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A Combinatorial Decomposition of Euclidean Spaces $\mathbf{R}^n$ with Contribution to Visibility

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Abstract: The visibility of human beings only allows them to find objects in $\mathbf{R}^3$ at a time t. That is why physicists prefer to adopt the Euclidean space $\mathbf{R}^3$ being physical space of particles until last century. Recent progress shows the geometrical space of physics maybe $\mathbf{R}^n$ for $n \geq 4$, for example, $n = 10$, or 11 in *string theory.* Then *how to we visualize an object in* $\mathbf{R}^n$ *for* $n \geq 4$? This paper presents a combinatorial model, i.e., *combinatorial Euclidean spaces* established on Euclidean spaces $\mathbf{R}^3$ and prove any such Euclidean space $\mathbf{R}^n$ with $n \geq 4$ can be decomposed into such combinatorial structure. We also discuss conditions for realization $\mathbf{R}^n$ in mathematics or physical space by combinatorics and show the space $\mathbf{R}^{10}$ in string theory is a special case in such model.

Key Words: Smarandache multi-space, combinatorial Euclidean space, combinatorial fan-space, spacetime, p-brane, parallel probe, ultimate theory for the Universe.

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§1. Introduction

A Euclidean space $\mathbf{R}^n$ is the point set $\{(x_1, x_2, \cdots, x_n)|\ x_i \in \mathbf{R}, 1 \leq i \leq n\}$ for an integer $n \geq 1$. The structure of our eyes determines that one can only detect particles in an Euclidean space $\mathbf{R}^3$, which gave rise to physicists prefer $\mathbf{R}^3$ as a physical space. In fact, as showed in the references [2], [18] and [21], our visible geometry is the *spherical geometry.* This means that we can only observe parts of a phenomenon in the Universe if its topological dimensional≥ 4 ([1], [14]). It should be noted that if parallel worlds [6], [20] exist the dimensional of Universe must≥ 4. Then,

Can we establish a model for detecting behaviors of particles in $\mathbf{R}^n$ *with* $n \geq 4$*?*

This paper suggests a combinatorial model and a system for visualizing phenomenons in the space $\mathbf{R}^n$ with $n \geq 4$. For this object, we establish the decomposition of $\mathbf{R}^n$ underlying a connected graph G in Sections 2 and 3, then show how to establish visualizing system in such combinatorial model and acquire its global properties, for example, the Einstein's gravitational equations in Section 4. The final sections discusses conditions of its physical realization. Terminologies and notations not defined here are followed in [1], [3] and [4] for topology, gravitational fields and graphs.

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§2. Combinatorial Euclidean Spaces

Definition 2.1([13]) *A combinatorial system* $\mathscr{C}_G$ *is a union of mathematical systems* $(\Sigma_1;\mathcal{R}_1)$,$(\Sigma_2;\mathcal{R}_2)$, $\cdots$, $(\Sigma_m;\mathcal{R}_m)$ *for an integer* m*, i.e.,*

$$\mathscr{C}_G = (\bigcup_{i=1}^{m}\Sigma_i;\ \bigcup_{i=1}^{m}\mathcal{R}_i)$$

with an underlying connected graph structure G*, i.e., a particular Smarandache multi-space([8]), where*

$$V(G) = \{\Sigma_1, \Sigma_2, \cdots, \Sigma_m\},$$
$$E(G) = \{\ (\Sigma_i, \Sigma_j)\ |\ \Sigma_i \bigcap \Sigma_j \neq \emptyset, 1 \le i, j \le m\}.$$

Definition 2.2 *A combinatorial Euclidean space is a combinatorial system* $\mathscr{C}_G$ *of Euclidean spaces* $\mathbf{R}^{n_1}$, $\mathbf{R}^{n_2}$, $\cdots$, $\mathbf{R}^{n_m}$ *with an underlying structure* G*, denoted by* $\mathscr{E}_G(n_1, \cdots, n_m)$ *and abbreviated to* $\mathscr{E}_G(r)$ *if* $n_1 = \cdots = n_m = r$.

It should be noted that a combinatorial Euclidean space is itself a Euclidean space. This fact enables us to decomposition a Euclidean space $\mathbf{R}^n$ into Euclidean spaces $\mathbf{R}^{n_1}$, $\mathbf{R}^{n_2}$, $\cdots$, $\mathbf{R}^{n_m}$ underlying a graph G but with less dimensions, which gives rise to a packing problem on Euclidean spaces following.

Problem 2.1 *Let* $\mathbf{R}^{n_1}$, $\mathbf{R}^{n_2}$, $\cdots$, $\mathbf{R}^{n_m}$ *be Euclidean spaces. In what conditions do they consist of a combinatorial Euclidean space* $\mathscr{E}_G(n_1, \cdots, n_m)$*?*

Notice that a Euclidean space $\mathbf{R}^n$ is an n-dimensional vector space with a normal basis $\bar{\epsilon}_1 = (1, 0, \cdots, 0)$, $\bar{\epsilon}_2 = (0, 1, 0 \cdots, 0)$, $\cdots$, $\bar{\epsilon}_n = (0, \cdots, 0, 1)$, namely, it has n orthogonal orientations. So if we think any Euclidean space $\mathbf{R}^n$ is a subspace of a Euclidean space $\mathbf{R}^{n_\infty}$ with a finite but sufficiently large dimension n_∞, then two Euclidean spaces $\mathbf{R}^{n_u}$ and $\mathbf{R}^{n_v}$ have a non-empty intersection if and only if they have common orientations. Whence, we only need to determine the number of different orthogonal orientations in $\mathscr{E}_G(n_1, \cdots, n_m)$.

Denoted by $X_{v_1}, X_{v_2}, \cdots, X_{v_m}$ consist of these orthogonal orientations in $\mathbf{R}^{n_{v_1}}$, $\mathbf{R}^{n_{v_2}}$, $\cdots$, $\mathbf{R}^{n_{v_m}}$, respectively. An intersection graph $G[X_{v_1}, X_{v_2}, \cdots, X_{v_m}]$ of $X_{v_1}, X_{v_2}, \cdots, X_{v_m}$ is defined by ([5])

$$V(G[X_{v_1}, X_{v_2}, \cdots, X_{v_m}]) = \{v_1, v_2, \cdots, v_m\},$$
$$E[X_{v_1}, X_{v_2}, \cdots, X_{v_m}] = \{(v_i, v_j)|X_{v_i} \bigcap X_{v_j} \neq \emptyset, 1 \le i \neq j \le m\}.$$

By definition, we know that

$$G \ \cong\ G[X_{v_1}, X_{v_2}, \cdots, X_{v_m}],$$

which transfers the Problem 2.1 of Euclidean spaces to a combinatorial one following.

Problem 2.2 *For given integers* κ, $m \ge 2$ *and* $n_1, n_2, \cdots, n_m$*, find finite sets* $Y_1, Y_2, \cdots, Y_m$ *with their intersection graph being* G *such that* $|Y_i| = n_i, 1 \le i \le m$*, and* $|Y_1 \cup Y_2 \cup \cdots \cup Y_m| = \kappa$.

2.1 The maximum dimension of $\mathscr{E}_G(n_1, \cdots, n_m)$

First, applying the *inclusion-exclusion principle*, we get the next counting result.

Theorem 2.1 *Let* $\mathscr{E}_G(n_1, \cdots, n_m)$ *be a combinatorial Euclidean space of* $\mathbf{R}^{n_1}$, $\mathbf{R}^{n_2}$, $\cdots$, $\mathbf{R}^{n_m}$ *with an underlying structure* G*. Then*

$$\dim\mathscr{E}_G(n_1, \cdots, n_m) = \sum_{\langle v_i \in V(G) | 1 \leq i \leq s \rangle \in CL_s(G)} (-1)^{s+1} \dim(\mathbf{R}^{n_{v_1}} \bigcap \mathbf{R}^{n_{v_2}} \bigcap \cdots \bigcap \mathbf{R}^{n_{v_s}}),$$

where n_{v_i} *denotes the dimensional number of the Euclidean space in* $v_i \in V(G)$ *and* $CL_s(G)$ *consists of all complete graphs of order* s *in* G*.*

Proof By definition, $\mathbf{R}^{n_u} \cap \mathbf{R}^{n_v} \neq \emptyset$ only if there is an edge $(\mathbf{R}^{n_u}, \mathbf{R}^{n_v})$ in G. This condition can be generalized to a more general situation, i.e., $\mathbf{R}^{n_{v_1}} \cap \mathbf{R}^{n_{v_2}} \cap \cdots \cap \mathbf{R}^{n_{v_l}} \neq \emptyset$ only if $\langle v_1, v_2, \cdots, v_l \rangle_G \cong K_l$.

In fact, if $\mathbf{R}^{n_{v_1}} \cap \mathbf{R}^{n_{v_2}} \cap \cdots \cap \mathbf{R}^{n_{v_l}} \neq \emptyset$, then $\mathbf{R}^{n_{v_i}} \cap \mathbf{R}^{n_{v_j}} \neq \emptyset$, which implies that $(\mathbf{R}^{n_{v_i}}, \mathbf{R}^{n_{v_j}}) \in E(G)$ for any integers $i, j,\ 1 \leq i, j \leq l$. Therefore, $\langle v_1, v_2, \cdots, v_l \rangle_G$ is a complete graph of order l in the intersection graph G.

Now we are needed to count these orthogonal orientations in $\mathscr{E}_G(n_1, \cdots, n_m)$. In fact, the number of different orthogonal orientations is

$$\dim\mathscr{E}_G(n_1, \cdots, n_m) = \dim(\bigcup_{v \in V(G)} \mathbf{R}^{n_v})$$

by previous discussion. Applying the inclusion-exclusion principle, we find that

$$\begin{aligned}
\dim\mathscr{E}_G(n_1, \cdots, n_m) &= \dim(\bigcup_{v \in V(G)} \mathbf{R}^{n_v}) \\
&= \sum_{\{v_1, \cdots, v_s\} \subset V(G)} (-1)^{s+1} \dim(\mathbf{R}^{n_{v_1}} \bigcap \mathbf{R}^{n_{v_2}} \bigcap \cdots \bigcap \mathbf{R}^{n_{v_s}}) \\
&= \sum_{\langle v_i \in V(G) | 1 \leq i \leq s \rangle \in CL_s(G)} (-1)^{s+1} \dim(\mathbf{R}^{n_{v_1}} \bigcap \mathbf{R}^{n_{v_2}} \bigcap \cdots \bigcap \mathbf{R}^{n_{v_s}}).
\end{aligned}$$

□

Notice that $\dim(\mathbf{R}^{n_{v_1}} \cap \mathbf{R}^{n_{v_2}} \cap \cdots \cap \mathbf{R}^{n_{v_s}}) = n_{v_1}$ if $s = 1$ and $\dim(\mathbf{R}^{n_{v_1}} \cap \mathbf{R}^{n_{v_2}}) \neq 0$ only if $(\mathbf{R}^{n_{v_1}}, \mathbf{R}^{n_{v_2}}) \in E(G)$. We get a more applicable formula for calculating $\dim\mathscr{E}_G(n_1, \cdots, n_m)$ on K_3-free graphs G by Theorem 2.1.

Corollary 2.1 *If* G *is* K_3*-free, then*

$$\dim\mathscr{E}_G(n_1, \cdots, n_m) = \sum_{v \in V(G)} n_v - \sum_{(u,v) \in E(G)} \dim(\mathbf{R}^{n_u} \bigcap \mathbf{R}^{n_v}).$$

Particularly, if $G = v_1 v_2 \cdots v_m$ *a circuit for an integer* $m \geq 4$*, then*

$$\dim\mathscr{E}_G(n_1, \cdots, n_m) = \sum_{i=1}^{m} n_{v_i} - \sum_{i=1}^{m} \dim(\mathbf{R}^{n_{v_i}} \bigcap \mathbf{R}^{n_{v_{i+1}}}),$$

where each index is modulo m*.*

Now we determine the maximum dimension of combinatorial Euclidean spaces of $\mathbf{R}^{n_1}$, $\mathbf{R}^{n_2}$, $\cdots$, $\mathbf{R}^{n_m}$ with an underlying structure G.

Theorem 2.2 *Let $\mathscr{E}_G(n_{v_1},\cdots,n_{v_m})$ be a combinatorial Euclidean space of $\mathbf{R}^{n_{v_1}}$, $\mathbf{R}^{n_{v_2}}$, $\cdots$, $\mathbf{R}^{n_{v_m}}$ with an underlying graph G, $V(G)=\{v_1,v_2,\cdots,v_m\}$. Then the maximum dimension $\dim_{max}\mathscr{E}_G(n_{v_1},\cdots,n_{v_m})$ of $\mathscr{E}_G(n_{v_1},\cdots,n_{v_m})$ is*

$$\dim_{max}\mathscr{E}_G(n_{v_1},\cdots,n_{v_m})=1-m+\sum_{v\in V(G)}n_v$$

with conditions $\dim(\mathbf{R}^{n_u}\cap\mathbf{R}^{n_v})=1$ *for* $\forall(u,v)\in E(G)$.

Proof Let $X_{v_1},X_{v_2},\cdots,X_{v_m}$ consist of these orthogonal orientations in $\mathbf{R}^{n_{v_1}}$, $\mathbf{R}^{n_{v_2}}$, $\cdots$, $\mathbf{R}^{n_{v_m}}$, respectively. Notice that

$$|X_{v_i}\bigcup X_{v_j}|=|X_{v_i}|+|X_{v_j}|-|X_{v_i}\bigcap X_{v_j}|$$

for $1\leq i\neq j\leq m$ by Theorem 1.5.1 in the case of $n=2$. We immediately know that $|X_{v_i}\cup X_{v_j}|$ attains its maximum value only if $|X_{v_i}\cap X_{v_j}|$ is minimum. Since X_{v_i} and X_{v_j} are nonempty sets, we find that the minimum value of $|X_{v_i}\cap X_{v_j}|=1$ if $(v_i,v_j)\in E(G)$.

The proof is finished by the inductive principle. Not loss of generality, assume $(v_1,v_2)\in E(G)$. Then we have known that $|X_{v_1}\bigcup X_{v_2}|$ attains its maximum

$$|X_{v_1}|+|X_{v_2}|-1$$

only if $|X_{v_1}\cap X_{v_2}|=1$. Since G is connected, not loss of generality, let v_3 be adjacent with $\{v_1,v_2\}$ in G. Then by

$$|X_{v_1}\bigcup X_{v_2}\bigcup X_{v_3}|=|X_{v_1}\bigcup X_{v_2}|+|X_{v_3}|-|(X_{v_1}\bigcup X_{v_2})\bigcap X_{v_3}|,$$

we know that $|X_{v_1}\cup X_{v_2}\cup X_{v_3}|$ attains its maximum value only if $|X_{v_1}\cup X_{v_2}|$ attains its maximum and $|(X_{v_1}\cup X_{v_2})\cap X_{v_3}|=1$ for $(X_{v_1}\cup X_{v_2})\cap X_{v_3}\neq\emptyset$. Whence, $|X_{v_1}\cap X_{v_3}|=1$ or $|X_{v_2}\cap X_{v_3}|=1$, or both. In the later case, there must be $|X_{v_1}\cap X_{v_2}\cap X_{v_3}|=1$. Therefore, the maximum value of $|X_{v_1}\cup X_{v_2}\cup X_{v_3}|$ is

$$|X_{v_1}|+|X_{v_2}|+|X_{v_3}|-2.$$

Generally, we assume the maximum value of $|X_{v_1}\cup X_{v_2}\cup\cdots\cup X_{v_k}|$ to be

$$|X_{v_1}|+|X_{v_2}|+\cdots+|X_{v_k}|-k+1$$

for an integer $k\leq m$ with conditions $|X_{v_i}\cap X_{v_j}|=1$ hold if $(v_i,v_j)\in E(G)$ for $1\leq i\neq j\leq k$. By the connectedness of G, without loss of generality, we choose a vertex v_{k+1} adjacent with $\{v_1,v_2,\cdots,v_k\}$ in G and find out the maximum value of $|X_{v_1}\cup X_{v_2}\cup\cdots\cup X_{v_k}\cup X_{v_{k+1}}|$. In fact, since

$$\begin{aligned}|X_{v_1}\cup X_{v_2}\cup\cdots\cup X_{v_k}\cup X_{v_{k+1}}| &= |X_{v_1}\cup X_{v_2}\cup\cdots\cup X_{v_k}|+|X_{v_{k+1}}|\\ &- |(X_{v_1}\cup X_{v_2}\cup\cdots\cup X_{v_k})\bigcap X_{v_{k+1}}|,\end{aligned}$$

we know that $|X_{v_1}\cup X_{v_2}\cup\cdots\cup X_{v_k}\cup X_{v_{k+1}}|$ attains its maximum value only if $|X_{v_1}\cup X_{v_2}\cup\cdots\cup X_{v_k}|$ attains its maximum and $|(X_{v_1}\cup X_{v_2}\cup\cdots\cup X_{v_k})\bigcap X_{v_{k+1}}|=1$ for $(X_{v_1}\cup X_{v_2}\cup\cdots\cup X_{v_k})\cap X_{v_{k+1}}\neq\emptyset$. Whence, $|X_{v_i}\cap X_{v_{k+1}}|=1$ if $(v_i,v_{k+1})\in E(G)$. Consequently, we find that the maximum value of $|X_{v_1}\cup X_{v_2}\cup\cdots\cup X_{v_k}\cup X_{v_{k+1}}|$ is

$$|X_{v_1}|+|X_{v_2}|+\cdots+|X_{v_k}|+|X_{v_{k+1}}|-k.$$

Notice that our process searching for the maximum value of $|X_{v_1}\cup X_{v_2}\cup\cdots\cup X_{v_k}|$ does not alter the intersection graph G of $X_{v_1},X_{v_2},\cdots,X_{v_m}$. Whence, by the inductive principle we finally get the maximum dimension $\dim_{max}\mathscr{E}_G$ of $\mathscr{E}_G$, that is,

$$\dim_{max}\mathscr{E}_G(n_{v_1},\cdots,n_{v_m})=1-m+n_1+n_2+\cdots+n_m$$

with conditions $\dim(\mathbf{R}^{n_u}\cap\mathbf{R}^{n_v})=1$ for $\forall(u,v)\in E(G)$. □

2.2 The minimum dimension of $\mathscr{E}_G(n_1,\cdots,n_m)$

Determining the minimum value $\dim_{min}\mathscr{E}_G(n_1,\cdots,n_m)$ of $\mathscr{E}_G(n_1,\cdots,n_m)$ is a difficult problem in general case. But for some graph families we can determine its minimum value.

Theorem 2.3 *Let $\mathscr{E}_G(n_{v_1},n_{v_2},\cdots,n_{v_m})$ be a combinatorial Euclidean space of $\mathbf{R}^{n_{v_1}}$, $\mathbf{R}^{n_{v_2}}$, $\cdots$, $\mathbf{R}^{n_{v_m}}$ with an underlying graph G, $V(G)=\{v_1,v_2,\cdots,v_m\}$ and $\{v_1,v_2,\cdots,v_l\}$ an independent vertex set in G. Then*

$$\dim_{min}\mathscr{E}_G(n_{v_1},\cdots,n_{v_m})\geq\sum_{i=1}^{l}n_{v_i}$$

and with the equality hold if G is a complete bipartite graph $K(V_1,V_2)$ with partite sets $V_1=\{v_1,v_2,\cdots,v_l\}$, $V_2=\{v_{l+1},v_{l+2},\cdots,v_m\}$ and

$$\sum_{i=1}^{l}n_{v_i}\geq\sum_{i=l+1}^{m}n_{v_i}.$$

Proof Similarly, we use $X_{v_1},X_{v_2},\cdots,X_{v_m}$ to denote these orthogonal orientations in $\mathbf{R}^{n_{v_1}}$, $\mathbf{R}^{n_{v_2}}$, $\cdots$, $\mathbf{R}^{n_{v_m}}$, respectively. By definition, we know that

$$X_{v_i}\bigcap X_{v_j}=\emptyset,\quad 1\leq i\neq j\leq l$$

for $(v_i,v_j)\notin E(G)$. Whence, we get that

$$|\bigcup_{i=1}^{m}X_{v_i}|\geq|\bigcup_{i=1}^{l}X_{v_i}|=\sum_{i=1}^{l}n_{v_i}.$$

By the assumption,

$$\sum_{i=1}^{l}n_{v_i}\geq\sum_{i=l+1}^{m}n_{v_i},$$

we can partition $X_{v_1},X_{v_2},\cdots,X_{v_m}$ to

$$X_{v_1} = (\bigcup_{i=l+1}^{m} Y_i(v_1)) \bigcup Z(v_1),$$

$$X_{v_2} = (\bigcup_{i=l+1}^{m} Y_i(v_2)) \bigcup Z(v_2),$$

$$\cdots\cdots\cdots\cdots\cdots,$$

$$X_{v_l} = (\bigcup_{i=l+1}^{m} Y_i(v_l)) \bigcup Z(v_l)$$

such that $\sum_{k=1}^{l} |Y_i(v_k)| = |X_{v_i}|$ for any integer $i,\ l+1 \leq i \leq m$, where $Z(v_i)$ maybe an empty set for integers $i,\ 1 \leq i \leq l$. Whence, we can choose

$$X'_{v_i} = \bigcup_{k=1}^{l} Y_i(v_k)$$

to replace each X_{v_i} for any integer $i,\ 1 \leq i \leq m$. Notice that the intersection graph of $X_{v_1}, X_{v_2}, \cdots, X_{v_l}, X'_{v_{l+1}}, \cdots, X'_{v_m}$ is still the complete bipartite graph $K(V_1, V_2)$, but

$$|\bigcup_{i=1}^{m} X_{v_i}| = |\bigcup_{i=1}^{l} X_{v_i}| = \sum_{i=1}^{l} n_i.$$

Therefore, we get that

$$\dim_{min}\mathscr{E}_G(n_{v_1}, \cdots, n_{v_m}) = \sum_{i=1}^{l} n_{v_i}$$

in the case of complete bipartite graph $K(V_1, V_2)$ with partite sets $V_1 = \{v_1, v_2, \cdots, v_l\}$, $V_2 = \{v_{l+1}, v_{l+2}, \cdots, v_m\}$ and

$$\sum_{i=1}^{l} n_{v_i} \geq \sum_{i=l+1}^{m} n_{v_i}. \qquad \square$$

Although the lower bound of $\dim\mathscr{E}_G(n_{v_1}, \cdots, n_{v_m})$ in Theorem 2.3 is sharp, but it is not better if G is given in some cases. Consider a complete system of r-subsets of a set with less than $2r$ elements. We know the next conclusion if $G = K_m$.

Theorem 2.4 *For any integer $r \geq 2$, let $\mathscr{E}_{K_m}(r)$ be a combinatorial Euclidean space of $\underbrace{\mathbf{R}^r, \cdots, \mathbf{R}^r}_{m}$, and there exists an integer $s,\ 0 \leq s \leq r-1$ such that*

$$\binom{r+s-1}{r} < m \leq \binom{r+s}{r}.$$

Then

$$\dim_{min}\mathscr{E}_{K_m}(r) = r + s.$$

Proof We denote by $X_1, X_2, \cdots, X_m$ these sets consist of orthogonal orientations in m Euclidean spaces $\mathbf{R}^r$. Then each $X_i,\ 1 \leq i \leq m$, is an r-set. By assumption,

$$\binom{r+s-1}{r} < m \leq \binom{r+s}{r}$$

and $0 \leq s \leq r-1$, we know that two r-subsets of an $(r+s)$-set must have a nonempty intersection. So we can determine these m r-subsets $X_1, X_2, \cdots, X_m$ by using the complete system of r-subsets in an $(r+s)$-set, and these m r-subsets $X_1, X_2, \cdots, X_m$ can not be chosen in an $(r+s-1)$-set. Therefore, we find that $|\bigcup\limits_{i=1}^{m} X_i| = r+s$, i.e., if $0 \leq s \leq r-1$, then $\dim_{min}\mathscr{E}_{K_m}(r) = r+s$. □

For general combinatorial spaces $\mathscr{E}_{K_m}(n_{v_1}, \cdots, n_{v_m})$ of $\mathbf{R}^{n_{v_1}}$, $\mathbf{R}^{n_{v_2}}$, $\cdots$, $\mathbf{R}^{n_{v_m}}$, we get their minimum dimension if n_{v_m} is large enough.

Theorem 2.5 *Let $\mathscr{E}_{K_m}$ be a combinatorial Euclidean space of $\mathbf{R}^{n_{v_1}}$, $\mathbf{R}^{n_{v_2}}$, $\cdots$, $\mathbf{R}^{n_{v_m}}$, $n_{v_1} \geq n_{v_2} \geq \cdots \geq n_{v_m} \geq \lceil \log_2(\frac{m+1}{2^{n_{v_1}-n_{v_2}}-1}) \rceil + 1$ and $V(K_m) = \{v_1, v_2, \cdots, v_m\}$. Then*

$$\dim_{min}\mathscr{E}_{K_m}(n_{v_1}, \cdots, n_{v_m}) = n_{v_1} + \lceil \log_2(\frac{m+1}{2^{n_{v_1}-n_{v_2}-1}}) \rceil.$$

Proof Let $X_{v_1}, X_{v_2}, \cdots, X_{v_m}$ be sets consist of these orthogonal orientations in $\mathbf{R}^{n_{v_1}}$, $\mathbf{R}^{n_{v_2}}$, $\cdots$, $\mathbf{R}^{n_{v_m}}$, respectively and

$$2^{s-1} < \frac{m}{2^{k+1}-1} + 1 \leq 2^s$$

for an integer s, where $k = n_{v_1} - n_{v_2}$. Then we find that

$$\lceil \log_2(\frac{m+1}{2^{n_{v_1}-n_{v_2}-1}}) \rceil = s.$$

We construct a family $\{Y_{v_1}, Y_{v_2}, \cdots, Y_{v_m}\}$ with none being a subset of another, $|Y_{v_i}| = |X_{v_i}|$ for $1 \leq i \leq m$ and its intersection graph is still K_m, but with

$$|Y_{v_1} \bigcup Y_{v_2} \bigcup \cdots \bigcup Y_{v_m}| = n_{v_1} + s.$$

In fact, let $X_{v_1} = \{x_1, x_2, \cdots, x_{n_{v_2}}, x_{n_{v_2}+1}, \cdots, x_{n_{v_1}}\}$ and $U = \{u_1, u_2, \cdots, u_s\}$, such as those shown in Fig.2.1 for $s = 1$ and $n_{v_1} = 9$.

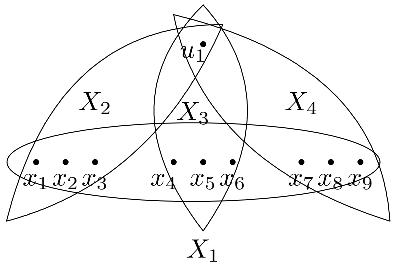

Fig.2.1

Choose g elements $x_{i_1}, x_{i_2}, \cdots, x_{i_g} \in X_{v_1}$ and $h \geq 1$ elements $u_{j_1}, u_{j_2}, \cdots, u_{j_h} \in U$. We construct a finite set

$$X_{g.h} = \{x_{i_1}, x_{i_2}, \cdots, x_{i_g}, u_{j_1}, u_{j_2}, \cdots, u_{j_h}\}$$

with a cardinal $g + h$. Let $g + h = |X_{v_1}|, |X_{v_2}|, \cdots, |X_{v_m}|$, respectively. We consequently find such sets $Y_{v_1}, Y_{v_2}, \cdots, Y_{v_m}$. Notice that there are no one set being a subset of another in the family $\{Y_{v_1}, Y_{v_2}, \cdots, Y_{v_m}\}$. So there must have two elements in each Y_{v_i}, $1 \le i \le m$ at least such that one is in U and another in $\{x_{n_{v_2}}, x_{n_{v_2}+1}, \cdots, x_{n_{v_1}}\}$. Now since $n_{v_m} \ge \lceil \log_2(\frac{m+1}{2^{n_{v_1}-n_{v_2}}-1})\rceil + 1$, there are

$$\sum_{i=1}^{k+1}\sum_{j=1}^{s}\begin{pmatrix} k+1 \\ i \end{pmatrix}\begin{pmatrix} s \\ j \end{pmatrix} = (2^{k+1}-1)(2^s-1) \ge m$$

different sets $Y_{v_1}, Y_{v_2}, \cdots, Y_{v_m}$ altogether with $|X_{v_1}| = |Y_{v_1}|, \cdots, |X_{v_m}| = |Y_{v_m}|$. None of them is a subset of another and their intersection graph is still K_m. For example,

$$X_{v_1}, \quad \{u_1, x_1, \cdots, x_{n_{v_2}-1}\},$$

$$\{u_1,\ x_{n_{v_2}-n_{v_3}+2},\ \cdots,\ x_{n_{v_2}}\},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots,$$

$$\{u_1, x_{n_{v_{k-1}}-n_{v_k}+2},\ \cdots,\ x_{n_{v_k}}\}$$

are such sets with only one element u_1 in U. See also in Fig.4.1.1 for details. It is easily to know that

$$|Y_{v_1}\bigcup Y_{v_2}\bigcup\cdots\bigcup Y_{v_m}| = n_{v_1} + s = n_{v_1} + \lceil \log_2(\frac{m+1}{2^{n_{v_1}-n_{v_2}}-1})\rceil$$

in our construction.

Conversely, if there exists a family $\{Y_{v_1}, Y_{v_2}, \cdots, Y_{v_m}\}$ such that $|X_{v_1}| = |Y_{v_1}|, \cdots, |X_{v_m}| = |Y_{v_m}|$ and

$$|Y_{v_1}\bigcup Y_{v_2}\bigcup\cdots\bigcup Y_{v_m}| < n_{v_1} + s,$$

then there at most

$$\sum_{i=1}^{k+1}\sum_{j=1}^{s}\begin{pmatrix} k+1 \\ i \end{pmatrix}\begin{pmatrix} s-1 \\ j \end{pmatrix} = (2^{k+1}-1)(2^{s-1}-1) < m$$

different sets in $\{Y_{v_1}, Y_{v_2}, \cdots, Y_{v_m}\}$ with none being a subset of another. This implies that there must exists integers $i, j, 1 \le i \ne j \le m$ with $Y_{v_i} \subset Y_{v_j}$, a contradiction. Therefore, we get the minimum dimension $\dim_{min}\mathscr{E}_{K_m}$ of $\mathscr{E}_{K_m}$ to be

$$\dim_{min}\mathscr{E}_{K_m}(n_{v_1}, \cdots, n_{v_m}) = n_{v_1} + \lceil \log_2(\frac{m+1}{2^{n_{v_1}-n_{v_2}}-1})\rceil.$$

□

As we introduce in Section 1, the combinatorial space of $\mathbf{R}^3$ is particularly interested in physics. In the case of K_m, we can determine its minimum dimension.

Theorem 2.5 *Let* $\mathscr{E}_{K_m}(3)$ *be a combinatorial Euclidean space of* $\underbrace{\mathbf{R}^3, \cdots, \mathbf{R}^3}_{m}$. *Then*

$$\dim_{min}\mathscr{E}_{K_m}(3) = \begin{cases} 3, & \text{if} \quad m = 1, \\ 4, & \text{if} \quad 2 \le m \le 4, \\ 5, & \text{if} \quad 5 \le m \le 10, \\ 2 + \lceil \sqrt{m} \rceil, & \text{if} \quad m \ge 11. \end{cases}$$

Proof Let $X_1, X_2, \cdots, X_m$ be these sets consist of orthogonal orientations in m Euclidean spaces $\mathbf{R}^3$, respectively and $|X_1 \cup X_2 \cup \cdots \cup X_m| = l$. Then each X_i, $1 \le i \le m$, is a 3-set.

In the case of $m \le 10 = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$, any s-sets have a nonempty intersection. So it is easily to check that

$$\dim_{min}\mathscr{E}_{K_m}(3) = \begin{cases} 3, & \text{if} \quad m = 1, \\ 4, & \text{if} \quad 2 \le m \le 4, \\ 5, & \text{if} \quad 5 \le m \le 10. \end{cases}$$

We only consider the case of $m \ge 11$. Let $X = \{u, v, w\}$ be a chosen 3-set. Notice that any 3-set will intersect X with 1 or 2 elements. Our discussion is divided into three cases.

Case 1 *There exist* 3*-sets* X_1', X_2', X_3' *such that* $X_1' \cap X = \{u, v\}$, $X_2' \cap X = \{u, w\}$ *and* $X_3' \cap X = \{v, w\}$ *such as those shown in Fig.*2.2*, where each triangle denotes a* 3*-set.*

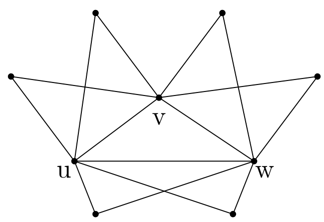

Fig.2.2

Notice that there are no 3-sets X' such that $|X' \cap X| = 1$ in this case. Otherwise, we can easily find two 3-sets with an empty intersection, a contradiction. Counting such 3-sets, we know that there are at most $3(v-3)+1$ 3-sets with their intersection graph being K_m. Thereafter, we know that

$$m \le 3(l-3)+1, \quad i.e., \quad l \ge \lceil \frac{m-1}{3} \rceil + 3.$$

Case 2 *There are* 3*-sets* X_1', X_2' *but no* 3*-set* X_3' *such that* $X_1' \cap X = \{u, v\}$, $X_2' \cap X = \{u, w\}$ *and* $X_3' \cap X = \{v, w\}$ *such as those shown in Fig.*2.3*, where each triangle denotes a* 3*-set.*

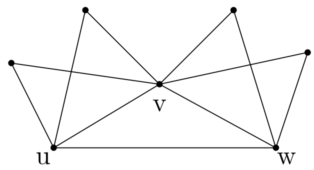

Fig.2.3

In this case, there are no 3-sets X' such that $X' \cap X = \{u\}$ or $\{w\}$. Otherwise, we can easily find two 3-sets with an empty intersection, a contradiction. Enumerating such 3-sets, we know that there are at most

$$2(l-1) + \binom{l-3}{2} + 1$$

3-sets with their intersection graph still being K_m. Whence, we get that

$$m \leq 2(l-1) + \binom{l-3}{2} + 1, \quad i.e., \quad l \geq \lceil \frac{3+\sqrt{8m+17}}{2} \rceil.$$

Case 3 *There are a* 3*-set* X_1' *but no* 3*-sets* X_2', X_3' *such that* $X_1' \cap X = \{u, v\}$, $X_2' \cap X = \{u, w\}$ *and* $X_3' \cap X = \{v, w\}$ *such as those shown in Fig.*2.4*, where each triangle denotes a* 3*-set.*

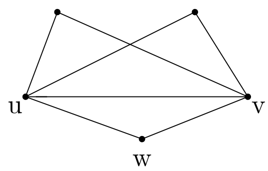

Fig.2.4

Enumerating 3-sets in this case, we know that there are at most

$$l - 2 + 2\binom{l-2}{2}$$

such 3-sets with their intersection graph still being K_m. Therefore, we find that

$$m \leq l - 2 + 2\binom{l-2}{2}, \quad i.e., \quad l \geq 2 + \lceil \sqrt{m} \rceil.$$

Combining these Cases $1-3$, we know that

$$l \geq \min\{\lceil \frac{m-1}{3} \rceil + 3, \lceil \frac{3+\sqrt{8m+17}}{2} \rceil, 2 + \lceil \sqrt{m} \rceil\} = 2 + \lceil \sqrt{m} \rceil.$$

Conversely, there 3-sets constructed in Case 3 show that there indeed exist 3-sets $X_1, X_2, \cdots, X_m$ whose intersection graph is K_m, where

$$m = l - 2 + 2\begin{pmatrix} l-2 \\ 2 \end{pmatrix}.$$

Therefore, we get that

$$\dim_{min}\mathscr{E}_{K_m}(3) = 2 + \lceil\sqrt{m}\rceil$$

if $m \geq 11$. This completes the proof. □

§3. A Combinatorial Model of Euclidean Spaces $\mathbf{R}^n$ with $n \geq 4$

A *combinatorial fan-space* $\widetilde{\mathbf{R}}(n_1, \cdots, n_m)$ is the combinatorial space $\mathscr{E}_{K_m}(n_1, \cdots, n_m)$ of $\mathbf{R}^{n_1}$, $\mathbf{R}^{n_2}$, $\cdots$, $\mathbf{R}^{n_m}$ such that for any integers $i, j,\ 1 \leq i \neq j \leq m$,

$$\mathbf{R}^{n_i}\bigcap\mathbf{R}^{n_j} = \bigcap_{k=1}^{m}\mathbf{R}^{n_k},$$

which is in fact a *p-brane* with $p = \dim \bigcap\limits_{k=1}^{m} \mathbf{R}^{n_k}$ in string theory ([15]-[17]), seeing Fig.3.1 for details.

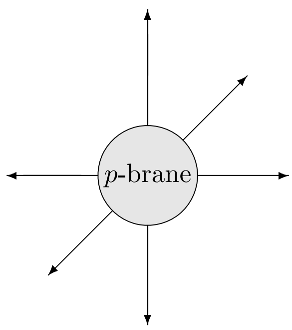

Fig.3.1

For $\forall p \in \widetilde{\mathbf{R}}(n_1, \cdots, n_m)$ we can present it by an $m \times n_m$ coordinate matrix $[\overline{x}]$ following with $x_{il} = \frac{x_l}{m}$ for $1 \leq i \leq m, 1 \leq l \leq \widehat{m}$,

$$[\overline{x}] = \begin{bmatrix} x_{11} & \cdots & x_{1\widehat{m}} & x_{1(\widehat{m})+1)} & \cdots & x_{1n_1} & \cdots & 0 \\ x_{21} & \cdots & x_{2\widehat{m}} & x_{2(\widehat{m}+1)} & \cdots & x_{2n_2} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & \\ x_{m1} & \cdots & x_{m\widehat{m}} & x_{m(\widehat{m}+1)} & \cdots & \cdots & x_{mn_m-1} & x_{mn_m} \end{bmatrix}.$$

By definition, we know the following result.

Theorem 3.1 *Let* $\widetilde{\mathbf{R}}(n_1, \cdots, n_m)$ *be a fan-space. Then*

$$\dim\widetilde{\mathbf{R}}(n_1, \cdots, n_m) = \widehat{m} + \sum_{i=1}^{m}(n_i - \widehat{m}),$$

where

$$\widehat{m} = \dim(\bigcap_{k=1}^{m} \mathbf{R}^{n_k}). \qquad \square$$

The *inner product* $\langle (A), (B) \rangle$ of (A) and (B) is defined by

$$\langle (A), (B) \rangle = \sum_{i,j} a_{ij} b_{ij}.$$

Then we know the next result by definition.

Theorem 3.2 *Let* $(A), (B), (C)$ *be* $m \times n$ *matrixes and* α *a constant. Then*

(1) $\langle \alpha A, B \rangle = \alpha \langle B, A \rangle$;

(2) $\langle A + B, C \rangle = \langle A, C \rangle + \langle B, C \rangle$;

(3) $\langle A, A \rangle \geq 0$ with equality hold if and only if $(A) = O_{m \times n}$.

Theorem 3.3 *Let* $(A), (B)$ *be* $m \times n$ *matrixes. Then*

$$\langle (A), (B) \rangle^2 \leq \langle (A), (A) \rangle \langle (B), (B) \rangle$$

and with equality hold only if $(A) = \lambda(B)$, *where* λ *is a real constant.*

Proof If $(A) = \lambda(B)$, then $\langle A, B \rangle^2 = \lambda^2 \langle B, B \rangle^2 = \langle A, A \rangle \langle B, B \rangle$. Now if there are no constant λ enabling $(A) = \lambda(B)$, then $(A) - \lambda(B) \neq O_{m \times n}$ for any real number λ. According to Theorem 3.2, we know that

$$\langle (A) - \lambda(B), (A) - \lambda(B) \rangle > 0,$$

i.e.,

$$\langle (A), (A) \rangle - 2\lambda \langle (A), (B) \rangle + \lambda^2 \langle (B), (B) \rangle > 0.$$

Therefore, we find that

$$\Delta = (-2 \langle (A), (B) \rangle)^2 - 4 \langle (A), (A) \rangle \langle (B), (B) \rangle < 0,$$

namely,

$$\langle (A), (B) \rangle^2 < \langle (A), (A) \rangle \langle (B), (B) \rangle. \qquad \square$$

Theorem 3.4 *For a given integer sequence* $n_1, n_2, \cdots, n_m, m \geq 1$ *with* $0 < n_1 < n_2 < \cdots < n_m$, $(\widetilde{\mathbf{R}}(n_1, \cdots, n_m); d)$ *is a metric space.*

Proof We only need to verify that each condition for a metric space is hold in $(\widetilde{\mathbf{R}}(n_1, \cdots, n_m); d)$. For two point $p, q \in \widetilde{\mathbf{R}}(n_1, \cdots, n_m)$, by definition we know that

$$d(p,q)=\sqrt{\langle [p]-[q],[p]-[q]\rangle}\geq 0$$

with equality hold if and only if $[p]=[q]$, namely, $p=q$ and

$$d(p,q)=\sqrt{\langle [p]-[q],[p]-[q]\rangle}=\sqrt{\langle [q]-[p],[q]-[p]\rangle}=d(q,p).$$

Now let $u\in\widetilde{\mathbf{R}}(n_1,\cdots,n_m)$. By Theorem 3.3, we then find that

$$\begin{aligned}&(d(p,u)+d(u,p))^2\\&=\langle [p]-[u],[p]-[u]\rangle+2\sqrt{\langle [p]-[u],[p]-[u]\rangle\langle [u]-[q],[u]-[q]\rangle}\\&+\langle [u]-[q],[u]-[q]\rangle\\&\geq\langle [p]-[u],[p]-[u]\rangle+2\langle [p]-[u],[u]-[q]\rangle+\langle [u]-[q],[u]-[q]\rangle\\&=\langle [p]-[q],[p]-[q]\rangle=d^2(p,q).\end{aligned}$$

Whence, $d(p,u)+d(u,p)\geq d(p,q)$ and $(\widetilde{\mathbf{R}}(n_1,\cdots,n_m);d)$ is a metric space. □

According to Theorem 3.1, a combinatorial fan-space $\widetilde{R}(n_1,n_2,\cdots,n_m)$ can be turned into a Euclidean space $\mathbf{R}^n$ with $n=\widehat{m}+\sum\limits_{i=1}^{m}(n_i-\widehat{m})$. Now the inverse question is that *for a Euclidean space* $\mathbf{R}^n$*, weather there exist a combinatorial Euclidean space* $\mathscr{E}_G(n_1,\cdots,n_m)$ *of Euclidean spaces* $\mathbf{R}^{n_1}$*,* $\mathbf{R}^{n_2}$*,* $\cdots$*,* $\mathbf{R}^{n_m}$ *such that* $\dim\mathbf{R}^{n_1}\cup\mathbf{R}^{n_2}\cup\cdots\cup\mathbf{R}^{n_m}=n$? We get the following decomposition result of Euclidean spaces.

Theorem 3.5 *Let* $\mathbf{R}^n$ *be a Euclidean space,* $n_1,n_2,\cdots,n_m$ *integers with* $\widehat{m}<n_i<n$ *for* $1\leq i\leq m$ *and the equation*

$$\widehat{m}+\sum_{i=1}^{m}(n_i-\widehat{m})=n$$

hold for an integer $\widehat{m},1\leq\widehat{m}\leq n$*. Then there is a combinatorial fan-space* $\widetilde{\mathbf{R}}(n_1,n_2,\cdots,n_m)$ *such that*

$$\mathbf{R}^n\cong\widetilde{\mathbf{R}}(n_1,n_2,\cdots,n_m).$$

Proof Not loss of generality, assume the normal basis of $\mathbf{R}^n$ is $\overline{\epsilon}_1=(1,0,\cdots,0)$, $\overline{\epsilon}_2=(0,1,0\cdots,0)$, $\cdots$, $\overline{\epsilon}_n=(0,\cdots,0,1)$. Then its coordinate system of $\mathbf{R}^n$ is $(x_1,x_2,\cdots,x_n)$. Since

$$n-\widehat{m}=\sum_{i=1}^{m}(n_i-\widehat{m}),$$

choose

$$\begin{aligned}&\mathbf{R}_1=\langle\overline{\epsilon}_1,\overline{\epsilon}_2,\cdots,\overline{\epsilon}_{\widehat{m}},\overline{\epsilon}_{\widehat{m}+1},\cdots,\overline{\epsilon}_{n_1}\rangle\,;\\&\mathbf{R}_2=\langle\overline{\epsilon}_1,\overline{\epsilon}_2,\cdots,\overline{\epsilon}_{\widehat{m}},\overline{\epsilon}_{n_1+1},\overline{\epsilon}_{n_1+2},\cdots,\overline{\epsilon}_{n_2}\rangle\,;\\&\mathbf{R}_3=\langle\overline{\epsilon}_1,\overline{\epsilon}_2,\cdots,\overline{\epsilon}_{\widehat{m}},\overline{\epsilon}_{n_2+1},\overline{\epsilon}_{n_2+2},\cdots,\overline{\epsilon}_{n_3}\rangle\,;\\&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\,;\end{aligned}$$

$$\mathbf{R}_m = \left\langle \bar{\epsilon}_1, \bar{\epsilon}_2, \cdots, \bar{\epsilon}_{\widehat{m}}, \bar{\epsilon}_{n_{m-1}+1}, \bar{\epsilon}_{n_{m-1}+2}, \cdots, \bar{\epsilon}_{n_m} \right\rangle.$$

Calculation shows that $\dim \mathbf{R}_i = n_i$ and $\dim(\bigcap_{i=1}^{m} \mathbf{R}_i) = \widehat{m}$. Whence $\widetilde{\mathbf{R}}(n_1, n_2, \cdots, n_m)$ is a combinatorial fan-space. Whence,

$$\mathbf{R}^n \cong \widetilde{\mathbf{R}}(n_1, n_2, \cdots, n_m). \qquad \square$$

Notice that a combinatorial fan-space $\widetilde{\mathbf{R}}(n_1, n_2, \cdots, n_m)$ is in fact $\mathscr{E}_{K_m}(n_1, n_2, \cdots, n_m)$. Let $n_i = 3$ for $1 \leq i \leq m$. We get a result following by Theorem 3.5.

Corollary3.1 *Let $\mathbf{R}^n$ be a Euclidean space with $n \geq 4$. Then there is a combinatorial Euclidean space $\mathscr{E}_{K_m}(3)$ such that*

$$\mathbf{R}^n \cong \mathscr{E}_{K_m}(3)$$

with $m = \frac{n-1}{2}$ or $m = n - 2$.

§4. A Particle in Euclidean Spaces $\mathbf{R}^n$ with $n \geq 4$

Corollary 3.1 asserts that an Euclidean space $\mathbf{R}^n$ can be really decomposed into 3-dimensional Euclidean spaces $\mathbf{R}^3$ underlying a complete graph K_m with $m = \frac{n-1}{2}$ or $m = n-2$. This suggests that we can visualize a particle in Euclidean space $\mathbf{R}^n$ by detecting its partially behavior in each $\mathbf{R}^3$. That is to say, we are needed to establish a *parallel probe* for Euclidean space $\mathbf{R}^n$ if $n \geq 4$.

Generally, a *parallel probe* on a combinatorial Euclidean space $\mathscr{E}_G(n_1, n_2, \cdots, n_m)$ is the set of probes established on each Euclidean space $\mathbf{R}^{n_i}$ for integers $1 \leq i \leq m$, particularly for $\mathscr{E}_G(3)$ which one can detects a particle in its each space $\mathbf{R}^3$ such as those shown in Fig.4.1 in where $G = K_4$ and there are four probes P_1, P_2, P_3, P_4.

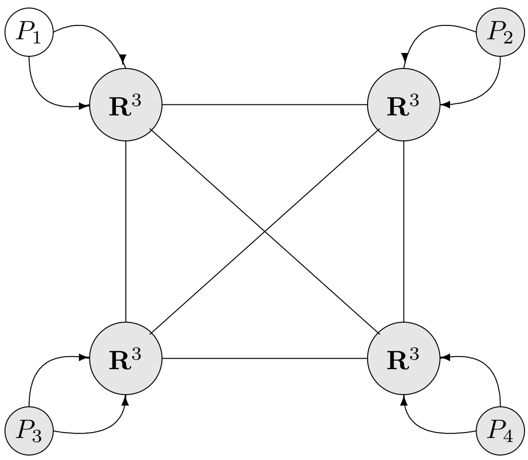

Fig.4.1

Notice that data obtained by such parallel probe is a set of local data $F(x_{i1}, x_{i2}, x_{i3})$ for $1 \leq i \leq m$ underlying G, i.e., the detecting data in a spatial $\bar{\epsilon}$ should be same if $\bar{\epsilon} \in \mathbf{R}^3_u \cap \mathbf{R}^3_v$, where $\mathbf{R}^3_u$ denotes the $\mathbf{R}^3$ at $u \in V(G)$ and $(\mathbf{R}^3_u, \mathbf{R}^3_v) \in E(G)$.

For data not in the $\mathbf{R}^3$ we lived, it is reasonable that we can conclude that all are the same as we obtained. Then we can analyze the global behavior of a particle in Euclidean space $\mathbf{R}^n$ with $n \geq 4$.

Then *how to apply this speculation*? Let us consider the gravitational field with dimensional$\geq$ 4. We know the Einstein's gravitation field equations in $\mathbf{R}^3$ are

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \kappa T_{\mu\nu},$$

where $R_{\mu\nu} = R^{\alpha}_{\mu\alpha\nu} = g^{\alpha\beta} R_{\alpha\mu\beta\nu}$, $R = g^{\mu\nu} R_{\mu\nu}$ are the respective *Ricci tensor*, *Ricci scalar curvature* and

$$\kappa = \frac{8\pi G}{c^4} = 2.08 \times 10^{-48} cm^{-1} \cdot g^{-1} \cdot s^2$$

Now for a gravitational field $\mathbf{R}^n$ with $n \geq 4$, we decompose it into dimensional 3 Euclidean spaces $\mathbf{R}^3_u$, $\mathbf{R}^3_v$, $\cdots$, $\mathbf{R}^3_w$. Then we find Einstein's gravitational equations shown in [4] as follows:

$$R_{\mu_u\nu_u} - \frac{1}{2} g_{\mu_u\nu_u} R = -8\pi G \mathscr{E}_{\mu_u\nu_u},$$

$$R_{\mu_v\nu_v} - \frac{1}{2} g_{\mu_v\nu_v} R = -8\pi G \mathscr{E}_{\mu_v\nu_v},$$

$$\cdots\cdots\cdots\cdots\cdots,$$

$$R_{\mu_w\nu_w} - \frac{1}{2} g_{\mu_w\nu_w} R = -8\pi G \mathscr{E}_{\mu_w\nu_w}$$

for each $\mathbf{R}^3_u$, $\mathbf{R}^3_v, \cdots$, $\mathbf{R}^3_w$. If we decompose $\mathbf{R}^n$ into a combinatorial Euclidean fan-space $\widetilde{R}(\underbrace{3, 3, \cdots, 3}_{m})$, then $u, v, \cdots, w$ can be abbreviated to $1, 2 \cdots, m$. In this case, these gravitational equations can be represented by

$$R_{(\mu\nu)(\sigma\tau)} - \frac{1}{2} g_{(\mu\nu)(\sigma\tau)} R = -8\pi G \mathscr{E}_{(\mu\nu)(\sigma\tau)}$$

with a coordinate matrix

$$[\overline{x}_p] = \begin{bmatrix} x^{11} & \cdots & x^{1\widehat{m}} & \cdots & x^{13} \\ x^{21} & \cdots & x^{2\widehat{m}} & \cdots & x^{23} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ x^{m1} & \cdots & x^{m\widehat{m}} & \cdots & x^{m3} \end{bmatrix}$$

for a point $p \in \mathbf{R}^n$, where $\widehat{m} = \dim(\bigcap\limits_{i=1}^{m} \mathbf{R}^{n_i})$ a constant for $\forall p \in \bigcap\limits_{i=1}^{m} \mathbf{R}^{n_i}$ and $x^{il} = \frac{x^l}{m}$ for $1 \leq i \leq m, 1 \leq l \leq \widehat{m}$. Because the local behavior is that of the projection of the global. Whence, the following principle for determining behavior of particles in $\mathbf{R}^n$, $n \geq 4$ hold.

Projective Principle *A physics law in a Euclidean space* $\mathbf{R}^n \cong \widetilde{R}(\underbrace{3,3,\cdots,3}_{m})$ *with* $n \geq 4$ *is invariant under a projection on* $\mathbf{R}^3$ *in* $\widetilde{R}(\underbrace{3,3,\cdots,3}_{m})$.

Applying this principe enables us to find a spherically symmetric solution of Einstein's gravitational equations in Euclidean space $\mathbf{R}^n$.

§5. Discussions

A simple calculation shows that the dimension of the combinatorial Euclidean fan-space $\widetilde{R}(\underbrace{3,3,\cdots,3}_{m})$ in Section 3 is

$$\dim\widetilde{R}(\underbrace{3,3,\cdots,3}_{m}) = 3m + (1-m)\widehat{m}, \qquad (4-1)$$

for example, $\dim\widetilde{R}(\underbrace{3,3,\cdots,3}_{m}) = 6,\ 9,\ 12$ if $\widehat{m} = 0$ and $5,\ 7,\ 9$ if $\widehat{m} = 1$ and $m = 2,\ 3,\ 4$ with an additional time dimension t.

We have discussed in Section 1 that the visible geometry is the spherical geometry of dimensional 3. That is why the sky looks like a spherical surface. In these geometrical elements, such as those of point, line, ray, block, body,$\cdots$, etc., we can only see the image of bodies on our spherical surface, i.e., surface blocks.

Then *what is the geometry of transferring information*? Here, the term *information* includes information known or not known by human beings. So the geometry of transferring information consists of all possible transferring routes. In other words, a combinatorial geometry of dimensional≥ 1. Therefore, not all information transferring can be seen by our eyes. But some of them can be felt by our six organs with the helps of apparatus if needed. For example, the *magnetism* or *electromagnetism* can be only detected by apparatus. Consider $\widehat{m}$ the discussion is divided into two cases, which lead to two opposite conclusions following.

Case 1. $\widehat{m} = 3$.

In this case, by the formula $(4-1)$ we get that $\dim\widetilde{R}(\underbrace{3,3,\cdots,3}_{m}) = 3$, i.e., all Euclidean spaces $\mathbf{R}_1^3, \mathbf{R}_2^3, \cdots, \mathbf{R}_m^3$ are in one $\mathbf{R}^3$, which is the most enjoyed case by human beings. If it is so, all the behavior of Universe can be realized finally by human beings, particularly, the observed interval is ds and all natural things can be come true by experiments. This also means that the discover of science will be ended, i.e., we can find an ultimate theory for the Universe - the *Theory of Everything*. This is the earnest wish of Einstein himself beginning, and then more physicists devoted all their lifetime to do so in last century.

Case 2. $\widehat{m} \leq 2$.

If the Universe is so, then $\dim\widetilde{R}(\underbrace{3,3,\cdots,3}_{m}) \geq 4$. In this case, the observed interval in the field $\mathbf{R}^3_{human}$ where human beings live is

$$ds^2_{human} = a(t,r,\theta,\phi)dt^2 - b(t,r,\theta,\phi)dr^2 - c(t,r,\theta,\phi)d\theta^2 - d(t,r,\theta,\phi)d\phi^2.$$

by Schwarzschild metrics in R^3. But we know the metric in $\widetilde{R}(\underbrace{3,3,\cdots,3}_{m})$ should be $ds_{\widetilde{R}}$. Then

how to we explain the differences $(ds_{\widetilde{R}} - ds_{human})$ *in physics?*

Notice that one can only observes the line element ds_{human}, i.e.,, a projection of $ds_{\widetilde{R}}$ on $\mathbf{R}^3_{human}$ by the projective principle. Whence, all contributions in $(ds_{\widetilde{R}} - ds_{human})$ come from the spatial direction not observable by human beings. In this case, it is difficult to determine the exact behavior and sometimes only partial information of the Universe, which means that each law on the Universe determined by human beings is an approximate result and hold with conditions.

Furthermore, if $\widehat{m} \leq 2$ holds, because there are infinite underlying connected graphs, i.e., there are infinite combinations of $\mathbf{R}^3$, one can not find an ultimate theory for the Universe, which means the discover of science for human beings will endless forever, i.e., there are no a *Theory of Everything*.

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