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Euclidean Pseudo-Geometry on $\mathbf{R}^n$

Linfan MAO

(Chinese Academy of Mathematics and System Science, Beijing 100080, P.R.China)

E-mail: maolinfan@163.com

Abstract: A Smarandache geometry is a geometry which has at least one Smarandachely denied axiom (1969), i.e., validated and invalided, or only invalided but in multiple distinct ways. Iseri constructed Smarandache 2-manifolds in Euclidean plane $\mathbf{R}^2$ in [1], and later Mao generalized his result to surfaces by *map geometry* in [4]. Then *can we construct Smarandache n-manifolds for $n \geq 3$*? The answer is YES. Not like the technique used in [6], we show how to construct Smarandache geometries in $\mathbf{R}^n$ by an algebraic methods, which was applied in [3] for $\mathbf{R}^2$ first, and then give a systematic way for constructing Smarandache n-manifolds.

Key Words: Smarandache geometries, Euclidean pseudo-geometry, combinatorial system.

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§1. Introduction

As it is usually cited in references, a *Smarandache geometry* is defined as follows.

Definition 1.1 *An axiom is said to be Smarandachely denied if the axiom behaves in at least two different ways within the same space, i.e., validated and invalided, or only invalided but in multiple distinct ways.*

*A Smarandache geometry is a geometry which has at least one Smarandachely denied axiom(*1969*).*

This anti-mathematical or multiple approach on sciences can be used to abstract systems. In the reference [8], we formally generalized it to the conceptions of Smarandachely systems as follows.

Definition 1.2 *A rule in a mathematical system* $(\Sigma; \mathcal{R})$ *is said to be Smarandachely denied if it behaves in at least two different ways within the same set* Σ*, i.e., validated and invalided, or only invalided but in multiple distinct ways.*

A Smarandache system $(\Sigma; \mathcal{R})$ *is a mathematical system which has at least one Smarandachely denied rule in* $\mathcal{R}$*.*

As its a simple or concrete example, a question raised in [4] is to construct a Smarandache geometry on R^n for $n \geq 2$. Certainly, the case of $n = 2$ has be solved by Iseri [1] and Mao [3]-

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[4]. The main purpose of this paper is to give an algebraic approach for constructing Euclidean pseudo-geometry on $\mathbf{R}^n$ for any integer $n \geq 2$, which also refines the definition of pseudo manifold geometry introduced in [6].

§2. Euclidean pseudo-geometry

Let $\mathbf{R}^n$ be an n-dimensional Euclidean space with a normal basis $\overline{\epsilon}_1 = (1, 0, \cdots, 0)$, $\overline{\epsilon}_2 = (0, 1, \cdots, 0)$, $\cdots$, $\overline{\epsilon}_n = (0, 0, \cdots, 1)$. An *orientation* $\overrightarrow{X}$ is a vector $\overrightarrow{OX}$ with $\|\overrightarrow{OX}\| = 1$ in $\mathbf{R}^n$, where $O = (0, 0, \cdots, 0)$. Usually, an orientation $\overrightarrow{X}$ is denoted by its projections of $\overrightarrow{OX}$ on each $\overline{\epsilon}_i$ for $1 \leq i \leq n$, i.e.,

$$\overrightarrow{X} = (\cos(\overrightarrow{OX}, \overline{\epsilon}_1), \cos(\overrightarrow{OX}, \overline{\epsilon}_2), \cdots, \cos(\overrightarrow{OX}, \overline{\epsilon}_n)),$$

where $(\overrightarrow{OX}, \overline{\epsilon}_i)$ denotes the angle between vectors $\overrightarrow{OX}$ and $\overline{\epsilon}_i$, $1 \leq i \leq n$. All possible orientations $\overrightarrow{X}$ in $\mathbf{R}^n$ consist of a set $\mathscr{O}$.

A *pseudo-Euclidean space* is a pair $(\mathbf{R^n}, \omega|_{\overrightarrow{O}})$, where $\omega|_{\overrightarrow{O}} : \mathbf{R}^n \to \mathscr{O}$ is a continuous function, i.e., a straight line with an orientation $\overrightarrow{O}$ will has an orientation $\overrightarrow{O} + \omega|_{\overrightarrow{O}}(\overline{u})$ after it passing through a point $\overline{u} \in \mathbf{E}$. It is obvious that $(\mathbf{E}, \omega|_{\overrightarrow{O}}) = \mathbf{E}$, namely the Euclidean space itself if and only if $\omega|_{\overrightarrow{O}}(\overline{u}) = \overline{0}$ for $\forall \overline{u} \in \mathbf{E}$.

We have known that a straight line L passing through a point $(x_1^0, x_2^0, \cdots, x_n^0)$ with an orientation $\overrightarrow{O} = (X_1, X_2, \cdots, X_n)$ is defined to be a point set $(x_1, x_2, \cdots, x_n)$ determined by an equation system

$$\begin{cases} x_1 = x_1^0 + tX_1 \\ x_2 = x_2^0 + tX_2 \\ \cdots\cdots\cdots\cdots \\ x_n = x_n^0 + tX_n \end{cases}$$

for $\forall t \in \mathbf{R}$ in analytic geometry on $\mathbf{R}^n$, or equivalently, by the equation system

$$\frac{x_1 - x_1^0}{X_1} = \frac{x_2 - x_2^0}{X_2} = \cdots = \frac{x_n - x_n^0}{X_n}.$$

Therefore, we can also determine its equation system for a straight line L in a pseudo-Euclidean space $(\mathbf{R}^n, \omega)$. By definition, a straight line L passing through a Euclidean point $\overline{x}^0 = (x_1^0, x_2^0, \cdots, x_n^0) \in \mathbf{R}^n$ with an orientation $\overrightarrow{O} = (X_1, X_2, \cdots, X_n)$ in $(\mathbf{R}^n, \omega)$ is a point set $(x_1, x_2, \cdots, x_n)$ determined by an equation system

$$\begin{cases} x_1 = x_1^0 + t(X_1 + \omega_1(\overline{x}^0)) \\ x_2 = x_2^0 + t(X_2 + \omega_2(\overline{x}^0)) \\ \cdots\cdots\cdots\cdots \\ x_n = x_n^0 + t(X_n + \omega_n(\overline{x}^0)) \end{cases}$$

for $\forall t \in \mathbf{R}$, or equivalently,

$$\frac{x_1 - x_1^0}{X_1 + \omega_1(\overline{x}^0)} = \frac{x_2 - x_2^0}{X_2 + \omega_2(\overline{x}^0)} = \cdots = \frac{x_n - x_n^0}{X_n + \omega_n(\overline{x}^0)},$$

where $\omega|_{\overrightarrow{O}}(\overline{x}^0) = (\omega_1(\overline{x}^0), \omega_2(\overline{x}^0), \cdots, \omega_n(\overline{x}^0))$. Notice that this equation system dependent on $\omega|_{\overrightarrow{O}}$, it maybe not a linear equation system.

Similarly, let $\overrightarrow{O}$ be an orientation. A point $\overline{u} \in \mathbf{R}^n$ is said to be *Euclidean* on orientation $\overrightarrow{O}$ if $\omega|_{\overrightarrow{O}}(\overline{u}) = \overline{0}$. Otherwise, let $\omega|_{\overrightarrow{O}}(\overline{u}) = (\omega_1(\overline{u}), \omega_2(\overline{u}), \cdots, \omega_n(\overline{u}))$. The point $\overline{u}$ is *elliptic* or *hyperbolic* determined by the following inductive programming.

STEP 1. If $\omega_1(\overline{u}) < 0$, then $\overline{u}$ is elliptic; otherwise, hyperbolic if $\omega_1(\overline{u}) > 0$;

STEP 2. If $\omega_1(\overline{u}) = \omega_2(\overline{u}) = \cdots = \omega_i(\overline{u} = 0$, but $\omega_{i+1}(\overline{u} < 0$ then $\overline{u}$ is elliptic; otherwise, hyperbolic if $\omega_{i+1}(\overline{u}) > 0$ for an integer $i, 0 \leq i \leq n-1$.

Denote these elliptic, Euclidean and hyperbolic point sets by

$$\overrightarrow{V}_{eu} = \{\ \overline{u} \in \mathbf{R}^n \mid \overline{\mathrm{u}} \text{ an Euclidean point }\},$$
$$\overrightarrow{V}_{el} = \{\ \overline{v} \in \mathbf{R}^n \mid \overline{\mathrm{v}} \text{ an elliptic point }\}.$$
$$\overrightarrow{V}_{hy} = \{\ \overline{v} \in \mathbf{R}^n \mid \overline{\mathrm{w}} \text{ a hyperbolic point }\}.$$

Then we get a partition

$$\mathbf{R}^n = \overrightarrow{V}_{eu} \bigcup \overrightarrow{V}_{el} \bigcup \overrightarrow{V}_{hy}$$

on points in $\mathbf{R}^n$ with $\overrightarrow{V}_{eu} \cap \overrightarrow{V}_{el} = \emptyset$, $\overrightarrow{V}_{eu} \cap \overrightarrow{V}_{hy} = \emptyset$ and $\overrightarrow{V}_{el} \cap \overrightarrow{V}_{hy} = \emptyset$. Points in $\overrightarrow{V}_{el} \cap \overrightarrow{V}_{hy}$ are called *non-Euclidean points*.

Now we introduce a linear order $\prec$ on $\mathscr{O}$ by the dictionary arrangement in the following.

For $(x_1, x_2, \cdots, x_n)$ and $(x_1', x_2', \cdots, x_n') \in \mathscr{O}$, if $x_1 = x_1', x_2 = x_2', \cdots, x_l = x_l'$ and $x_{l+1} < x_{l+1}'$ for any integer $l, 0 \leq l \leq n-1$, then define $(x_1, x_2, \cdots, x_n) \prec (x_1', x_2', \cdots, x_n')$.

By this definition, we know that

$$\omega|_{\overrightarrow{O}}(\overline{u}) \prec \omega|_{\overrightarrow{O}}(\overline{v}) \prec \omega|_{\overrightarrow{O}}(\overline{w})$$

for $\forall \overline{u} \in \overrightarrow{V}_{el}, \overline{v} \in \overrightarrow{V}_{eu}$, $\overline{w} \in \overrightarrow{V}_{hy}$ and a given orientation $\overrightarrow{O}$. This fact enables us to find an interesting result following.

Theorem 2.1 *For any orientation $\overrightarrow{O} \in \mathscr{O}$ in a pseudo-Euclidean space $(\mathbf{R}^n, \omega|_{\overrightarrow{O}})$, if $\overrightarrow{V}_{el} \neq \emptyset$ and $\overrightarrow{V}_{hy} \neq \emptyset$, then $\overrightarrow{V}_{eu} \neq \emptyset$.*

Proof By assumption, $\overrightarrow{V}_{el} \neq \emptyset$ and $\overrightarrow{V}_{hy} \neq \emptyset$, we can choose points $\overline{u} \in \overrightarrow{V}_{el}$ and $\overline{w} \in \overrightarrow{V}_{hy}$. Notice that $\omega|_{\overrightarrow{O}} : \mathbf{R}^n \to \mathscr{O}$ is a continuous and $(\mathscr{O}, \prec)$ a linear ordered set. Applying the *generalized intermediate value theorem* on continuous mappings in topology, i.e.,

Let $f : X \to Y$ be a continuous mapping with X a connected space and Y a linear ordered set in the order topology. If $a, b \in X$ and $y \in Y$ lies between $f(a)$ and $f(b)$, then there exists $x \in X$ such that $f(x) = y$.

we know that there is a point $\overline{v} \in \mathbf{R}^n$ such that

$$\omega|_{\overrightarrow{O}}(\overline{v}) = \overline{0},$$

i.e., $\overline{v}$ is a Euclidean point by definition. □

Corollary 2.1 *For any orientation $\overrightarrow{O} \in \mathscr{O}$ in a pseudo-Euclidean space $(\mathbf{R}^n, \omega|_{\overrightarrow{O}})$, if $\overrightarrow{V}_{eu} = \emptyset$, then either points in $(\mathbf{R}^n, \omega|_{\overrightarrow{O}})$ is elliptic or hyperbolic.*

Certainly, a pseudo-Euclidean space $(\mathbf{R}^n, \omega|_{\overrightarrow{O}})$ is a Smarandache geometry sometimes explained in the following.

Theorem 2.2 *A pseudo-Euclidean space $(\mathbf{R}^n, \omega|_{\overrightarrow{O}})$ is a Smarandache geometry if $\overrightarrow{V}_{eu}, \overrightarrow{V}_{el} \neq \emptyset$, or $\overrightarrow{V}_{eu}, \overrightarrow{V}_{hy} \neq \emptyset$, or $\overrightarrow{V}_{el}, \overrightarrow{V}_{hy} \neq \emptyset$ for an orientation $\overrightarrow{O}$ in $(\mathbf{R}^n, \omega|_{\overrightarrow{O}})$.*

Proof Notice that $\omega|_{\overrightarrow{O}}(\overline{u}) = \overline{0}$ is an axiom in $\mathbf{R}^n$, but a Smarandache denied axiom if $\overrightarrow{V}_{eu}, \overrightarrow{V}_{el} \neq \emptyset$, or $\overrightarrow{V}_{eu}, \overrightarrow{V}_{hy} \neq \emptyset$, or $\overrightarrow{V}_{el}, \overrightarrow{V}_{hy} \neq \emptyset$ for an orientation $\overrightarrow{O}$ in $(\mathbf{R}^n, \omega|_{\overrightarrow{O}})$ for $\omega|_{\overrightarrow{O}}(\overline{u}) = \overline{0}$ or $\neq \overline{0}$ in the former two cases and $\omega|_{\overrightarrow{O}}(\overline{u}) \prec \overline{0}$ or $\succ \overline{0}$ both hold in the last one. Whence, we know that $(\mathbf{R}^n, \omega|_{\overrightarrow{O}})$ is a Smarandache geometry by definition. □

Notice that there infinite points on a segment of a straight line in $\mathbf{R}^n$. Whence, a necessary for the existence of a straight line is there exist infinite Euclidean points in $(\mathbf{R}^n, \omega|_{\overrightarrow{O}})$. We find a necessary and sufficient result for the existence of a curve C in $(\mathbf{R}^n, \omega|_{\overrightarrow{O}})$ following.

Theorem 2.3 *A curve $C = (f_1(t), f_2(t), \cdots, f_n(t))$ exists in a pseudo-Euclidean space $(\mathbf{R}^n, \omega|_{\overrightarrow{O}})$ for an orientation $\overrightarrow{O}$ if and only if*

$$\frac{df_1(t)}{dt}|_{\overline{u}} = \sqrt{(\frac{1}{\omega_1(\overline{u})})^2 - 1},$$

$$\frac{df_2(t)}{dt}|_{\overline{u}} = \sqrt{(\frac{1}{\omega_2(\overline{u})})^2 - 1},$$

$$\cdots\cdots\cdots\cdots,$$

$$\frac{df_n(t)}{dt}|_{\overline{u}} = \sqrt{(\frac{1}{\omega_n(\overline{u})})^2 - 1}.$$

for $\forall \overline{u} \in C$, where $\omega|_{\overrightarrow{O}} = (\omega_1, \omega_2, \cdots, \omega_n)$.

Proof Let the angle between $\omega|_{\overrightarrow{O}}$ and $\overline{\epsilon}_i$ be θ_i, $1 \leq \theta_i \leq n$.

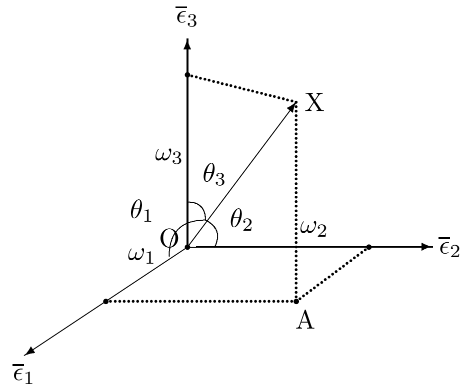

Fig.2.1

Then we know that

$$\cos\theta_i = \omega_i, \quad 1 \leq i \leq n.$$

According to the geometrical implication of differential at a point $\overline{u} \in \mathbf{R}^n$, seeing also Fig.2.1, we know that

$$\frac{df_i(t)}{dt}|_{\overline{u}} = tg\theta_i = \sqrt{(\frac{1}{\omega_i(\overline{u})})^2 - 1}$$

for $1 \leq i \leq n$. Therefore, if a curve $C = (f_1(t), f_2(t), \cdots, f_n(t))$ exists in a pseudo-Euclidean space $(\mathbf{R}^n, \omega|_{\overrightarrow{O}})$ for an orientation $\overrightarrow{O}$, then

$$\frac{df_i(t)}{dt}|_{\overline{u}} = \sqrt{(\frac{1}{\omega_2(\overline{u})})^2 - 1}, \quad 1 \leq i \leq n$$

for $\forall\overline{u} \in C$. On the other hand, if

$$\frac{df_i(t)}{dt}|_{\overline{v}} = \sqrt{(\frac{1}{\omega_2(\overline{v})})^2 - 1}, \quad 1 \leq i \leq n$$

hold for points $\overline{v}$ for $\forall t \in \mathbf{R}$, then all points $\overline{v}, t \in \mathbf{R}$ consist of a curve $C = (f_1(t), f_2(t), \cdots, f_n(t))$ in $(\mathbf{R}^n, \omega|_{\overrightarrow{O}})$ for the orientation $\overrightarrow{O}$. □

Corollary 2.2 *A straight line L exists in $(\mathbf{R}^n, \omega|_{\overrightarrow{O}})$ if and only if $\omega|_{\overrightarrow{O}}(\overline{u}) = \overline{0}$ for $\forall\overline{u} \in L$ and $\forall\overrightarrow{O} \in \mathscr{O}$.*

§3. Application to Smarandache n-manifolds

Application of the definition of pseudo-Euclidean space $\mathbf{R}^n$ enables us to formally define a dimensional n pseudo-manifold in [6] following, which makes its structure clear.

Definition 3.1 *An n-dimensional pseudo-manifold $(M^n, \mathcal{A}^\omega)$ is a Hausdorff space such that each points p has an open neighborhood U_p homomorphic to a pseudo-Euclidean space $(\mathbf{R}^n, \omega|_{\overrightarrow{O}})$,* *where* $\mathcal{A} = \{(U_p, \varphi_p^\omega)|p \in M^n\}$ *is its atlas with a homomorphism* $\varphi_p^\omega : U_p \to (\mathbf{R}^n, \omega|_{\overrightarrow{O}})$ *and a chart* (U_p, φ_p^ω).

Applications of this definition will rebuilt pseudo-manifold geometries constructed in [6], which will appear in a forthcoming book *Combinatorial Geometry with Applications to Field Theory* of the author in 2009.

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