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Retraction Effect on Some Geometric Properties of Geometric Figures

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Abstract: In this paper, we introduce the effect of some types of retractions on some geometric properties of some geometric figures, which makes the geometric figure that is not manifold to be a manifold. The limit of retractions of some geometric figures is deduced and the types of retractions which fail to change the geometric figure to be a manifold are discussed. Theorems governing these types of retractions are deduced.

Key Words: Manifolds, geometric figures, retraction.

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§1. Introduction

The folding of a manifold into another manifold or into itself are presented by EL-Ghoul [4, 7-9], EL-Kholy [12], El-Ahmady [1,2] and in [14], the deformation retract and the topological folding of a manifold are introduced in [1,4,6,7], the retraction of the manifolds are introduced in [5,8,10]. In this paper we have presented the effect of retraction on some geometric properties of some geometric figures, which makes some geometric figures which is not manifolds to be manifolds, also the limit of these retractions is discussed, the types of retractions, which fail to make the non-manifold to be a manifold will be presented, the end of limits of retractions of any geometric figure of dimension n is presented, we introduce a type of retraction ,which makes the non-simple closed curve in R^3 to be a knot, the effect of retraction on some geometric properties of some geometric figures as dimension is discussed, the theorems governing these types of retractions are presented.

§2. Definitions and background

1. Let M and N be two smooth manifolds of dimensions m and n respectively. A map $f : M \to N$ is said to be an isometric folding of M into N if and only if for every piecewise geodesic path $\gamma : I \to M$ the induced path $f \circ \gamma : I \to N$ is piecewise geodesic and of the same length as γ. If f does not preserve the length, it is called topological folding [14].
2. A subset A of a topological space X is called a retract of X, if there exists a continuous map

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$r : X \to A$ (called retraction) such that $r(a) = a \forall a \in A$ [5,13] .
3. An n-dimensional manifold is a Hausdorff topological space, such that each point has an open neighborhood homeomorphic to an open n-dimension disc [13, 15] .
4. A knot is a subset of 3-space that is homeomorphic to the unit circle in R^3 [16].

§3. The main results

Aiming to our study, we will introduce the following:

our goal is to study the effect of some retractions on the geometric properties of some geometric figures, which are not manifolds as some non simple closed curves and we introduce some types of retractions which makes the geometric figure, which is not manifold to be a manifold and the types of retractions which fail to change the geometric figure to be a manifold.

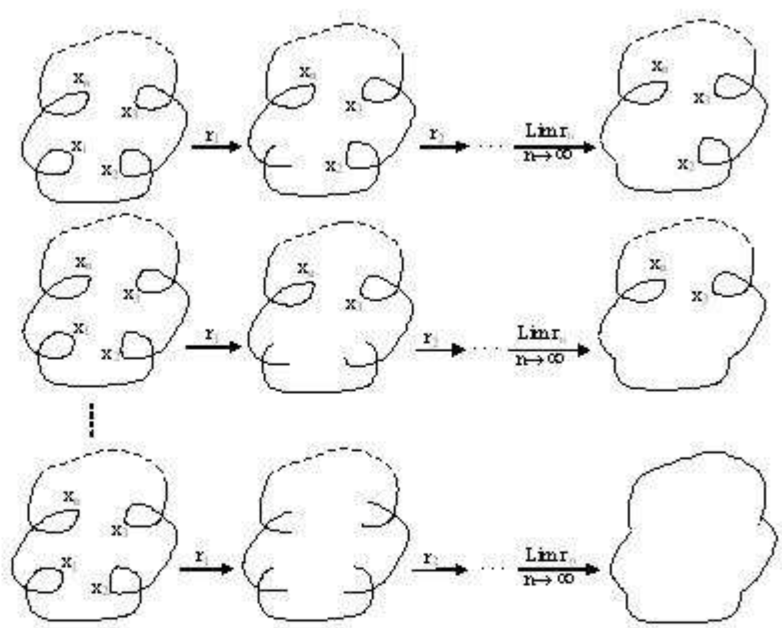

Fig.1

Proposition 3.1 *There is a type of retraction which makes the non-simple closed curve , which is not manifold to be a manifold.*

Proof Let $r : X - \{x_i\} \to X_1$, be a retraction map of X−$\{x_i\}$ into X_1, where X is a non-simple closed curve self-intersection at n-points , since X be a non-simple closed curve self-intersection at n-points, and the neighborhoods of the intersection points different from the neighborhoods of the other points of the curve X, then X is not manifold, let x_i, $i = 1, 2, \cdots, m$ are any points on the loops of the intersection points of X respectively, when the number of the points m is less than the number of the intersection points i.e. $m <$n, then the limit of the retractions of X is not a manifold, when the number of the points m is equal to the number of the intersection points , i.e. $m = n$, then the limit of retractions of X is a simple closed curve, which is a manifold and when m¿n, then the limit of the retractions of X is a 0-manifold, see Fig.1. □

Proposition 3.2 *There is a type of retraction which makes the non-simple closed curve to be a disjoint union of points which is a manifold.*

Proof Let $r : X \setminus \{x_i\} \to X_2$, be a retraction map of $X \setminus \{x_i\}$ into X_2, where X is a non-simple closed curve self-intersection at n-points, when the number of points x_i, $i = 1, 2, \cdots, m$ is less than the number of intersection points n i.e., $m < n$, then the limit of retractions of X is not a manifold, when the number of the points m is equal the number of the intersection points n then the limit of retractions of X is a disjoint union of points, which is a manifold and when X lies in R^3, we have the same results, see Fig.2. □

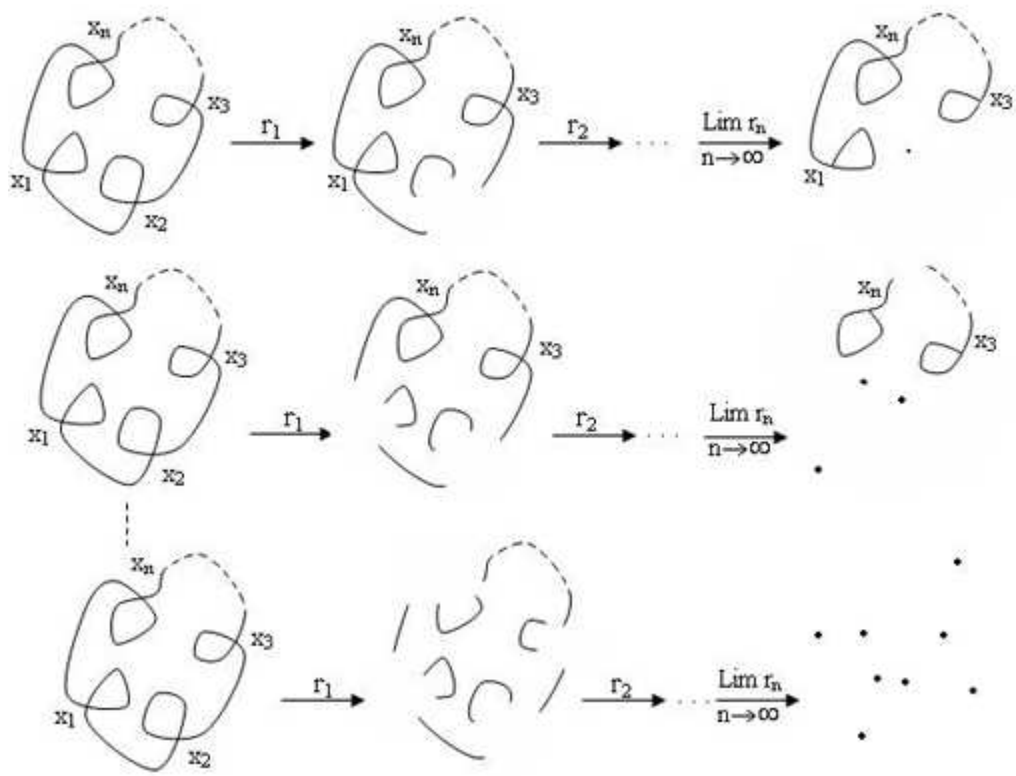

Fig.2

Proposition 3.3 *Let $r : X \setminus \{p_i\} \to X_3, i = 1, 2, ..., m$ be a retraction map, where X is a non-simple closed curve, p_i are the points on X,which lie between any two consecutive inter-section points of X respectively, then the limit of retractions of X is a manifold.*

Proof Let $r : X \setminus \{p_i\} \to X_3, i = 1, 2, ..., m$ be a retraction map of $X \setminus \{p_i\}$ into X_3, where X is a non-simple closed curve self-intersection at n-points and p_1, p_2, p_3,$\cdots$, p_m are the points on X, which lie between any two consecutive intersection points of X respectively, when the number of points m is less than the number of intersection points m i.e., $m < n$, then the limit of retractions of X is not a manifold, when the number of the points m is equal to the number of points n i.e. $m = n$, then the limit of retractions of X is a disjoint union of loops which is a manifold see Fig.3. □

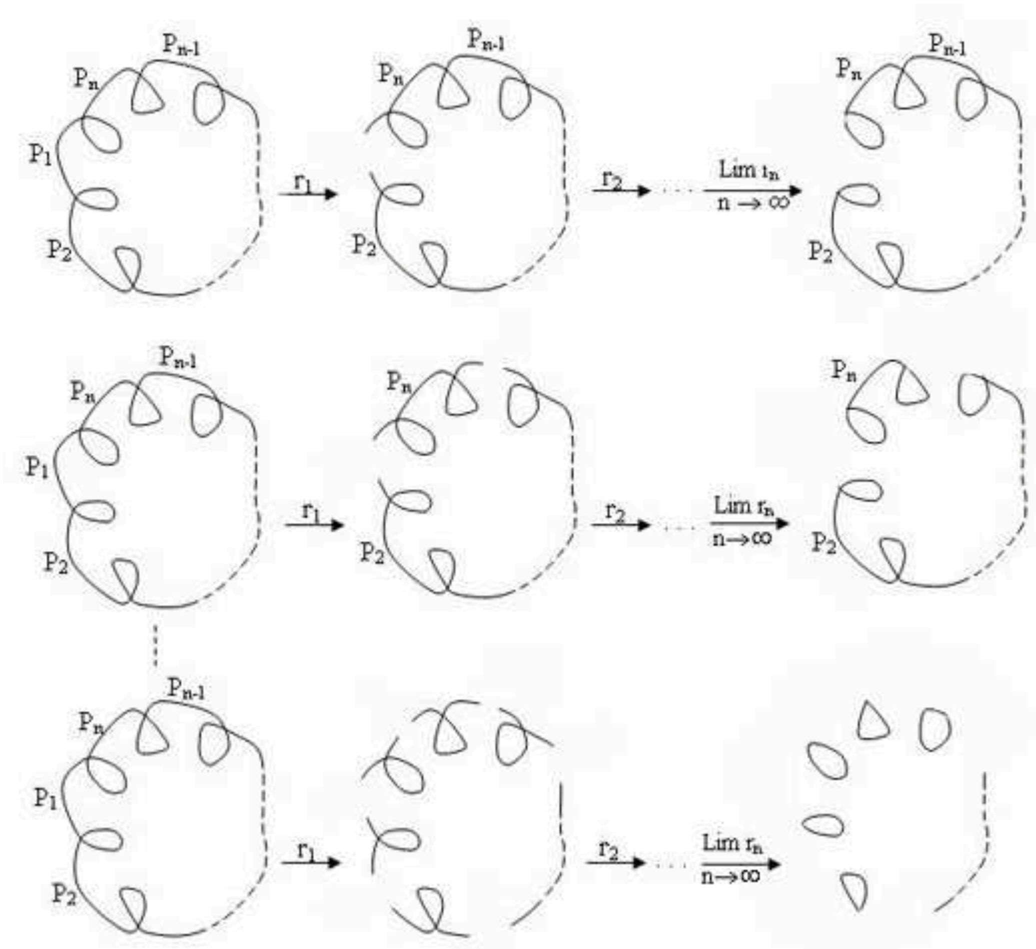

Fig.3

Proposition 3.4 *If $r : X \setminus \{p_i, k_i\} \to X^*$ be a retraction map of X where X is a non-simple closed curve, p_i, k_i are defined as any two points of each loop of the loops of X, then the limit of retraction of X is not a manifold.*

Proof Let $r : X \setminus \{p_i, k_i\} \to X^*$, be a retraction map of $X \setminus \{p_i, k_i\}$ into X^*, where X is a non-simple closed curve self-intersection at n-points of the curve X, let p_i and k_i, $i = 1, 2, \cdots, m$ are the points of each loop of the loops of the curve X i.e., the retraction by removing two points p_i and k_i from each loop respectively ,when the number of points $\{p_i, k_i\}$ is less than n i.e., $m < n$, then the limit of retractions of X is not a manifold, when the number of points $\{p_i, k_i\}$ is equal to the number of points n i.e., $m = n$, then the limit of retractions of X is not a manifold and when X lies in R^3, we have the same results, see Fig.4. □

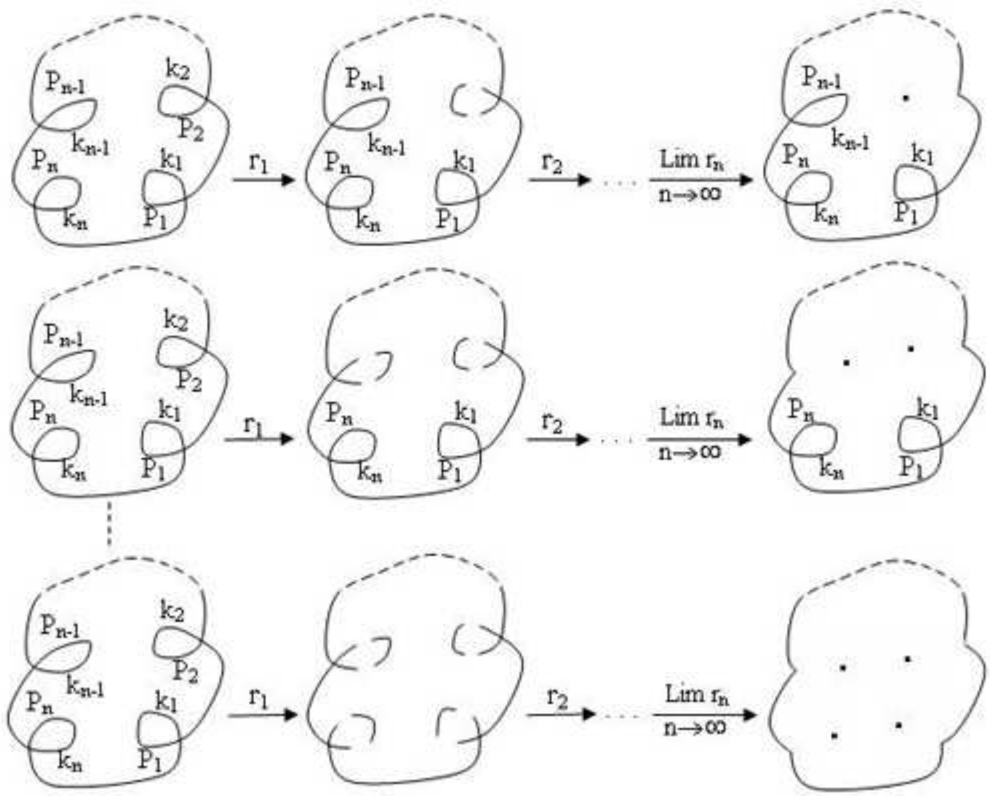

Fig.4

Proposition 3.5 *There is a type of retraction which make the non-simple curve with boundaries which is not a manifold to be a manifold with boundaries.*

Proof Let $r : X \setminus \{x_i\} \to X^r$ be a retraction map of $X \setminus \{x_i\}$ into X^r, where X is a non-simple curve with boundaries b_1 and b_2, which is self-intersection at n-points, let x_i, $i = 1, 2, \cdots, m$ are the points on the loops of the intersection points of X respectively, when the number of the points m are less than n i.e., $m < n$, then the limit of retractions of X is not a manifold, when the number of the points m is equal to the number of the intersection points n, then the limit of retractions of X is a simple curve with boundaries b_1 and b_2, which is a manifold with boundaries and when $m > n$, then the limit of retractions of X is a manifold see Fig.5, when X lies in R^3, we have the same results. $\square$

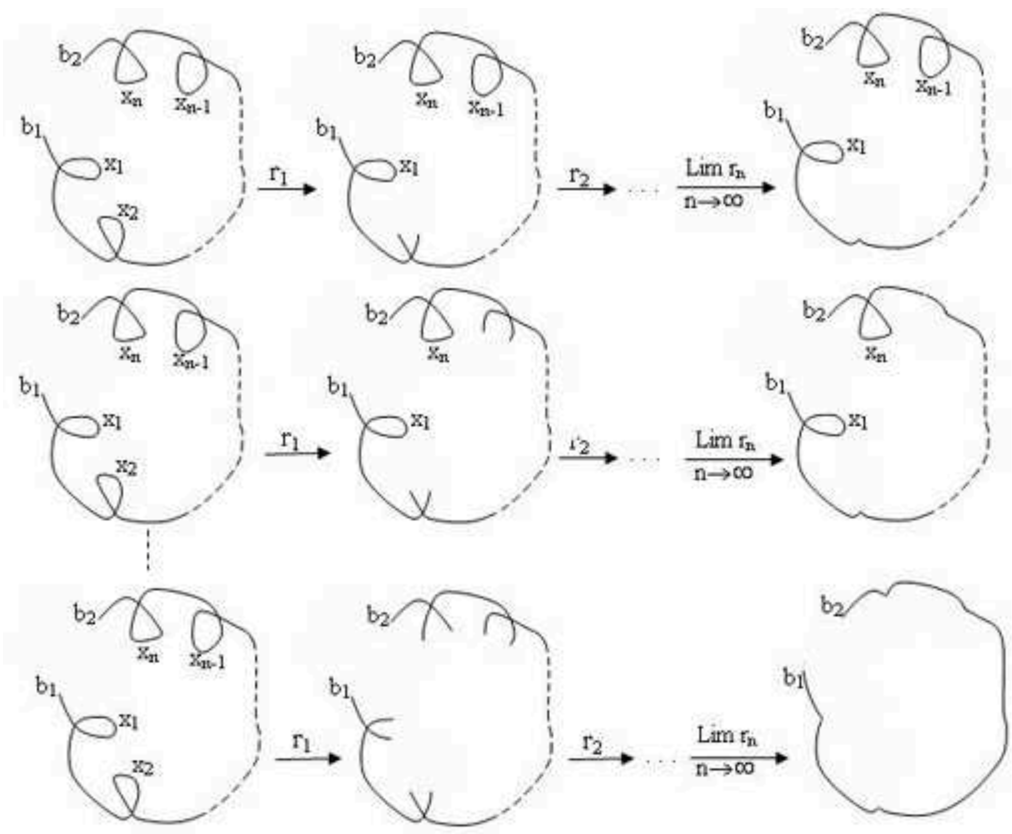

Fig.5

Proposition 3.6 *If $r : X \setminus \{x\} \to X^{/}$ be a retraction map of X, where X is a non-simple curve with boundaries, and x is a point between one point of the boundary and the nearest intersection point, then the limit of retractions of X is not a manifold.*

Proof Let $r : X \setminus \{x\} \to X^{/}$, be a retraction map of $X \setminus \{x\}$ into $X^{/}$, where X is a non-simple curve with boundaries b_1 and b_2 self-intersection at n-points, where x is the point between the boundary b_2 and the nearest intersection point of X, then the limit of retractions of X is not a manifold, when we define the retraction map $r : X \setminus \{x\} \to X^{/}$, where x is the point between the boundary $b_1$and the nearest intersection point of X then the limit of retractions of X is not a manifold, see Fig.6. □

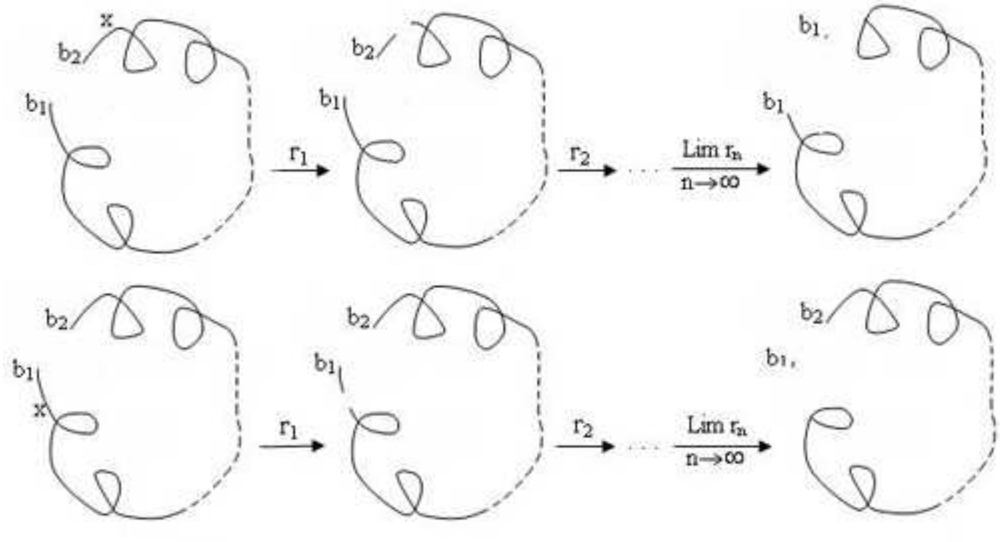

Fig.6

Proposition 3, 7 *If $r : X \setminus \{b_i\} \to X_b, i = 1, 2$ be a retraction map of X, where X is a non simple curve with boundaries b_1 and b_2, then the limit of retractions of X is not a manifold.*

Proof Let $r : X \setminus \{b_i\} \to X_b$ be a retraction map of $X \setminus \{b_i\}$ into X_b, where X is a non-simple curve self-intersection at n-points, $b_i,\ i = 1, 2$ are the boundaries of $X, i = 1, 2$, then the retraction of X is not a manifold and the limit of retractions of X is not a manifold, see Fig.7. □

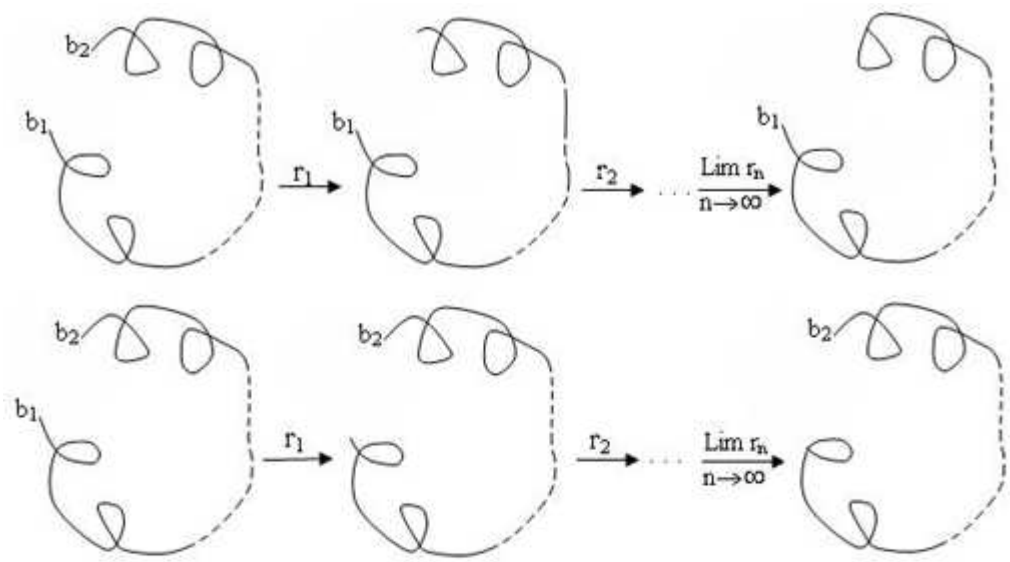

Fig.7

Proposition 3, 8 *There is a type of retraction which makes the non-simple closed curve in R^3 to be a knot.*

Proof Let $r : X \setminus \{x_i\} \to X^*$, be a retraction map of $X \setminus \{x_i\}$ into X^*, where X is a non-simple closed curve in R^3 self-intersection at n-points, X is not a manifold, let x_i, $i = 1, 2, \cdots, m$ are the points on the loops of the intersection points of X respectively, when the number of the points m are less than the number of the intersection points n i.e., $m < n$, then the limit of the retractions of X is not a knot, when the number of the points m is equal to the number of points n of X, i.e., $m = n$, then the limit of retractions of X is a simple closed curve homeomorphic to a circle in R^3 which is a knot, which is also a manifold and when the number of points m¿n, then the limit of the retractions of X is not a knot, but it is a manifold, see Fig.8. □

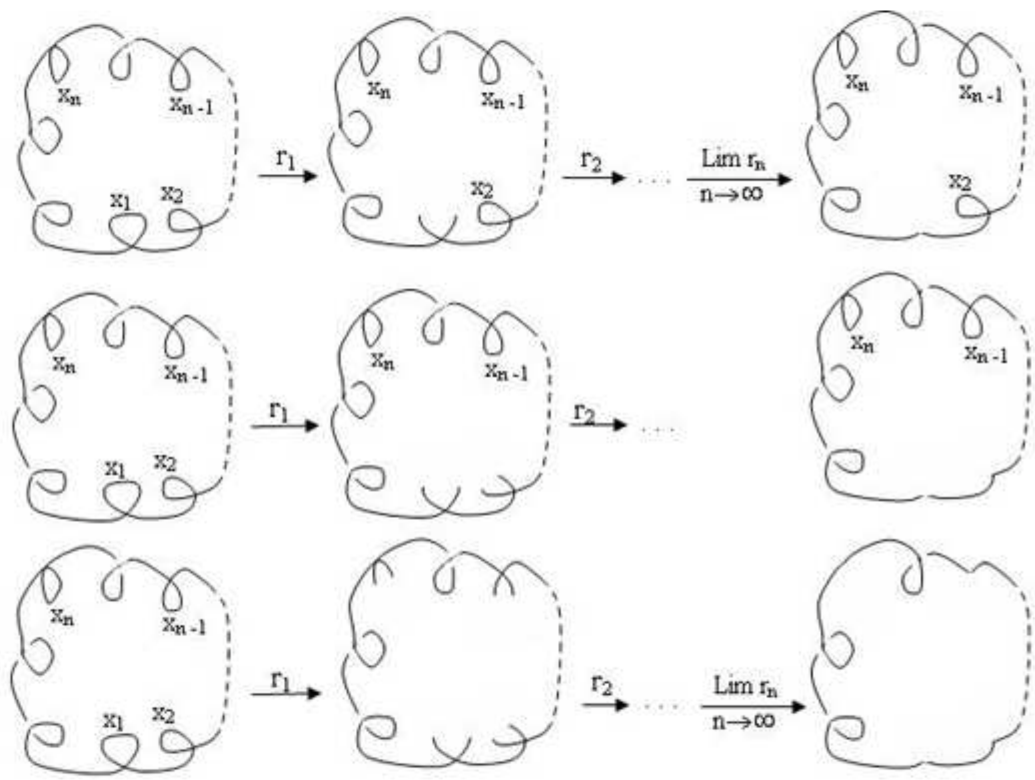

Fig.8

Proposition 3.9 *Let A be a subset of a topological space X and $r : X \to A$ is a retraction map of X into A, $A = r(X)$, then $dim(X) = dim(r(X)), dim(X) \geq dim(lim\ r(X)), dim(X) \neq dim(lim\ r(X))$ and $dim(r(X)) \geq dim(limr(X))$.*

Proposition 3.10 *The limit of retractions of any geometric figure in R^n, which is not a manifold is not necessary be a manifold, but the end of the limits of retractions of any n-geometric figure is a manifold.*

Proof Let $r : M^n \to M_1^n$ be a retraction map of M^n into M_1^n, M^n is a geometric figure of dimension n, M^n is not a manifold, then the limit of retractions of the geometric figure M^n is M^{n-1} and it may be a manifold or not, there is at least one point , which their neighborhood is not homeomorphic to the other points of M^{n-1}, the limit of retractions of M^{n-1} is M^{n-2} and it may be manifold or not, by using a sequence of retractions of M^n as shown in the following, then we find that the end of limits of retractions of M^n is a 0-manifold.

$$M^n \stackrel{r_1^1}{\to} M_1^n \stackrel{r_2^1}{\to} M_2^n \stackrel{r_2^1}{\to} ...M_{n-1}^n \stackrel{\lim\limits_{n\to\infty} r_n^1}{\to} M^{n-1},$$

$$M^{n-1} \stackrel{r_1^2}{\to} M_1^{n-1} \stackrel{r_2^2}{\to} M_2^{n-1} \stackrel{r_3^2}{\to} ...M_{n-1}^{n-1} \stackrel{\lim\limits_{n\to\infty} r_n^2}{\to} M^{n-2},$$

$$M^{n-2} \stackrel{r_1^3}{\to} M_1^{n-2} \stackrel{r_2^3}{\to} M_2^{n-2} \stackrel{r_3^3}{\to} ...M_{n-1}^{n-2} \stackrel{\lim\limits_{n\to\infty} r_n^3}{\to} M^{n-3},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots ,$$

$$M^1 \stackrel{r_1^n}{\to} M_1^1 \stackrel{r_2^n}{\to} M_2^1 \stackrel{r_3^n}{\to} ...M_{n-1}^1 \stackrel{\lim\limits_{n\to\infty} r_n^n}{\to} M^0.$$

Then the end of limits of retractions of any n-geometric figure is *a* 0-manifold. □

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