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# Notes on the Curves in Lorentzian Plane $\mathbf{L}^2$

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**Abstract** In this study, position vector of a Lorentzian plane curve (space-like or time-like, i.e.) is investigated. First, a system of differential equation whose solution gives the components of the position vector on the Frenet axis is constructed. By means of solution of mentioned system, position vector of all such curves according to Frenet frame is obtained. Thereafter, it is proven that, position vector and curvature of a Lorentzian plane curve satisfy a vector differential equation of third order. Moreover, using this result, position vector of such curves with respect to standard frame is presented. By this way, we present a short contribution to *Smarandache geometries*.

**Key Words** Classical differential geometry, Smarandache geometries, Lorentzian plane, position vector.

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## §1. Introduction

In recent years, the theory of degenerate submanifolds is treated by the researchers and some of classical differential geometry topics are extended to Lorentzian manifolds. For instance in [1], author deeply studies theory of the curves and surfaces and also presents mathematical principles about theory of Relativitiy. Also, T. Ikawa [4] presents some characterizations of the theory of curves in an indefinite-Riemannian manifold.

F. Smarandache in [2], defined a geometry which has at least one Smarandachely denied axiom, i.e., an axiom behaves in at least two different ways within the same space, i.e., validated and invalided, or only invalided but in multiple distinct ways and a Smarandache n-manifold is a n- manifold that support a Smarandache geometry.

Since, following these constructions, nearly all existent geometries, such as those of Euclid geometry, Lobachevshy- Bolyai geometry, Riemann geometry, Weyl geometry, K a hler geometry and Finsler geometry, ...,etc., are their sub-geometries (further details, see [3].

In the presented paper, we have determined position vector of a Lorentzian plane curve. First, using Frenet formula, we have constructed a system of differential equation. Solution of it yields components of the position vector on Frenet axis. Thereafter, again, using Frenet equations, we have constructed a vector differential equation with respect to position vector. Moreover, its solution has given us position vector the curve according to standard Euclidean frame. Since, we get a short contribution about Smarandache geometries.

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## §2. Preliminaries

To meet the requirements in the next sections, here, the basic elements of the theory of curves in the Lorentzian plane are briefly presented (A more complete elementary treatment can be found in [1], [4], [5]).

Let $\mathrm{L}^2$ be the Lorentzian plane with metric

$$g = dx_1^2 - dx_2^2, \tag{1}$$

where $x_1$ and $x_2$ are rectangular coordinate system. A vector $a$ of $\mathrm{L}^2$ is said to be space-like if $g(a,a) > 0$ or $a = 0$, time-like if $g(a,a) < 0$ and null if $g(a,a) = 0$ for $a \neq 0$. A curve $x$ is a smooth mapping $x : I \to \mathrm{L}^2$ from an open interval $I$ onto $\mathrm{L}^2$. Let $s$ be an arbitrary parameter of $x$. By $x = (x_1(s), x_2(s))$, we denote the orthogonal coordinate representation of $x$. The vector

$$\frac{dx}{ds} = \left(\frac{dx_1}{ds}, \frac{dx_2}{ds}\right) = t \tag{2}$$

is called the tangent vector field of the curve $x = x(s)$. If tangent vector field $t$ of $x(s)$ is a space-like, time-like or null, then, the curve $x(s)$ is called space-like, time-like or null, respectively.

In the rest of the paper, we shall consider non-null curves. When the tangent vector field $t$ is non-null, we can have the arc length parameter $s$ and have the Frenet formula

$$\begin{bmatrix} \dot{t} \\ \dot{n} \end{bmatrix} = \begin{bmatrix} 0 & \kappa \\ \kappa & 0 \end{bmatrix} \begin{bmatrix} t \\ n \end{bmatrix} \tag{3}$$

where $\kappa = \kappa(s)$ is the curvature of the unit spped curve $x = x(s)$. The vector field $n$ is called the normal vector field of the curve $x(s)$. Remark that, we have the same representation of the Frenet formula regardless of whether the curve is space-like ot time-like. And, if $\phi(s)$ is the slope angle of the curve, then we have

$$\frac{d\phi}{ds} = \kappa(s). \tag{4}$$

## §3. Position vector of a Lorentzian plane curve

Let $x = x(s)$ be an unit speed curve on the plane $\mathrm{L}^2$. Then, we can write position vector of $x(s)$ with respect to Frenet frame as

$$x = x(s) = \delta t + \lambda n \tag{5}$$

where $\delta$ and $\lambda$ are arbitrary functions of $s$. Differentiating both sides of (5) and using Frenet equations, we have a system of ordinary differential equations as follows:

$$\begin{array}{c} \frac{d\delta}{ds} + \lambda\kappa - 1 = 0 \\ \frac{d\lambda}{ds} + \delta\kappa = 0 \end{array} . \tag{6}$$

Using $(6)_1$ in $(6)_2$, we write

$$\frac{d}{ds}\left[\frac{1}{\kappa}\left(1-\frac{d\delta}{ds}\right)\right]+\delta\kappa=0. \tag{7}$$

This differential equation of second order, according to $\delta$, is a characterization for the curve $x=x(s)$. Using an exchange variable $\phi=\int_0^s \kappa ds$ in (7), we easily arrive

$$\frac{d^2\delta}{d\phi^2}-\delta=\frac{d\rho}{d\phi}, \tag{8}$$

where $\kappa=\frac{1}{\rho}$. By the method of variation of parameters and hyperbolic functions, solution of (8) yields

$$\delta=\cosh\phi\left[A-\int_0^\phi \rho\sinh\phi d\phi\right]+\sinh\phi\left[B+\int_0^\phi \rho\cosh\phi d\phi\right]. \tag{9}$$

Here $A,B\in R$. Rewriting the exchange variable, that is,

$$\delta=\cosh\int_0^s \kappa ds\left[A-\int_0^\phi\left(\sinh\int_0^s \kappa ds\right)ds\right]+\sinh\int_0^s \kappa ds\left[B+\int_0^\phi\left(\cosh\int_0^s \kappa ds\right)ds\right]. \tag{10}$$

Denoting differentation of equation (10) as $\frac{d\delta}{ds}=\xi(s)$, we have

$$\lambda=\rho(\xi(s)-1). \tag{11}$$

Since, we give the following theorem.

**Theorem** 3.1 *Let $x=x(s)$ be an arbitrary unit speed curve (space-like or time-like, i.e.) in Lorentzian plane. Position vector of the curve $x=x(s)$ with respect to Frenet frame can be composed by the equations* (10) *and* (11).

## §4. Vector differential equation of third order characterizes Lorentzian plane curves

**Theorem** 4.1 *Let $x=x(s)$ be an arbitrary unit speed curve (space-like or time-like, i.e.) in Lorentzian plane. Position vector and curvature of it satisfy a vector differential equation of third order.*

*Proof* Let $x=x(s)$ be an arbitrary unit speed curve (space-like or time-like, i.e.) in Lorentzian plane. Then formula (3) holds. Using $(3)_1$ in $(3)_2$, we easily have

$$\frac{d}{ds}\left(\frac{1}{\kappa}\frac{dt}{ds}\right)-\kappa t=0, \tag{12}$$

where $\frac{dx}{ds}=t=\dot{x}$. Consequently, we write

$$\frac{d}{ds}\left(\frac{1}{\kappa}\frac{d^2x}{ds^2}\right)-\kappa\frac{dx}{ds}=0. \tag{13}$$

Formula (13) completes the proof.

Let us solve equation (12) with respect to $t$. Here, we know, $t = (t_1, t_2) = (\dot{x}_1, \dot{x}_2)$. Using the exchange variable $\phi = \int_0^s \kappa ds$ in (), we obtain

$$\frac{d^2t}{d\phi^2} - t = 0 \tag{14}$$

or in parametric for

$$\begin{array}{l} \frac{d^2t_1}{d\phi^2} - t_1 = 0 \\ \frac{d^2t_2}{d\phi^2} - t_2 = 0 \end{array} . \tag{15}$$

It follows that

$$\begin{array}{l} t_1 = \varepsilon_1 e^{\phi} - \varepsilon_2 e^{-\phi} \\ t_2 = \varepsilon_3 e^{\phi} - \varepsilon_4 e^{-\phi} \end{array} , \tag{16}$$

where $\varepsilon_i \in R$ for $1 \leq i \leq 4$. Therefore, we get

$$\begin{array}{l} t_1 = \gamma_1 \cosh \int_0^s \kappa ds + \gamma_2 \sinh \int_0^s \kappa ds \\ t_2 = \gamma_3 \cosh \int_0^s \kappa ds + \gamma_4 \sinh \int_0^s \kappa ds \end{array} . \tag{17}$$

Finally, we give the following theorem.

**Theorem** 4.2 *Let $x = x(s)$ be an arbitrary unit speed curve (space-like ot time-like, i.e.) in Lorentzian plane. Position vector of it with respect to standard frame can be expressed as*

$$x = x(s) = \begin{pmatrix} \int_0^s \left\{ \gamma_1 \cosh \int_0^s \kappa ds + \gamma_2 \sinh \int_0^s \kappa ds \right\} ds, \\ \int_0^s \left\{ \gamma_3 \cosh \int_0^s \kappa ds + \gamma_4 \sinh \int_0^s \kappa ds \right\} ds \end{pmatrix} \tag{18}$$

*for the real numbers $\gamma_1, ..., \gamma_4$.* □

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