International J.Math. Combin. Vol.1 (2009), 50-60

Smarandachely k-Constrained labeling of Graphs

ShreedharK[1], B. Sooryanarayana[2] and RaghunathP[3]

[1]Department of Mathematics, K.V.G.College of Engineering, Karnataka, INDIA, 574 327

[2]Department of Math.& Comput., Dr.Ambedkar Institute of Technology, Karnataka, INDIA, 560 056

[3]Dept. of Master of Computer Science, Reva Institute of Technology, Karnataka , INDIA, 560 064

Email: shreedhar.k@rediffmail.com, sooryanarayan.mat@dr-ait.org, p_raghunath1@yahoo.co.in

Abstract A *Smarandachely* $k-$ *constrained labeling* of a graph $G(V,E)$ is a bijective mapping $f : V \cup E \to \{1,2,..,|V|+|E|\}$ with the additional conditions that $|f(u)-f(v)| \geq k$ whenever $uv \in E$, $|f(u)-f(uv)| \geq k$ and $|f(uv)-f(vw)| \geq k$ whenever $u \neq w$, for an integer $k \geq 2$. A graph G which admits a such labeling is called a *Smarandachely* $k-$ *constrained total graph*, abbreviated as $k-CTG$. The minimum number of isolated vertices required for a given graph G to make the resultant graph a $k-CTG$ is called the $k-$ *constrained number* of the graph G and is denoted by $t_k(G)$. Here we obtain $t_k(K_{1,n}) = n(k-2)$, for all $k \geq 3$ and $n \geq 4$ and also prove that wheels, cycles, paths, complete graphs and Cartesian product of any two non trivial graphs etc., are CTG's for some k .In addition we pose some open problems.

Key Words: Smarandachely k-constrained labeling, Smarandachely k-constrained total graph.

AMS(2000): 05C78

§1. Introduction

All the graphs considered in this paper are simple, finite and undirected. For standard terminology and notations we refer [2], [3]. There are several types of graph labelings studied by various authors. We refer [1] for the entire survey on graph labeling. Here we introduce a new labeling and call it as Smarandachely k-constrained labeling. Let $G = (V,E)$ be a graph. A bijective mapping $f : V \cup E \to \{1,2,...,|V|+|E|\}$ is called a *Smarandachely* $k-$ *constrained labeling* of G if it satisfies the following conditions for every $u,v,w \in V$:

(i) $|f(u)-f(v)| \geq k$ whenever $uv \in E$;
(ii) $|f(u)-f(uv)| \geq k$;
(iii) $|f(uv)-f(vw)| \geq k$ whenever $u \neq w$.

A graph G which admits such a labeling is called a *Smarandachely k-constrained total graph*, abbreviated as $k-CTG$. We note here that every graph G need not be a $k-CTG$ (e.g. the path P_2). However, with the addition of some isolated vertices, we can always make

[1]Received January 10, 2009. Accepted February 12, 2009.

the resultant graph a $k-CTG$. The minimum n such that the graph $G \cup \overline{K}_n$ is a $k-CTG$ is called *k-constrained number of the graph* G and denoted by $t_k(G)$, the corresponding labeling is called a *minimum k-constrained total labeling of* G. Further it follows from the definitions that if G is a $k-CTG$, then its total graph $T(G)$ is $k-chromatic$ (i.e. minimum span of $L(k,1)$ labeling of $T(G)$ is $|V(T(G))|$) and vice-versa.

If G and H are any two graphs, then $G \cup H$, $G+H$, and $G \times H$ respectively denote the Union, Sum and Cartesian product of G and H. For any real number n, $\lceil n \rceil$ and $\lfloor n \rfloor$ are respectively denote the smallest integer greater than or equal to n and the greatest integer less than or equal to n.

In this paper we obtain $t_k(K_{1,n}) = n(k-2)$, for all $k \geq 3$ and is $n(k-2)+1$ if $n=3$ or $k=2$, and also prove that wheels, cycles, paths, complete graphs and Cartesian product of any two non trivial graphs etc., are CTG's for some k. In addition we pose some open problems.

§2. Results and Open problems on 2-CTG

Observation 2.1 *Every totally disconnected graph is trivially a $k-CTG$, for all $k \geq 1$ and every graph is trivially a $1-CTG$.*

Observation 2.2 *No nontrivial connected $2-CTG$ of order less than 4, and P_4 is the smallest such connected graph.*

Observation 2.3 *If G_1 and G_2 are $k-CTG$'s, then their union is again a $k-CTG$.*

Theorem 2.4 *For a path P_n on n vertices,*
$$t_2(P_n) = \begin{cases} 2 & if \quad n=2, \\ 1 & if \quad n=3, \\ 0 & else. \end{cases}$$

Proof Let $V(P_n) = \{v_1, v_2, ..., v_n\}$ and $E(P_n) = \{v_i v_{i+1} | 1 \leq i \leq n-1\}$. Consider a total labeling $f : V \cup E \longrightarrow \{1, 2, 3, ..., 2n-1\}$ defined as $f(v_1) = 2n-3$; $f(v_2) = 2n-1$; $f(v_1 v_2) = 2$; $f(v_2 v_3) = 4$; and $f(v_k) = 2k-5$, $f(v_k v_{k+1}) = 2k$, for all $k \geq 3$. This function f serves as a Smarandachely 2-constrained labeling for P_n, for $n \geq 4$. Further, the cases $n=2$ and $n=3$ are easy to prove. □

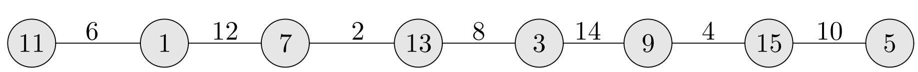

Figure 1: A 2-constrained labeling of a path P_7.

Corollary 2.5 *For every $n \geq 4$, the cycle C_n is a 2-CTG and when $n=3$, $t_2(C_n) = 2$.*

Proof If $n \geq 4$, then the result follows immediately by joining end vertices of P_n by an edge $v_1 v_n$, and, extending the total labeling f of the path as in the proof of the Theorem 2.4 above to include $f(v_1 v_2) = 2n$.

Consider the case $n = 3$. If the integers a and $a+1$ are used as labels, then one of them is assigned for a vertex and other is to the edge not incident with that vertex. But then, $a+2$

can not be used to label the vertex or an edge in C_3. Therefore, for each three consecutive integers we should leave at least one integer to label C_3. Hence the span of any Smarandachely 2-constrained labeling of C_3 should be at least 8. So $t_2(C_3) \geq 2$. Now from the Figure 3 it is clear that $t_2(C_3) \leq 2$. Thus $t_2(C_3) = 2$.

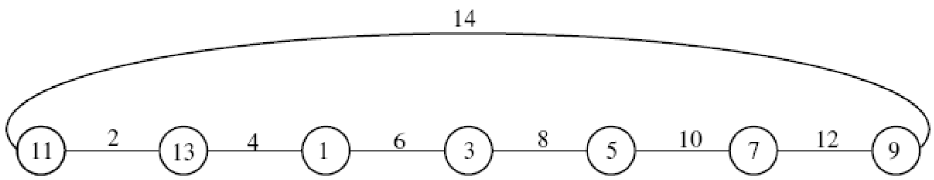

Figure 2: A 2-constrained labeling of a path C_7

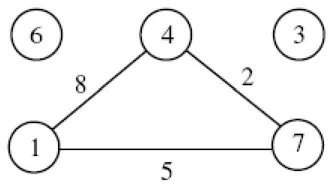

Figure 3: A 2-constrained labeling of a path $C_3 \cup \overline{K}_2$

Lemma 2.6 *For any integer $n \geq 3$, $t_2(K_{1,n}) = 1$.*

Proof Since each edge is incident with the central vertex and every other vertex is adjacent to the central vertex, no two consecutive integers can be used to label the central vertex and an edge (or a vertex) of the star. Hence $t_2(K_{1,n}) \geq 1$. Now to prove the opposite inequality, let $\acute{G} = K_{1,n} \cup K_1$, v_0 be the central vertex and $v_1, v_2, ..., v_n$ be the end vertices of the star $K_{1,n}$. Let v_{n+1} be the isolated vertex of $\acute{G}$.

We now define $f : \acute{G} \to \{1, 2, ..., 2n+2\}$ as follows:

$f(v_0) = 2n+2$; $f(v_1) = 2n-1$; $f(v_{n+1}) = 2n+1$; $f(v_k) = 2k-3$ for all $k, 2 \leq k \leq n$; $f(v_0v_i) = 2i$, for all $i, 1 \leq i \leq n$.

The function f defined above is clearly a Smarandachely 2-constrained labeling of $\acute{G}$. So $t_2(K_{1,n}) \leq 1$. Hence the result. □

Lemma 2.7 *The graph $K_{2,n}$ is a 2-CTG if and only if $n \geq 2$.*

Proof When $n = 1$ or $n = 2$ the result follows respectively from Theorem 2.4 and Corollary 2.5. For $n \geq 3$, let $H_1 = \{v_1, v_2, ..., v_n\}$ and $H_2 = \{u_1, u_2\}$ be the bipartitions of the graph $K_{2,n}$. Define a total labeling f as follows:

$f(u_1) = 2n+1$; $f(u_2) = 2n+2$; $f(v_1) = 2n-1$; $f(v_{i+1}) = 2i-1$, for all $i, 1 \leq i \leq n-1$; and for all odd j, $f(u_1v_j) = 2(n+1)+j$,$f(u_2v_j) = 2j$; and for all even j, $f(u_1v_j) = 2j$, $f(u_2v_j) = 2(n+1)+j$, $1 \leq j \leq n$. Since f assigns no two consecutive integers for the adjacent or incident pairs, it is a Smarandachely 2-constrained labeling with span $3n+2$. Hence $K_{2,n}$ is a 2-CTG. □

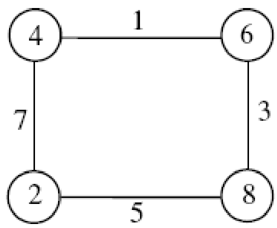

Figure 4: A 2-constrained total labeling of l

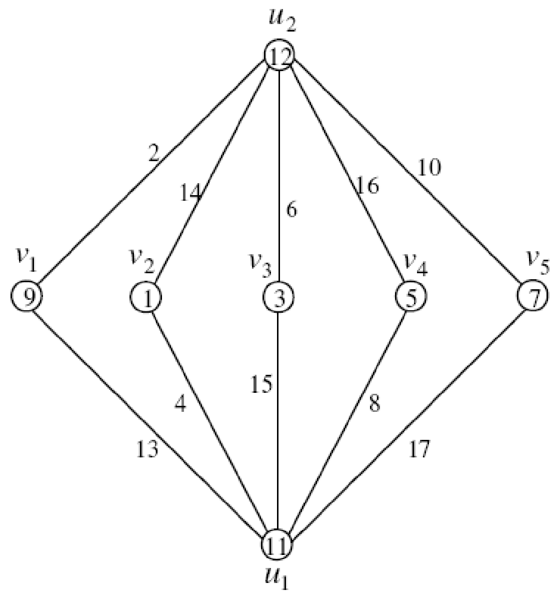

Figure 5: A 2-constrained total labeling of $K_{2,5}$

A function $f : E \rightarrow \{1, 2, ..., |E|\}$ is called a *k-constrained edge labeling of a graph $G(V, E)$* if $|f(e_1) - f(e_2)| \geq k$ whenever the edges e_1 and e_2 are adjacent in G. A graph G which admits a k-constrained edge labeling is called a *$k-$ constrained edge labeled graph* ($k - CEG$).

Lemma 2.8 *For any two positive integers $m, n \geq 3$, the complete bipartite graph $K_{m,n}$ is a 2-CEG.*

Proof Without loss of generality, we assume that $m \geq n$. Let $U = \{u_0, u_1, u_3, ..., u_{m-1}\}$ and $V = \{v_0, v_1, v_2, ..., v_{n-1}\}$ be the bipartitions of $K_{m,n}$.

Case(i): $m \not\equiv 2(modn)$

Define a function $f : E(K_{m.n}) \rightarrow \{1, 2, 3, ..., mn\}$, by

$f(u_i v_{i+k(mod\ \ n)}) = km + i + 1$, for all i and k, where $0 \leq i \leq m-1$ and $0 \leq k \leq n-1$.

The function f defined above is clearly a bijection. Further, the two distinct edges $u_i v_j$ and $u_l v_k$ are adjacent only if $i = l$ or $j = k$, but not both. So for $0 \leq j, k \leq n-1$, we have $|f(u_i v_j) - f(u_i v_k)| = |[(j-i)m + i + 1] - [(k-i)m + i + 1]| = |(j-k)m| = |(j-k)|m \geq m \geq 2$, whenever $j \neq k$. And if $j = k$, then $l \neq i$ and hence $|f(u_i v_j) - f(u_l v_j)| = |1 + i + m(i-j) - 1 - l - m(j-l)| = |(i-l) + m(j-i-j+l)| = |(i-l)(1-m)| = |m-1||l-i| \geq 2$ (since $m \geq 3$). Therefore the function f is a valid 2-constrained edge labeling.

Case(ii): $m \equiv 2(\text{mod } n)$

Relabel the vertices $v_0, v_1, v_2, ..., v_{n-1}$ in V respectively as $v_0, v_{n-1}, v_1, v_{n-2}, v_2, ..., v_{\lfloor \frac{n}{2} \rfloor}$.

Then the function f defined in the above case (i) serves again as a valid 2-constrained edge labeling. □

Theorem 2.9 *For the given positive integers m and n, with $m \geq n$*

$$t_2(K_{m,n})=\begin{cases} 2 \;\; if \;\; n=1 \;\; and \;\; m=1, \\ 1 \;\; if \;\; n=1 \;\; and \;\; m \geq 2, \\ 0 \;\; else. \end{cases}$$

Proof For $n = 1$ and $m = 1$ or 2, the result follows from Theorem 2.4. And the case $n = 1$ and $m \geq 3$ follows from Lemma 2.6. We now take the case $n > 1$. When $n = 2$, $m \geq 2$, the result follows by Lemma 2.7. If $m, n \geq 3$, then by Lemma 2.8, there exists a 2-constrained edge labeling $f : E(K_{m,n}) \to \{1, 2, ..., mn\}$. Let $U = \{u_0, u_1, ..., u_{m-1}\}$ and $V = \{v_0, v_1, v_2, ..., v_{n-1}\}$ be the bipartitions of $K_{m,n}$. We now consider a function $g : V(K_{m,n}) \cup E(K_{m,n}) \to \{1, 2, 3, ..., m + n + mn\}$, defined as follows:

$g(u_i) = i + 1$,
$g(v_j) = mn + m + j + 1$, and
$g(u_iv_j) = f(u_iv_j) + m$,

for all i, j such that $0 \leq i \leq m - 1, \;\; 0 \leq j \leq n - 1$.

The function g so defined is a Smarandachely 2-constrained labeling of $K_{m,n}$ for $m, n \geq 3$. Hence the result. □

Theorem 2.10 *If G_1 and G_2 are any two nontrivial connected graphs which are 2-CTG's, then $G_1 + G_2$ is a 2-CTG.*

Proof Let $G_1(V_1, E_1)$ be a graph of order m and size q_1 and $G_2(V_2, E_2)$ be a graph of order n and size q_2 . Let $u_0, u_1, ..., u_{m-1}$ be the vertices of G_1 and $v_0, v_1, v_2, ..., v_{n-1}$ be the vertices of G_2 . Since G_1 and G_2 are 2-CTG's, there exist Smarandachely 2-constrained labelings, $f_1 : V(G_1) \cup E(G_1) \to \{1, 2, 3, ..., m + q_1\}$, and $f_2 : V(G_2) \cup E(G_2) \to \{1, 2, 3, ..., n + q_2\}$ for G_1 and G_2 respectively.

Let $G = G_1 + G_2$ and G^* be the graph obtained from G by deleting all the edges of G_1 as well as G_2. Then G^* is a complete bipartite graph $K_{m,n}$ and $G = G_1 \cup G_2 \cup G^*$. Since both the graphs G_1 and G_2 are 2-CTG's, we have both m and n are at least 4, and hence by Lemma 2.8, there exists a 2-constrained edge labeling $g : E(G^*) \to \{1, 2, ..., mn\}$ for G^*. Since G_1 is Smarandachely 2-constrained total graph, the maximum label assigned to a vertex or edge is $m + q_1$. Let u_i be the vertex of G_1 such that $m + q_1$ is assigned for the vertex u_i or to an edge incident with the vertex u_i in G_1 by the function f_1. If g is not assigned 1 for the edge incident with u_i of G^* , then just super impose the vertex u_i of G_1 with the vertex u_i of G^* for all $i, 0 \leq i \leq m - 1$. Else if g is assigned 1 for an edge incident with u_i then re-label the vertex u_i of G^* as $u_{i+1(mod\ m)}$ for every $i, 0 \leq i \leq m - 1$, before the superimposition. Repeat the process of superimposition of the vertex v_i of G^* with the corresponding vertex v_i of G_2 in the similar manner depending on whether the largest assignment of g to an edge of G^* adjacent to the smallest assignment 1 of G_2 assigned by the function f_2 or not. Now extend these functions to the function $f : VG \cup E(G) \to \{1, 2, 3, ..., m+n+q_1+q_2+mn\}$, by defining it as follows:

$$f(x)=\begin{cases} f_1(x), & if \quad x \in V(G_1) \cup E(G_1), \\ f_2(x)+m(n+q_1), & if \quad x \in V(G_2) \cup E(G_2), \\ g(x)+m+q_1 & if \quad x = u_iv_j \quad for \quad all \quad i,j, \quad 0 \le i \le m-1, \quad 0 \le j \le n-1. \end{cases}$$

The function f defined above serves as a Smarandachely 2-constrained labeling. □

Corollary 2.11 *For every integer $n \geq 4$, the complete graph K_n is a 2-CTG.*

Proof Follows from the following four Figures 6 to 9 and by Theorem 2.10 (since every other complete graph is a successive sum of two or more of these graphs). □

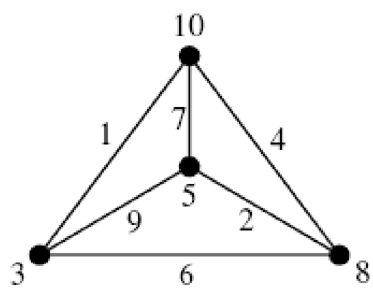

Figure 6: A 2-constrained labeling of K_4

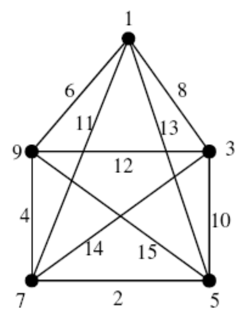

Figure 7: A 2-constrained labeling of K_5

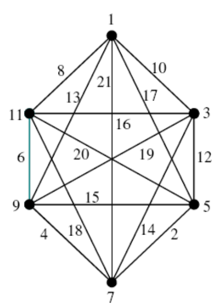

Figure 8: A 2-constrained labeling of K_6

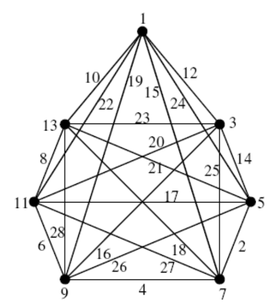

Figure 9: A 2-constrained labeling of K_7

Theorem 2.12 *For any integer $n \geq 3$, the wheel $W_{1,n}$ is a 2-CTG.*

Proof Let v_0 be the central vertex and $v_1, ..., v_n$ be the rim vertices of $W_{1,n}$. Define a total labeling f on $W_{1,n}$ as; (i)$f(v_0) = 3n+1$; (ii) For all $i, 1 \leq i \leq n$, $f(v_i) = 2i - 3(mod\ 2n)$; (iii) $f(v_0v_i) = 2i$; and (iv) For all $l, 0 \leq l \leq n$, $f(v_{1+lk(mod\ n)}v_{2+lk(mod\ n)}) = 2n+l+1$, where k is any integer such that $2 \leq k < n-1$ and $gcd(n,k) = 1$. The existence of such k for a given integer n is obvious for all n except $n = 3, 4$ and 6. For $n = 3$, the result follows by Corollary **??**. The required labeling for the special cases $n = 4$ and $n = 6$ are shown in Figures 10 and 11 below. □

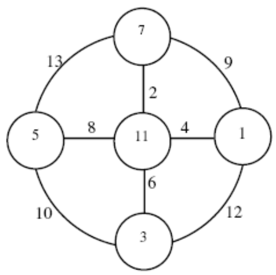

Figure 10: A 2-constrained labeling of $W_{1,4}$

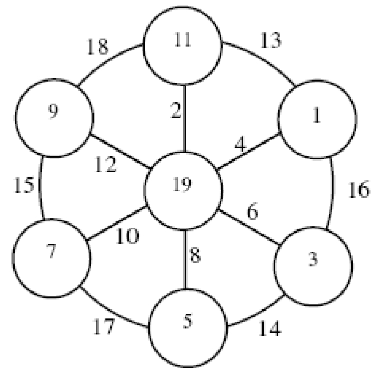

Figure 11:A 2-constrained labeling of $W_{1,6}$

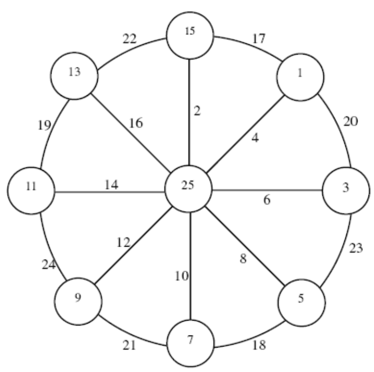

Figure 12:A 2-constrained labeling of $W_{1,8}$

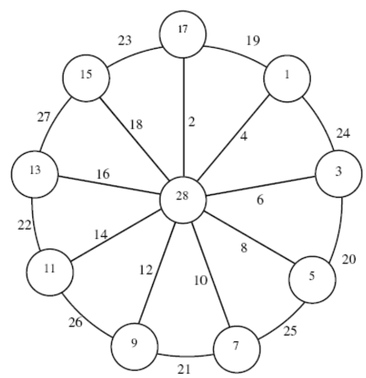

Figure 13: A 2-constrained labeling of $W_{1,9}$

We end up this section with the following open problem.

Problem 2.13 *Determine the graph of order at least 4 which is not a 2-CTG?*

§3. Results on k-CTG

We now prove the results of previous sections for general cases and give some open problems.

Observation 3.1 *G is a k-CTG $\Rightarrow$ G is a $(k-1)$-CTG.*

Lemma 3.2 *If the path P_n on n vertices is a k-CTG for some $k \geq 2$, then $k \leq \frac{2n-3}{2}$.*

Proof The result is obvious for the case $n \leq 4$. In fact, if $n \leq 4$, $2n-3 \leq 5 \Rightarrow k = 1$ or 2, so the result follows by Theorem 2.4. Now assume that $n \geq 5$. Let f be any Smarandachely k-constrained labeling of the path P_n. Then the span of f is $2n-1$. Further f assigns the integer 1 to a vertex or an edge.

Case (i) $f(v_i) = 1$, for some $i, 1 \leq i \leq n$.

Subcase (i) $i \neq 1$ (or $i \neq n$)

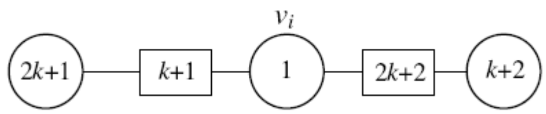

Figure 14: A minimum possible assignment for three consecutive vertices of a path.

The minimum assignment for the neighboring vertices of v_i is shown in the Figure 14. Since span of f is $2n-1$, we get $2k+2 \leq 2n-1$. Hence the result is true in this case.

Subcase (ii) $i = 1$ (or $i = n$)

In this case for the internal (other than the end vertex) vertex v_j, $f(v_j) \geq 2$, and hence for the minimum assignment for the neighboring vertices as well as the incident edges we get(again referring the same Figure 14 with label 1 as $f(v_j)$) $2k+f(v_j)+1 \leq 2n-1 \Rightarrow 2k \leq 2n-2-f(v_j) < 2n-3$.

Case (ii) $f(v_iv_{i+1}) = 1$, for $i, 1 \leq i \leq n-1$.

Result follows immediately by the Figure 14 treating rectangular boxes as vertices and circles as edges. □

The following theorem extends Theorem 2.9 up to certain k.

Theorem 3.3 *The path P_n on n vertices is a k-CTG whenever $2 \leq k \leq n - \lceil \frac{(n+1)}{3} \rceil$.*

Proof In view of observation 3.1, it suffices to define a total labeling f for $k = n - \lceil \frac{(n+1)}{3} \rceil$. Let us first denote the vertices and edges of the path simultaneously by the integers $1, 2, 3, ..., 2n-1$ as $v_1 = 1, v_1v_2 = 2, v_2 = 3, v_2v_3 = 4, v_3 = 5, ..., v_i = 2i-1, v_iv_{i+1} = 2i, v_{i+1} = 2i+1, ..., v_{n-1}v_n = 2(n-1), v_n = 2n-1$. Define an automorphism on $Z_{2n}/\{0\}$ as $f(1) = n+1+\lfloor \frac{(n-2)}{3} \rfloor$,$f(2) = n+1-\lceil \frac{(n+1)}{3} \rceil$, $f(3) = 1$ and for all $i, 4 \leq i \leq 2n-1$, $f(i) = f(i-3)+1$. The function f defined above is a Smarandachely $(n - \lceil \frac{(n+1)}{3} \rceil)$-constrained labeling for P_n. □

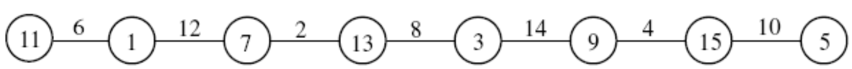

Figure 15: A 5-constrained total labeling of the path P_8.

Problem 3.4 *For any integers $n, k \geq 3$, determine the value of $t_k(P_n)$.*

Corollary 3.5 *The cycle C_n on n vertices are k-CTG's for every $2 \leq k \leq n - \lceil \frac{(n+1)}{3} \rceil$.*

Proof Let $v_0, v_1, \cdots, v_{n-1}$ be the vertices of C_n such that $v_iv_{i\oplus_n 1} \in V(C_n)$. Now for each $i, 0 \leq i \leq n-1$, denote the vertices and edges of C_n consecutively as $v_0 = 0, v_0v_1 = 1, v_1 =$

$2, v_1v_2 = 3, v_2 = 4, \cdots, v_{i-1} = 2(i-1), v_{i-1}v_i = 2i-1, v_i = 2i, \cdots, v_{n-2}v_{n-1} = 2n-3, v_{n-1} = 2n-2, v_{n-1}v_0 = 2n-1$. We now define a function f as follows:

Case (i) $3 \nmid (n-2)$.

Define: $f(0) = 1, f(1) = n+1-\lfloor \frac{n}{3} \rfloor, f(2) = n+1+\lceil \frac{n}{3} \rceil, f(i) = f(i-3)+1$, for all $i, 3 \leq i \leq 2n-1$. The function f is a Smarandachely $(n - \lceil \frac{(n+1)}{3} \rceil)$-constrained labeling of C_n.

Case (ii) $3 \mid (n-2)$.

Define: $f(0) = 1, f(1) = n+1+\lceil \frac{n}{3} \rceil, f(2) = n+1-\lfloor \frac{n}{3} \rfloor, f(i) = f(i-3)+1$, for all $i, 3 \leq i \leq 2n-1$. The function f is again a Smarandachely $(n - \lceil \frac{(n+1)}{3} \rceil)$-constrained labeling of C_n. □

Problem 3.6 *For any integers $n, k \geq 3$, determine the value of $t_k(C_n)$.*

Observation 3.7 *We are not sure about the range of k, that is, k may exceed $(n - \lceil \frac{(n+1)}{3} \rceil)$ for some path or cycle on n vertices. However achieving the maximum value of k may be tedious for a general graph (even for a path itself).*

Problem 3.8 *For a given integer $k \geq 2$, determine the bounds for a graph G to be a k-CTG.*

Problem 3.9 *For given positive integers m, n and k, does there exist a connected graph G with n vertices such that $t_k(G) = m$?*

Following theorem is a partial answer to the above Problem 3.9, which is also an extension of Lemma 2.6.

Theorem 3.10 *If $k \geq 3$ is any integer and $n \geq 3$, then,*

$$t_k(K_{1,n}) = \begin{cases} 3k-5, & if \quad n = 3, \\ n(k-2), & otherwise. \end{cases}$$

Proof For any Smarandachely k-constrained labeling f of a star $K_{1,n}$, the span of f, after labeling an edge by the least positive integer a is at least $a + nk$. Further, the span is minimum only if $a = 1$. Thus, as there are only $n+1$ vertices and n edges, for any minimum total labeling we require at least $1 + nk - (2n+1) = n(k-2)$ isolated vertices if $n \geq 4$ and at least $1 + nk - 2n = n(k-2) + 1$ if $n = 3$. In fact, for the case $n = 3$, as the central vertex is incident with each edge and edges are mutually adjacent, by a minimum k-constrained total labeling, the edges as well the central vertex can be labeled only by the set $\{1, 1+k, 1+2k, 1+3k\}$. Suppose the label 1 is assigned for the central vertex, then to label the end vertex adjacent to edge labeled $1+2k$ is at least $(1+3k)+1$ (since it is adjacent to 1, it can not be less than $1+k$). Thus at most two vertices can only be labeled by the integers between 1 and $1+3k$. Similar argument holds for the other cases also.

Therefore, $t(K_{1,n}) \geq n(k-2)$ for $n \geq 4$ and $t(K_{1,n}) \geq n(k-2)+1$ for $n = 3$.

To prove the reverse inequality, we define a k-constrained total labeling for all $k \geq 3$, as follows:

(1) When $n = 3$, the labeling is shown in the Figure 16 below

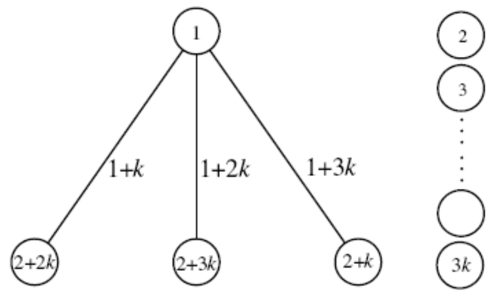

Figure 16: A k-constrained total labeling of $K_{1,3} \cup \overline{K}_{3k-5}$.

(2) When $n \geq 4$, define a total labeling f as $f(v_0v_j) = 1 + (j-1)k$ for all $j, 1 \leq j \leq n$. $f(v_0) = 1 + nk$, $f(v_1) = 2 + (n-2)k$, $f(v_2) = 3 + (n-2)k$,and for $3 \leq i \leq (n-1)$,

$$f(v_{i+1}) = \begin{cases} f(v_i) + 2, & if \quad f(v_i) \equiv 0 (mod \quad k), \\ f(v_i) + 1, & otherwise. \end{cases}$$

and the rest all unassigned integers between 1 and $1 + nk$ to the $n(k-2)$ isolated vertices, where v_0 is the central vertex and $v_1, v_2, v_3, ..., v_n$ are the end vertices.

The function so defined is a Smarandachely k-constrained labeling of $K_{1,n} \cup \bar{K}_{n(k-2)}$, for all $n \geq 4$. □

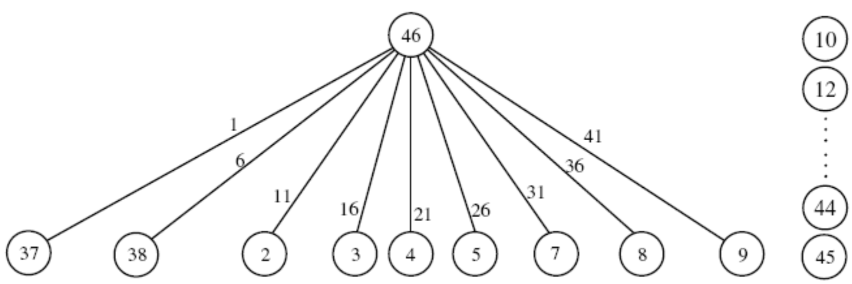

Figure 17: A 5-constrained total labeling of $K_{1,9} \cup \overline{K}_{27}$.

Theorem 3.11 *Let G_1 and G_2 be any two connected non-trivial graphs of order m and n respectively. Then their Cartesian product graph $G_1 \times G_2$ is a k-CTG for every $k \leq min\{m, n\}$.*

Proof Let $u_1, u_2, ..., u_m$ be the vertices of G_1 and $v_1, v_2, ..., v_n$ be the vertices of G_2 . Let $G = G_1 \times G_2$. Define a total labeling f on G as follows:

If $u_iu_j \in E(G_1)$, then label the corresponding edge $\{(u_i, v_1), (u_j, v_1)\}$ in G by the integer 1, the edge $\{(u_i, v_2), (u_j, v_2)\}$ by the integer 2, . . . so on, the edge $\{(u_i, v_l), (u_j, v_l)\}$ by the integer l, for all $l, 1 \leq l \leq n$. Label the vertex (u_i, v_l) by $n + l$ and the vertex (u_j, v_l) by $2n + l$ for all $l, 1 \leq l \leq n$. Next choose the new edge (if it exists) incident with either u_i or u_j, label the corresponding edges to this edge in $G_1 \times G_2$ by next n integers respectively as above and then continue the labeling for the corresponding unlabeled end vertices of these edges (if they exist). Repeat the process until all the edges as well as the vertices of each copy of G_1 in $G_1 \times G_2$ is labeled.

Since G_2 is connected, for each $s, 1 \le s \le m$, there exists an edge $\{(u_s, v_1), (u_s, v_i)\}$, for some $i, 1 \le i \le n$. Label the edge $\{(u_1, v_1), (u_1, v_i)\}$ by $n(m + q_1) + 1$ and then the parallel edges $\{(u_s, v_1), (u_s, v_i)\}$ by $n(m+q_1)+s$, for each $s, 2 \le s \le m$. Repeat the process of labeling by the next integers for each possible i, then repeat for next s. Continue this process for the possible edges $\{(u_s, v_2), (u_s, v_i)\}$, $2 \le i \le n$, then to $\{(u_s, v_3), (u_s, v_i)\}$, $3 \le i \le n$, . . . so on $\{(u_s, v_{n-1}), (u_s, v_n)\}$ (if no such edge exists at any stage then skip that step). Since the difference between two adjacent edges (as well as adjacent vertices and incident pairs) is at least $min\{m, n\}$, f is a Smarandachely $Min\{m, n\}$-constrained labeling of G. □

The illustration of the proof of the theorem is shown in the following figure.

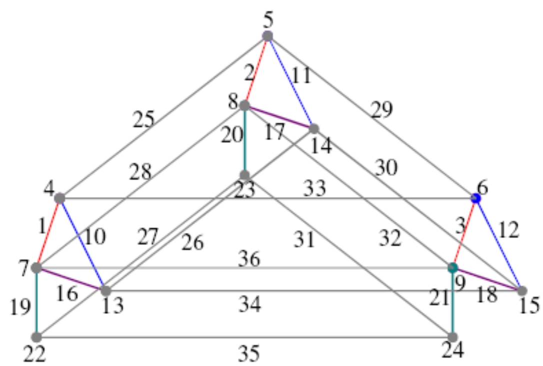

Figure 18: A 3-constrained total labeling of Cartesian product of graphs.

Problem 3.12 *Determine* $t_k(K_{m,n})$*, for any integer* $k \ge 3$.

Problem 3.13 *For any integer* $n \ge 4$*, determine* $t_k(K_n)$.

Problem 3.14 *Determine* $t_k(W_{1,n})$*, for any integer* $k \ge 3$.

Acknowledgment

We are very much thankful to the Principals, Prof. Jnanesh N.A., K.V.G. College of Engineering, Prof. Martin Jebaraj, Dr. Ambedkar Institute of Technology and Prof. Rana Prathapa Reddy, Reva Institute of Technology for their constant support and encouragement during the preparation of this paper.

References

[1] J. A. Gallian, A dynamic survey of graph labeling, *The Electronic Journal of Combinatorics*, # DS6,16(2009),1-219.

[2] Buckley F and Harary F, *Distance in Graphs*, Addison-Wesley, 1990.

[3] Hartsfield Gerhard and Ringel, *Pearls in Graph Theory*, Academic Press, USA, 1994.