtriangledown\mathcal{M} = \sum\limits_{M\in\mathcal{M}} \{\bigtriangledown_i M \mid i = 0, 1, 2, \cdots, l(M)\}, \end{cases} \tag{32}$$

where $\bigtriangledown_i M$ is the resultant map of splitting the root-vertex of M into two vertices v'_r and v''_r with a new edge $\langle v'_r, v''_r\rangle$ as the root-edge of the new map $\bigtriangledown_i M$ such that the valency of its root-vertex $val(\bigtriangledown_i M) = i + 1$.

Further,write that

$$\begin{cases} \mathcal{M}^{(e)} = \{M \in \mathcal{M} \mid val(M) \equiv 0(\text{mod}2)\}; \\ \mathcal{M}^{(o)} = \{M \in \mathcal{M} \mid val(M) \equiv 1(\text{mod}2)\}. \end{cases} \tag{33}$$

It is clear that $\mathcal{M}^{(e)}$ and $\mathcal{M}^{(o)}$ stand for maps in $\mathcal{M}$ with the valency of root-vertex of the maps being even and odd, respectively.

Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be two sets of maps. For two maps $M_1 \in \mathcal{M}_1$ and $M_2 \in \mathcal{M}_2$, let $M_1 \dot{+} M_2$ be the map $M_1 \bigcup M_2$ such that

(i) $M_1 \bigcap M_2$ is only a vertex as the root-vertex of $M_1 \dot{+} M_2$;

(ii) M_1 is inside one of the faces incident with the root-vertex of M_2;

(iii) The root-edge of $M_1 \dot{+} M_2$ is the same as that of M_2;

(iv) The first occurrence of the edges in M_1 incident with the root-vertex of $M_1\dot{+}M_2$ is the root-edge of M_1 when one moves around the root-vertex of $M_1\dot{+}M_2$ in the rotational direction starting from the root-edge of $M_1\dot{+}M_2$.

For the maps $M_i \in \mathcal{M}_i$, $i = 1, 2, \cdots, k$, we define that

$$\begin{cases} M_1\dot{+}M_2\dot{+}\cdots\dot{+}M_k = \left(M_1\dot{+}M_2\dot{+}\cdots\dot{+}M_{k-1}\right)\dot{+}M_k; \\ \mathcal{M}_1 \odot \mathcal{M}_2 \odot \cdots \odot \mathcal{M}_k = \left\{M_1\dot{+}M_2\dot{+}\cdots\dot{+}M_k \mid M_i \in \mathcal{M}_i, 1 \leqslant i \leqslant k\right\}, \\ \mathcal{M}^{\odot k} = \mathcal{M}_1 \odot \mathcal{M}_2 \odot \cdots \odot \mathcal{M}_k|_{\mathcal{M}_1=\mathcal{M}_2=\cdots=\mathcal{M}_k=\mathcal{M}}. \end{cases} \tag{34}$$

Now, we have to introduce another kind operation in order to finish the construction of the sets of maps as follows.

For two maps $M_1 \in \mathcal{M}_1$ and $M_2 \in \mathcal{M}_2$, let $M_1\hat{+}M_2$ be the resultant map of identifying the two root-edges of M_1 and M_2 such that M_1 is inside the non-root-face incident with the root-edge of M_2, or onto the non-root-side of M_2 if the root-edge of M_2 is a cut-edge. Of course, the root-edge of $M_1\hat{+}M_2$ has to be the identified edge and the non-root-face incident with the root-edge of $M_1\hat{+}M_2$ is the same as in M_1.

For the maps $M \in \mathcal{M}$ and $M_i \in \mathcal{M}_i$, $i = 1, 2, \cdots, k$, we define that

$$\begin{cases} M_1\hat{+}M_2\hat{+}\cdots\hat{+}M_k = \left(M_1\hat{+}M_2\hat{+}\cdots\hat{+}M_{k-1}\right)\hat{+}M_k; \\ \mathcal{M}_1 \oplus \mathcal{M}_2 \oplus \cdots \oplus \mathcal{M}_k = \left\{M_1\hat{+}M_2\hat{+}\cdots\hat{+}M_k \mid M_i \in \mathcal{M}_i, 1 \leqslant i \leqslant k\right\}; \\ \mathcal{M}^{\oplus k} = \mathcal{M}_1 \oplus \mathcal{M}_2 \oplus \cdots \oplus \mathcal{M}_k|_{\mathcal{M}_1=\mathcal{M}_2=\cdots=\mathcal{M}_k=\mathcal{M}}. \end{cases} \tag{35}$$

A map is called *Eulerian* if all its vertices are of even valency. It is well-known that a map is Eularian if and only if it has an Eularian circuit, a circuit containing each of the edges exactly once. A map is called *loopless* if there is no any *loop* in the map.

Let $\mathcal{E}_{nl}$ be *the set of all rooted loopless Eulerian planar maps* with the *vertex map* ϑ in $\mathcal{E}_{nl}$ as a special case. Of course, the loop map O is not in $\mathcal{E}_{nl}$. It is easily checked that no Eulerian maps has a separable edge.

The enumerating problems of rooted loopless Eulerian planar maps will be discussed here by using a new method witch is much simpler than that used in the past [4,5,6,9].

Let $\mathcal{E}_{\mathrm{nl}_0} = \{\vartheta\}$ and $\mathcal{E}_{\mathrm{nl}_i} = \{M \in \mathcal{E}_{\mathrm{nl}} - \vartheta \mid \mathrm{R(M)} \text{ is } i \text{ multi-edges in } M\}$, for $i \geqslant 1$. Then the set $\mathcal{E}_{\mathrm{nl}}$ can be partitioned into the following form

$$\mathcal{E}_{\mathrm{nl}} = \sum_{i\geqslant 0} \mathcal{E}_{\mathrm{nl}_i}, \quad \text{and} \quad \mathcal{E}_{\mathrm{nl}}(R) = \sum_{i\geqslant 0} \mathcal{E}_{\mathrm{nl}_i}(R), \tag{36}$$

where $\mathcal{E}_{\mathrm{nl}_0}(R) = \mathcal{E}_{\mathrm{nl}_0} = \{\vartheta\}$ and $\mathcal{E}_{\mathrm{nl}_1}(R) = \mathcal{E}_{\mathrm{nl}} - \mathcal{E}_{\mathrm{nl}_0}$. Let $\mathcal{E}_{in} = \mathcal{E}_{\mathrm{nl}}\dot{+}\{O\}$ be the set of all rooted *inner Eulerian planar maps* [9], then from (4) we have

$$\mathcal{E}_{\mathrm{nl}_i}(R) = \mathcal{E}_{in}^{\odot i-1} \odot \mathcal{E}_{\mathrm{nl}},$$

for $i \geqslant 2$.

If we write $\mathcal{E}_{\rm in}^{\odot 0}=\mathcal{E}_{\rm nl_0}=\{\vartheta\}$, then from (6) we have

$$\begin{aligned}\mathcal{E}_{\rm nl}(R) &= \sum_{i\geqslant 0}\mathcal{E}_{{\rm nl}_i}(R)=\mathcal{E}_{\rm nl_0}(R)+\mathcal{E}_{\rm nl_1}(R)+\sum_{i\geqslant 2}\mathcal{E}_{{\rm nl}_i}(R)\\ &= \mathcal{E}_{\rm nl_0}+(\mathcal{E}_{\rm nl}-\mathcal{E}_{\rm nl_0})+\sum_{i\geqslant 2}(\mathcal{E}_{\rm in}^{\odot i-1}\odot\mathcal{E}_{\rm nl})\\ &= \mathcal{E}_{\rm nl}+\sum_{i\geqslant 1}(\mathcal{E}_{\rm in}^{\odot i}\odot\mathcal{E}_{\rm nl})=\mathcal{E}_{\rm nl_0}\odot\mathcal{E}_{\rm nl}+\sum_{i\geqslant 1}(\mathcal{E}_{\rm in}^{\odot i}\odot\mathcal{E}_{\rm nl})\\ &= \mathcal{E}_{\rm in}^{\odot 0}\odot\mathcal{E}_{\rm nl}+\sum_{i\geqslant 1}(\mathcal{E}_{\rm in}^{\odot i}\odot\mathcal{E}_{\rm nl}).\end{aligned}$$

i.e.,

$$\mathcal{E}_{\rm nl}(R)=\sum_{i\geqslant 0}(\mathcal{E}_{\rm in}^{\odot i}\odot\mathcal{E}_{\rm nl}). \tag{37}$$

Now, In order to enumerate the maps in $\mathcal{E}_{\rm nl}$ conveniently, we need to reconstruct the set $\mathcal{E}_{\rm nl}$ according to the construction of $\mathcal{E}_{\rm nl}(R)$ in (7). Hence, we suppose that

$$\mathcal{F}=\sum_{i\geqslant 0}\left[\left(\widetilde{\bigtriangledown}\mathcal{E}_{\rm in}\right)^{\oplus i}\oplus(\bigtriangledown\mathcal{E}_{\rm nl})\right], \tag{38}$$

where $\left(\widetilde{\bigtriangledown}\mathcal{E}_{\rm in}\right)^{\oplus 0}$ is defined as $\mathcal{E}_{\rm nl_0}$.

In general, a map in $\mathcal{F}$ may be not Eulerian. It is obvious that $\mathcal{F}$ can be classified into two classes $\mathcal{F}^{(e)}$ and $\mathcal{F}^{(o)}$ where $\mathcal{F}^{(e)}$ is just what we need because the maps in it are all Eulerian, i.e., $\mathcal{F}^{(e)}\subseteq\mathcal{E}_{\rm nl}-\mathcal{E}_{\rm nl_0}$. Conversely, for any map $M\in\mathcal{E}_{\rm nl}-\mathcal{E}_{\rm nl_0}$, there is a set $\mathcal{E}_{{\rm nl}_i}$, $i\geqslant 1$ such that $M\in\mathcal{E}_{{\rm nl}_i}$, thus $M\bullet R\in\mathcal{E}_{{\rm nl}_i}(R)=\mathcal{E}_{\rm in}^{\odot i-1}\odot\mathcal{E}_{\rm nl}$. So we have $M\in\mathcal{E}_{{\rm nl}_i}=\left[(\widetilde{\bigtriangledown}\mathcal{E}_{\rm in})^{\oplus i-1}\oplus(\bigtriangledown\mathcal{E}_{\rm nl})\right]^{(e)}\subset\mathcal{F}^{(e)}$, i.e., $\mathcal{E}_{\rm nl}-\mathcal{E}_{\rm nl_0}\subseteq\mathcal{F}^{(e)}$. In the other words, we have

$$\mathcal{E}_{\rm nl}=\mathcal{E}_{\rm nl_0}+\mathcal{F}^{(e)}\quad\text{and}\quad\mathcal{F}^{(e)}=\mathcal{F}-\mathcal{F}^{(o)}. \tag{39}$$

In addition, it is not difficult to see that

$$\mathcal{E}_{\rm in}\langle R\rangle=\mathcal{E}_{\rm nl}. \tag{40}$$

§3. Equations with Vertex Partition

In this section we want to discuss the following generating function for the set $\mathcal{M}$ of some maps.

$$g_{\mathcal{M}}(x:\underline{y})=\sum_{M\in\mathcal{M}}x^{l(M)}\underline{y}^{\underline{n}(M)}, \tag{41}$$

in which $\underline{y}(M)$ and $\underline{n}(M)$ stand for infinite vectors, and

$$\underline{y}^{\underline{n}(M)}=\prod_{i\geqslant 1}y_i^{n_i(M)};\quad \underline{y}=(y_1,y_2,\cdots);\quad \underline{n}(M)=(n_1(M),n_2(M),\cdots),$$

where $l(M)=val(M)$ and $n_i(M)$ is the number of the non-root vertices of valency i, $i\geqslant 1$. The function (11) is said to be the *vertex partition function* of $\mathcal{M}$. Naturally, for a Eulerian planar map $M\in\mathcal{E}_{\rm nl}$, we may let $l(M)=val(M)=2m(M)$ and $n_{2j+1}(M)\equiv 0$ for $j\geqslant 0$.

For this reason we need to introduce the following *Blisard* -operator in y

$$\int_y y^i = y_i, \quad i \geqslant 1 \quad \text{and} \quad \int_y y^0 = 1,$$

which is a *linear operator* and for a function $f(z)$ we define that

$$\delta_{x,y} f = \frac{f(x) - f(y)}{x^2 - y^2}. \tag{42}$$

They are said to be (x, y)-deference of $f(z)$.

In the following, the new algebraic method is used for enumerating the set of maps in $\mathcal{F}^{(e)}$.

Lemma 3.1 *For the set $\mathcal{F}^{(e)}$, we have*

$$g_{\mathcal{F}^{(e)}}(x : \underline{y}) = \int_y \frac{x^2 y^2 \delta_{x,y}(f + z^2 f^2)}{1 - x^2 y^2 \delta_{x,y}(2f + z^2 f^2)}, \tag{43}$$

where $f = f(z) = g_{\mathcal{E}_{\mathrm{nl}}}(z : \underline{y})$.

Proof From the definitions (3), (8) and (11), we have

$$\begin{aligned}
g_{\mathcal{F}}(x : \underline{y}) &= \sum_{i \geqslant 0} x \int_y y \left(\sum_{M \in \mathcal{E}_{\mathrm{in}}} \sum_{j=1}^{l(M)-1} x^j y^{l(M)-j} \underline{y}^{\underline{n}(M)} \right)^i \sum_{M \in \mathcal{E}_{\mathrm{nl}}} \sum_{j=0}^{l(M)} x^j y^{l(M)-j} \underline{y}^{\underline{n}(M)} \\
&= x \int_y y \sum_{i \geqslant 0} \left(x y^{-1} \frac{g_{\mathcal{E}_{\mathrm{in}}}(y) - x^{-1} y g_{\mathcal{E}_{\mathrm{in}}}(x)}{1 - x y^{-1}} \right)^i \frac{f(y) - x y^{-1} f(x)}{1 - x y^{-1}} \\
&= \int_y \sum_{i \geqslant 0} \left(xy \frac{y f(y) - x f(x)}{y - x} \right)^{i+1} \\
&= \int_y \frac{xy(yf(y) - xf(x))}{y\,(1 + x^2 f(x)) - x\,(1 + y^2 f(y))} \\
&= x^2 \int_y y^2 \frac{\left(1 + y^2 f(y)\right) f(y) - \left(1 + x^2 f(x)\right) f(x)}{y^2\,(1 + x^2 f(x))^2 - x^2\,(1 + y^2 f(y))^2} + g_{\mathcal{F}^{(o)}}(x : \underline{y})
\end{aligned}$$

i.e.,

$$g_{\mathcal{F}}(x : \underline{y}) = \int_y \frac{x^2 y^2 \delta_{x,y}(f + z^2 f^2)}{1 - x^2 y^2 \delta_{x,y}(2f + z^2 f^2)} + g_{\mathcal{F}^{(o)}}(x : \underline{y}),$$

where $f = f(z) = g_{\mathcal{E}_{\mathrm{nl}}}(z : \underline{y})$ and

$$g_{\mathcal{F}^{(o)}}(x : \underline{y}) = \int_y \frac{xy \delta_{x,y}(z^2 f)}{1 - x^2 y^2 \delta_{x,y}(2f + z^2 f^2)}. \tag{44}$$

This lemma can be derived from (9) immediately. □

Theorem 3.1 *The generating function $f = f(z) = g_{\mathcal{E}_{\mathrm{nl}}}(z : \underline{y})$ with vertex partition satisfies the following enumerating equation*

$$f = \int_y \frac{1 - x^2 y^2 \delta_{x,y} f}{1 - x^2 y^2 \delta_{x,y}(2f + z^2 f^2)}, \tag{45}$$

This is a modification and simplification to the result (3.13) in [5].

Proof It is clear that $g_{\mathcal{E}_{\mathrm{nl}_0}}(x : \underline{y}) = 1$. So from (9) and (13), Eq.(15) is obtained by grouping the terms. □

§4. Equations with the Numbers of Vertices and Faces

In what following we want to study the following generating function for the set $\mathcal{M}$ of some maps.

$$f_{\mathcal{M}}(x, y, z) = \sum_{M \in \mathcal{M}} x^{l(M)} y^{n(M)} z^{q(M)}, \tag{46}$$

where $l(M) = val(M)$ and $n(M)$ and $q(M)$ are the numbers of non-root vertices and inner faces of $M \in \mathcal{M}$, respectively. It is clear that we may write $l(M) = val(M) = 2m(M)$ if $M \in \mathcal{E}_{\mathrm{nl}}$ is an Eulerian map.

In fact, this section will provide a functional equation satisfied by the generating function $f = f_{\mathcal{E}_{\mathrm{nl}}}(x, y, z)$ with the valency of root-vertex, the numbers of non-root vertices and inner faces of the maps in $\mathcal{E}_{\mathrm{nl}}$, respectively, as three parameters.

Summing the results as above, we can obtain the following results.

Lemma 4.1 *For the set $\mathcal{E}_{\mathrm{in}}$, we have*

$$f_{\mathcal{E}_{\mathrm{in}}}(x, y, z) = x^2 z f, \tag{47}$$

where $f = f_{\mathcal{E}_{\mathrm{nl}}}(x, y, z)$.

Proof The Lemma is obtained directly from (10) and (16). □

In the following, the algebraic method is used again for enumerating the set of maps in $\mathcal{F}^{(e)}$.

Lemma 4.2 *For the set $\mathcal{F}^{(e)}$, we have*

$$f_{\mathcal{F}^{(e)}}(x, y, z) = x^2 y \frac{(1 + zf^*) f^* - \left(1 + x^2 zf\right) f}{\left(1 + x^2 zf\right)^2 - x^2 \left(1 + zf^*\right)^2}, \tag{48}$$

where $f = f_{\mathcal{E}_{\mathrm{nl}}}(x, y, z)$ and $f^* = f_{\mathcal{E}_{\mathrm{nl}}}(1, y, z)$.

Proof By (8),(9),(12) and (16) we have

$$\begin{aligned} f_{\mathcal{F}}(x, y, z) &= xy \sum_{i \geqslant 0} \left(\sum_{M \in \mathcal{E}_{\mathrm{in}}} \sum_{j=1}^{2m(M)-1} x^j y^{n(M)} z^{q(M)} \right)^i \sum_{M \in \mathcal{E}_{\mathrm{nl}}} \sum_{j=0}^{2m(M)} x^j y^{n(M)} z^{q(M)} \\ &= xy \sum_{i \geqslant 0} \left(\frac{x f^*_{\mathcal{E}_{\mathrm{in}}} - f_{\mathcal{E}_{\mathrm{in}}}}{1 - x} \right)^i \frac{f^* - xf}{1 - x} = yz^{-1} \sum_{i \geqslant 0} \left(xz \frac{f^* - xf}{1 - x} \right)^{i+1} \\ &= \frac{xy(f^* - xf)}{1 - x\left(1 + zf^*\right) + x^2 zf}, \end{aligned}$$

i.e.,

$$f_{\mathcal{F}}(x,y,z) = x^2 y \frac{(1+zf^*)\,f^* - \left(1+x^2zf\right) f}{\left(1+x^2zf\right)^2 - x^2\left(1+zf^*\right)^2} + f_{\mathcal{F}^{(o)}}(x,y,z),$$

where $f^*_{\mathcal{E}_{\rm in}} = f_{\mathcal{E}_{\rm in}}(1,y,z)$ and

$$f_{\mathcal{F}^{(o)}}(x,y,z) = \frac{xy\left(f^* - x^2 f\right)}{\left(1+x^2zf\right)^2 - x^2\left(1+zf^*\right)^2}. \tag{49}$$

The Lemma is obtained directly from the definition of $\mathcal{F}^{(e)}$ in (9). □

Theorem 4.1 *The generating function $f = f_{\mathcal{E}_{\rm nl}}(x,y,z)$ with the valency of root-vertex, the numbers of non-root vertices and inner faces of the maps in $\mathcal{E}_{\rm nl}$, respectively, as three parameters satisfies the following cubic equation*

$$f = 1 + x^2 y \frac{(1+zf^*)\,f^* - \left(1+x^2zf\right) f}{\left(1+x^2zf\right)^2 - x^2\left(1+zf^*\right)^2}, \tag{50}$$

where $f^* = f(1,y,z)$.

Proof From (9) we have

$$f = f_{\mathcal{E}_{{\rm nl}_0}}(x,y,z) + f_{\mathcal{F}^{(e)}}(x,y,z),$$

where $f_{\mathcal{E}_{{\rm nl}_0}}(x,y,z) = 1$. By substituting (18) into the above formula Eq(20) holds. □

§5. Equations with the Edge Number and the Root-Face Valency

In this section we study the following generating function for the set $\mathcal{M}$ of some maps.

$$f_{\mathcal{M}}(x,y,z) = \sum_{M\in\mathcal{M}} x^{l(M)} y^{s(M)} z^{p(M)}, \tag{51}$$

where $l(M) = val(M)$ and $s(M)$ and $p(M)$ are the number of edges and the valency of root-face of $M \in \mathcal{M}$, respectively. we may also write $l(M) = val(M) = 2m(M)$ if $M \in \mathcal{E}_{\rm nl}$ is an Eulerian map.

In this section we provide a functional equation satisfied by the generating function $f = f_{\mathcal{E}_{\rm nl}}(x,y,z)$ with the valency of root-vertex, the number of edges the valency of the root-face of the maps in $\mathcal{E}_{\rm nl}$, respectively, as three parameters. Write that

$$h_{\mathcal{E}_{\rm nl}}(x,y) = f_{\mathcal{E}_{\rm nl}}(x,y,1), \quad F_{\mathcal{E}_{\rm nl}}(y,z) = f_{\mathcal{E}_{\rm nl}}(1,y,z), \quad H_{\mathcal{E}_{\rm nl}}(y) = f_{\mathcal{E}_{\rm nl}}(1,y,1).$$

Lemma 5.1 *For the set $\mathcal{E}_{\rm in}$, we have*

$$f_{\mathcal{E}_{\rm in}}(x,y,z) = x^2 yzf, \tag{52}$$

where $f = f_{\mathcal{E}_{\rm nl}}(x,y,z)$.

Proof The Lemma is obtained directly from (10) and (21). □

Lemma 5.2 *For the set $\mathcal{F}^{(e)}$, we have*

$$f_{\mathcal{F}^{(e)}}(x,y,z) = \frac{x^2yz[FH_0 - (1+x^2yh)f]}{(1+x^2yh)^2 - x^2H_0^2} - \frac{(1-z)x^2y^2zHFhf}{1-x^2y^2H^2h^2}, \quad (53)$$

where $h = h_{\mathcal{E}_{\rm nl}}(x,y)$, $F = F_{\mathcal{E}_{\rm nl}}(y,z)$, $H = H_{\mathcal{E}_{\rm nl}}(y)$ and $H_0 = 1 + yH$.

Proof By (8),(9),(12) and (21) we have

$$\begin{aligned} f_{\mathcal{F}}(x,y,z) &= xyz\sum_{i\geqslant 0}\left(\sum_{M\in\mathcal{E}_{\rm in}}\sum_{j=1}^{2m(M)-1}x^jy^{s(M)}\right)^i\sum_{M\in\mathcal{E}_{\rm nl}}\sum_{j=0}^{2m(M)}x^jy^{s(M)}z^{p(M)} \\ &- \sum_{k\geqslant 1}x^ky^k(z-z^2)fh^{k-1}H^{k-1}F \\ &= xyz\sum_{i\geqslant 0}\left(\frac{xH_{\mathcal{E}_{\rm in}} - h_{\mathcal{E}_{\rm in}}}{1-x}\right)^i\frac{F-xf}{1-x} - xyz(1-z)Ff\sum_{k\geqslant 1}(xyHh)^{k-1} \\ &= \frac{xyz(F-xf)}{1-x-xH_{\mathcal{E}_{\rm in}} + h_{\mathcal{E}_{\rm in}}} - \frac{(1-z)xyzFf}{1-xyHh}, \end{aligned}$$

where $H_{\mathcal{E}_{\rm in}} = yH$,$h_{\mathcal{E}_{\rm in}} = x^2yh$, i.e.,

$$f_{\mathcal{F}}(x,y,z) = \frac{x^2yz[FH_0 - (1+x^2yh)f]}{(1+x^2yh)^2 - x^2H_0^2} - \frac{(1-z)x^2y^2zHFhf}{1-x^2y^2H^2h^2} + f_{\mathcal{F}^{(o)}}(x,y,z),$$

where $H_0 = 1 + yH$ and

$$f_{\mathcal{F}^{(o)}}(x,y,z) = \frac{xyz[(1+x^2yh)F - x^2H_0f]}{(1+x^2yh)^2 - x^2H_0^2} - \frac{(1-z)xyzFf}{1-x^2y^2H^2h^2}. \quad (54)$$

The Lemma is obtained directly from the definition of $\mathcal{F}^{(e)}$ in (9). □

Theorem 5.1 *The generating function $f = f_{\mathcal{E}_{\rm nl}}(x,y,z)$ with the valency of root-vertex, the numbers of non-root vertices and inner faces of the maps in $\mathcal{E}_{\rm nl}$, respectively, as three parameters satisfies the following cubic equation*

$$f = 1 + \frac{x^2yz[H_0F - (1+x^2yh)f]}{(1+x^2yh)^2 - x^2H_0^2} - \frac{(1-z)x^2y^2zHFhf}{1-x^2y^2H^2h^2}, \quad (55)$$

where $h = h_{\mathcal{E}_{\rm nl}}(x,y)$, $F = F_{\mathcal{E}_{\rm nl}}(y,z)$, $H = H_{\mathcal{E}_{\rm nl}}(y)$ and $H_0 = 1 + yH$.

Proof From (9) we have

$$f = f_{\mathcal{E}_{\rm nl_0}}(x,y,z) + f_{\mathcal{F}^{(e)}}(x,y,z),$$

where $f_{\mathcal{E}_{\rm nl_0}}(x,y,z) = 1$. By substituting (23) into the above formula Eq(25) holds. □

Theorem 5.2 *The generating function $h = h_{\mathcal{E}_{\rm nl}}(x,y)$ with the valency of root-vertex and the number of edges of the maps in $\mathcal{E}_{\rm nl}$, respectively, as two parameters satisfies the following cubic equation*

$$h_0^3 - h_0^2 - (y + H_0^2)x^2h_0 + x^2H_0^2 + x^4yH_0 = 0, \quad (56)$$

where $H_0 = 1 + yH_{\mathcal{E}_{\rm nl}}(y)$ *and* $h_0 = 1 + x^2yh_{\mathcal{E}_{\rm nl}}(x, y)$.

This is a modification and simplification to the result (4.11) in [5].

Proof For any map $M \in \mathcal{E}_{\rm nl}$, since the number of vertices of M is $n(M) + 1$ and the number of faces of M is $q(M) + 1$, the number $s(M)$ of edges of M is $n(M) + q(M)$ by Eulerian formula. It follows from (16) and (21) that $h = h_{\mathcal{E}_{\rm nl}}(x, y) = f_{\mathcal{E}_{\rm nl}}(x, y, y)$. So if we take $z = y$, then Eq(20) becomes Eq(26) by grouping the terms where $H = f^*_{\mathcal{E}_{\rm nl}}(y, y) = f_{\mathcal{E}_{\rm nl}}(1, y, y)$.

Of course, Eq(26) may be also derived by substituting $y_{2i} = y^i$ into Eq(15) and replacing x^2 in it with x^2y since $s(M) = \sum_{i\geqslant 0} in_{2i}(M)$, or by substituting $z = 1$ into Eq(25). □

Note that Eq(20) and Eq(26) have been solved in the forms of parametric expressions or explicit formulae in [2] and [7], respectively.

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