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Equiparity Path Decomposition Number of a Graph

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Abstract A *decomposition* of a graph G is a collection ψ of edge-disjoint subgraphs $H_1, H_2, \ldots, H_n$ of G such that every edge of G belongs to exactly one H_i. If each H_i is a path in G, then ψ is called a *path partition* or *path cover* or *path decomposition* of G. Various types of path covers such as Smarandache path k-cover, simple path covers have been studied by several authors by imposing conditions on the paths in the path covers . Here we impose parity condition on lengths of the paths and define an equiparity path cover as follows. An *equiparity path decomposition* of a graph G is a path cover ψ of G such that the lengths of all the paths in ψ have the same parity. The minimum cardinality of a equiparity path decomposition of G is called the *equiparity path decomposition number* of G and is denoted by $\pi_P(G)$. In this paper we initiate a study of the parameter π_P and determine the value of π_P for some standard graphs. Further, we obtain some bounds for π_P and characterize graphs attaining the bounds.

Key words: Odd parity path decomposition, even parity path decomposition, equiparity path decomposition, equiparity path decomposition number, Smarandache path k-cover.

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§1. Introduction

By a graph, we mean a finite, undirected, non-trivial, connected graph without loops and multiple edges. The order and size of a graph are denoted by p and q respectively. For terms not defined here we refer to Harary [6].

Let $P = (v_1, v_2, \ldots, v_n)$ be a path in a graph $G = (V, E)$. The vertices $v_2, v_3, \ldots, v_{n-1}$ are called *internal vertices* of P and v_1 and v_n are called *external vertices* of P. The length of a path is denoted by $l(P)$. If the length of the path is odd(even) then we say that it is an *odd(even) path.*

A *subdivision graph* $S(G)$ of a graph G is obtained by subdividing each edge of G only

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once. Two graphs are said to be *homeomorphic* if both can be obtained from the same graph by a sequence of subdivision of edges. A cycle with exactly one chord is called a θ*-graph*. The length of a largest cycle of a graph is called *circumference* of a graph and it is denoted by c. For vertices x and y in a connected graph G, the *detour distance* $D(x, y)$ is the length of a longest $x-y$ path in G. The *detour diameter* D of G is defined to be $D = max\{D(x, y) : x, y \in V(G)\}$. An (n, t)*-kite* consists of a cycle of length n with a t-edge path(called tail) attached to one vertex of the cycle. An $(n, 1)$*-kite* is called a kite with tail length 1.

A *decomposition* of a graph G is a collection of edge-disjoint subgraphs $H_1, H_2, \ldots, H_r$ of G such that every edge of G belongs to exactly one H_i. If each $H_i \cong H$, then the decomposition is called *isomorphic decomposition* and we also say that G is H decomposable. If each H_i is a path, then ψ is called a *path partition* or *path cover* or *path decomposition* of G. The minimum cardinality of a path partition of G is called the path partition number of G and is denoted by $\pi(G)$ and any path partition ψ of G for which $|\psi| = \pi(G)$ is called a *minimum path partition* or π*-cover* of G. The parameter π was studied by Harary and Schwenk [7], Peroche [9], Stanton *et.al.,* [10] and Arumugam and Suresh Suseela [4].

A more general definition on graph covering using paths is given as follows.

Definition 1.1([2]) *For any integer $k \geq 1$, a Smarandache path k-cover of a graph G is a collection ψ of paths in G such that each edge of G is in at least one path of ψ and two paths of ψ have at most k vertices in common.*

Thus if $k = 1$ and every edge of G is in exactly one path in ψ, then a Smarandache path k-cover of G is a simple path cover of G.

Consider the following path decomposition theorems.

Theorem 1.2([5]) *For any connected graph $(p, q)-$graph G, if q is even, then G has a P_3-decomposition.*

Theorem 1.3([10]) *If G is a 3-regular $(p, q)-$graph, then G is P_4 decomposable and*

$$\pi(G) = \frac{q}{3} = \frac{p}{2}.$$

Theorem 1.4([10]) *A complete graph K_{2n} is hamilton path decomposable of length $2n-1$. The path partition number π of a complete graphs are given by a) $\pi(K_{2n}) = n$ and (b) $\pi(K_{2n+1}) = n + 1$.*

The Theorems 1.2, 1.3 and 1.4(a) give the path decomposition in which all the paths are of even (odd) length. The above results give the isomorphic path decomposition in which all the paths are of same parity. This observation motivates the following definition for non-isomorphic path decomposition also.

Definition 1,5 *An equiparity path decomposition(EQPPD) of a graph G is a path cover ψ of G such that the lengths of all the paths in ψ have the same parity.*

Since for any graph G, the edge set $E(G)$ is an equiparity path decomposition, the collection $\mathcal{P}_P$ of all equiparity path decompositions of G is non-empty. Let $\pi_P(G) = min\,|\psi|$. Then $\pi_P(G)$ is called the *equiparity path decomposition number* of G and any equiparity path decomposition ψ of G for which $|\psi| = \pi_P(G)$ is called a *minimum equiparity path decomposition* of G or π_P*-cover* of G.

If the lengths of all the paths in ψ are even(odd) then we say that ψ is an *even (odd) parity path decomposition*, shortly *EPPD (OPPD)*.

Remark 1.6 Let $\psi = \{P_1, P_2, \ldots, P_n\}$ be an EQPPD of a $(p, q)-$graph G such that $l(P_1) \leq l(P_2) \leq , \ldots , l(P_n)$. Since every edge of G is in exactly one path P_i , we have $\sum\limits_{i=1}^{n} l(P_i) = q$ and hence every EQPPD of G gives rise to a partition of an integer q into same parity,

Remark 1.7 If G is a graph of odd size, then any equiparity path decomposition ψ of a graph G is an odd parity path decomposition and consequently $\pi_P(G)$ is odd.

Remark 1.8 If an equiparity path decomposition ψ of a graph G is an even equiparity path decomposition, then q is even.

Various types of path decompositions and corresponding parameters have been studied by several authors by imposing conditions on the paths in the decomposition. Some such path decomposition parameters are graphoidal covering number [1], simple path covering number [2], simple graphoidal covering number [3], simple acyclic graphoidal covering number [3] and 2-graphoidal path covering number [8].

In this paper we initiate a study of the parameter π_P and determine the value of π_P for some standard graphs. Further, we obtain bounds for π_P and characterize graphs attaining the bounds.

§2. Main results

We first present a general result which is useful in determining the value of π_p.

Theorem 2.1 *For any EQPPD ψ of a graph G, let $t_\psi = \sum\limits_{P\in\psi} t(P)$, where $t(P)$ denotes the number of internal vertices of P and let $t = \max t_\psi$, where the maximum is taken over all equiparity path decompositions ψ of G. Then $\pi_p(G) = q - t$.*

Proof Let ψ be any EQPPD of G. Then

$$\begin{aligned} q &= \sum_{P\in\psi} |E(P)| = \sum_{P\in\psi} (t(P)+1) \\ &= \sum_{P\in\psi} t(P) + |\psi| = t_\psi + |\psi|\,. \end{aligned}$$

Hence $|\psi| = q - t_\psi$ so that $\pi_p = q - t$. □

Next we will find some bounds for π_P. First, we find a simple bound for π_P in terms of the size of G.

Theorem 2.2 *For any graph G of even size,$\pi_P(G) \le \frac{q}{2}$.*

Proof It follows from Theorem **??** that G has a P_3-decomposition, which is an EPPD and hence $\pi_P(G) \le \frac{q}{2}$. □

Remark 2.3 The bound given in Theorem 2.2 is sharp. For the cycle C_4 and the star $K_{1,n}$, where n is even, $\pi_P = \frac{q}{2}$.

The following problem naturally arises.

Problem 2.4 *Characterize graphs of an even size for which $\pi_P = \frac{q}{2}$.*

Now, we characterize graphs attaining the extreme bounds.

Theorem 2.5 *For a graph G, $1 \le \pi_P(G) \le q$. Then $\pi_P(G) = 1$ if and only if G is a path and $\pi_P(G) = q$ if and only if G is either K_3 or $K_{1,q}$ where q is odd.*

Proof The inequalities are trivial. Further , it is obvious that $\pi_P(G) = 1$ if and only if G is a path.

Now, suppose $\pi_P(G) = q > 1$. Then it follows from Theorem 2.2 that q is odd. Let P be a path of length greater than one in G. If the length of P is odd, then $\psi = \{P\} \bigcup \{E(G) \setminus E(P)\}$ is an OPPD of G so that $\pi_P(G) < q$, which is a contradiction. Thus every path of length greater than one is even and consequently every path in G is of length 1 or 2. Hence any two edges in G are adjacent, so that G is either a triangle or a star. Converse is obvious. □

The following theorem gives the lower and upper bounds for π_P in terms of π.

Theorem 2.6 *For any graph G, $\pi(G) \le \pi_P(G) \le 2\pi(G) - 1$.*

Proof Since every equiparity path decomposition is a path cover, we have $\pi(G) \le \pi_P(G)$.

Let ψ be a $\pi-$cover of G and let m and n be the number of even and odd paths in ψ respectively, Then $1 \le m, n \le \pi - 1$ and $m + n = \pi$. Then the path decomposition ψ_1 obtained from ψ by splitting each even path in ψ into two odd paths is an OPPD and hence

$$\pi_P(G) \le |\psi_1| = 2m + n = m + (m + n) \le \pi - 1 + \pi = 2\pi - 1.$$

□

Corollary 2.7 *For a graph G of odd size, if $\pi(G)$ is even, then $\pi(G) + 1 \le \pi_P(G)$.*

Proof Since $\pi(G)$ is even and q is odd, we have, $\pi(G) \ne \pi_P(G)$ and from Theorem **??** , we have $\pi(G) + 1 \le \pi_P(G)$. □

The above bounds will be very useful to find the value of π_P for some standard graphs.

Remark 2.8 It is obvious that $\pi_P(G) = \pi(G)$ if and only if there exists a π-cover of G in which lengths of all the paths have the same parity. Further, if $\pi_P(G) = 2\pi(G) - 1$, then every π-cover of G contains only one path of odd length.

From the above bounds the following problems will naturally arise.

Problem 2.9 *Characterize the class of graphs for which* $\pi_P(G) = \pi(G)$.

Problem 2.10 *Characterize the class of graphs for which* $\pi_P(G) = 2\pi(G) - 1$.

Problem 2.11 *Characterize the class of graphs for which* $\pi_P(G) = \pi(G) + 1$.

Corollary 2.12 *For a graph* G, *if* q *is even, then* $\pi_P(G) \leq q - 1$.

Proof From Theorem 1.2, it follows that $\pi(G) \leq \frac{q}{2}$. Then from Theorem 2.6, it follows that $\pi_P(G) \leq q - 1$. □

Now, we characterize graphs attaining the above bound.

Theorem 2.13 *For any graph* G, $\pi_P(G) = q - 1$ *if and only if* $G \cong P_3$.

Proof Suppose $\pi_P(G) = q - 1$. If G has a path P of length 3, then the path P together with the remaining edges form an OPPD ψ of G so that $\pi_P(G) \leq |\psi| = q - 2 < q - 1$, which is a contradiction. Thus every path in G is of length at most 2. Hence any two edges in G are adjacent , so that G is either a triangle or a star. From Theorem 2.5, it follows that G is neither a triangle nor a star of odd size. Thus G is a star of even size. Then clearly, $\pi_P(G) = \frac{q}{2}$. Thus $q = 2$ and hence $G \cong P_3$. The converse is obvious. □

Next we solve the following realization problem.

Theorem 2.14 *If* a *is a positive integer and for every odd* b *with* $a \leq b \leq 2a - 1$, *then there exists a connected graph* G *such that* $\pi(G) = a$ *and* $\pi_P(G) = b$.

Proof Now, suppose a is a positive integer and for every odd b with $a \leq b \leq 2a - 1$.

Case (i) a is odd.

We now construct a graph $G_r, r = 0, 1, 2, \ldots, \frac{r-1}{2}$ as follows. Let G_0 be a star graph with $v_1, v_2, \ldots, v_{2a-2}, v_{2a-1}$ as pendant vertices and v_{2a} as central vertex. Let G_r be a graph obtained from G_0 by subdividing $2r$ edges $v_1v_{2a}, v_2v_{2a}, \ldots, v_{2r}v_{2a}$ of G_0 once by the vertices $v'_1, v'_2, \ldots, v'_{2r}$, where $r = 1, 2, \ldots, \frac{a-1}{2}$ (Fig.1). Note that $p = 2a + 2r$ and $q = 2a - 1 + 2r$.

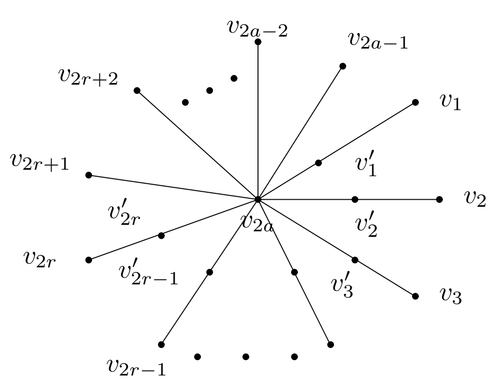

Fig.1

First we prove that $\pi(G_r) = a, (r = 0, 1, 2, \ldots, \frac{r-1}{2})$. Since every odd degree vertex of G_r is an end vertex of a path in any path cover of G_r, we have $\pi(G_r) \geq \frac{2a}{2} = a$. Now the paths $(v_i, v_i', v_{2a}, v_{2r+i})$, $1 \leq i \leq 2r$, $(v_{4r+1}, v_{2a}, v_{4r+2})$, $(v_{4r+3}, v_{2a}, v_{4r+4})$, $\cdots$, $(v_{2a-3}, v_{2a}, v_{2a-2})$, $(v_{2a-1}v_{2a})$ form a path cover for G_r so that $\pi(G_r) \leq 2r + \frac{(2a-1)-(4r+1)}{2} + 1 = a$. Hence $\pi(G_r) = a$.

Next we prove that $\pi_P(G_r) = b$, where $a \leq b \leq 2a-1$. Now the paths $P_i = (v_i, v_i', v_{2a}, v_{2r+i}), 1 \leq i \leq 2r$ and the remaining edges form an OPPD ψ of G_r such that $\pi_P(G_r) \leq |\psi| = 2r + (2a-1+2r-6r) = 2a-(2r+1)$. Now let ψ be any minimum EQPPD of G_r. Since q is odd, ψ is an OPPD. Now it is clear that any OPPD ψ of G_r contains either all the edges of G_r or paths of length 3 together with the remaining edges. Hence it follows that $|\psi| \geq 2r+(2a-1+2r-6r) = 2a-(2r+1)$ so that $\pi_P(G_r) \geq 2a - (2r+1)$. Thus $\pi_P(G_r) = 2a - (2r+1)$ where $r = 0, 1, 2, \ldots, \frac{a-1}{2}$. Let $b = 2a - (2r+1), r = 0, 1, 2, \ldots, \frac{a-1}{2}$. Then $a \leq b \leq 2a - 1$. Thus $\pi_P(G_r) = b$, where $a \leq b \leq 2a - 1$.

Case (ii) a is even.

Since b is odd, we have $a+1 \leq b \leq 2a-1$. Let G_0 be a star graph with $v_1, v_2, \ldots, v_{2a-1}, v_{2a}$ as pendant vertices and v_{2a+1} as central vertex with a subdivision of the edge v_1v_{2a+1} by a vertex v_1' . Let G_r be a graph obtained from G_0 by subdividing $2r$ edges $v_2v_{2a+1}, v_3v_{2a+1}, \ldots, v_{2r}v_{2a+1}$, $v_{2r+1}v_{2a+1}$ of G_0 once by the vertices $v_2', v_3', \ldots, v_{2r}', v_{2r+1}'$, where $r = 1, 2, \ldots, \frac{a-2}{2}$ (Fig. 2). Note that $p = 2a + 2r + 2$ and $q = 2a + 2r + 1$.

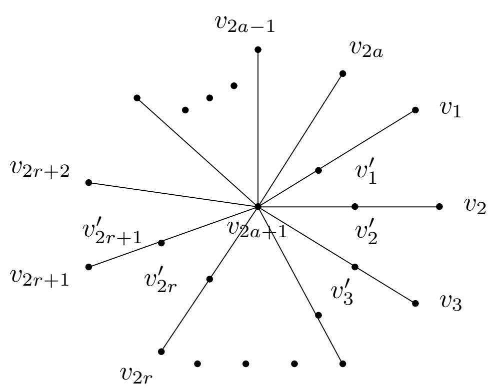

Fig.2

First we prove that $\pi(G_r) = a, (r = 0, 1, 2, \ldots, \frac{r-1}{2})$. Since every odd degree vertex of G_r is an end vertex of a path in any path cover of G_r, we have $\pi(G_r) \geq \frac{2a}{2} = a$. Now the paths $(v_i, v_i', v_{2a+1}, v_{2r+1+i})$, $1 \leq i \leq 2r+1$, $(v_{4r+3}, v_{2a+1}, v_{4r+4})$, $(v_{4r+5}, v_{2a+1}, v_{4r+5})$, $\cdots$, $(v_{2a-1}, v_{2a+1}, v_{2a})$ form a path cover for G_r so that $\pi(G_r) \leq 2r+1+\frac{(2a-1)-(4r+3)}{2} + 1 = a$. Hence $\pi(G_r) = a$.

Next we prove that $\pi_P(G_r) = b$, where $a+1 \leq b \leq 2a-1$. Now the paths $P_i = (v_i, v_i', v_{2a+1}, v_{2r+1+i}), 1 \leq i \leq 2r+1$ and the remaining edges form an OPPD ψ of G_r such that $\pi_P(G_r) \leq |\psi| = 2r+1+(2a+2r+1-6r-3) = 2a-(2r+1)$. Now let ψ be any minimum EQPPD of G_r. Since q is odd, ψ is an OPPD. Now it is clear that any OPPD ψ of G_r contains either all the edges of G_r or paths of length 3 together with the remaining edges. Hence it follows that $|\psi| \geq 2r+1+(2a+2r+1-6r-3) = 2a-(2r+1)$ so that $\pi_P(G_r) \geq 2a-(2r+1)$. Thus $\pi_P(G_r) = 2a-(2r+1)$ where $r = 0, 1, 2, \ldots, \frac{a-2}{2}$. Let $b = 2a-(2r+1), r = 0, 1, 2, \ldots, \frac{a-2}{2}$. Then $a+1 \leq b \leq 2a-1$. Thus $\pi_P(G_r) = b$, where $a+1 \leq b \leq 2a-1$. □

For the even number b, we make a problem as follows.

Problem 2.15 *If a is a positive integer and for every even b with $a \leq b \leq 2a-1$, then there exists a connected graph G such that $\pi(G) = a$ and $\pi_P(G) = b$.*

The following theorem gives the lower bound for π_P in terms of detour diameter D.

Theorem 2.16 *For any graph G, $\pi_P(G) \geq \lceil \frac{q}{D} \rceil$ where D is the detour diameter of G.*

Proof Let ψ be a minimum π_P-cover of G. Since every edge of G is in exactly one path in ψ we have $q = \sum\limits_{P\in\psi} |E(P)|$. Also $|E(P)| \leq D$ for each P in ψ. Hence $q \leq \pi_p D$ so that $\pi_P(G) \geq \lceil \frac{q}{D} \rceil$. □

The following theorem shows that the path covering number π of a graph G is same as the equiparity path decomposition numberπ_P of a subdivision graph of G.

Theorem 2.17 *For any graph G, $\pi(G) = \pi_P(S(G))$, where $S(G)$ is the subdivision graph of G.*

Proof As G and $S(G)$ are homeomorphic, $\pi(G) = \pi(S(G))$ and hence by Theorem 2.6, $\pi(G) \leq \pi_P(S(G))$. Now let $\psi = \{P_1, P_2, \cdots, P_\pi\}$ be a π-cover of G. Let $P_i^{'}, 1 \leq i \leq \pi$, be the path obtained from P_i by subdividing each edge P_i exactly once. Then $\psi' = \{P_1', P_2' \cdots, P_\pi'\}$ is an EPPD of $S(G)$ and hence $\pi_P(S(G)) \leq \pi(G)$. Thus $\pi(G) = \pi_P(S(G))$. □

In the following theorems we determine the value of the equiparity path decomposition number of several classes of graphs such as cycle, wheel, cubic graphs and complete graphs.

Theorem 2.18 *For a cycle C_p,*

$$\pi_P(C_p) = \begin{cases} 2 & if\ n\ is\ even, \\ 3 & if\ n\ is\ odd. \end{cases}$$

Proof Let $C = (v_1, v_2, \ldots, v_p, v_1)$.

If p even, then $\psi = \{(v_1, v_2, \ldots, v_{\frac{p}{2}}), (v_{\frac{p}{2}}, v_{\frac{p}{2}+1}, \ldots, v_p, v_1)\}$ is an EPPD, so that $\pi_P(C_p) \leq |\psi| = 2$ and further $\pi_P(C_p) \geq 2$ and hence $\pi_P(C_p) = 2$.

If p odd, then $\psi = \{(v_1, v_2, \ldots, v_{p-1}), (v_{p-1}, v_p), (v_p, v_1)\}$ is an OPPD, so that $\pi_P(C_p) \leq |\psi| = 3$. Since q is odd, it follows that $\pi_P(C_p)$ is odd. Then we have $\pi_P(C_p) \geq 3$. Hence $\pi_P(C_p) = 3$. □

Theorem 2.19 *For the wheel W_p on p vertices, we have $\pi_P(W_p) = \lfloor \frac{p}{2} \rfloor$.*

Proof Let $V(W_p) = \{v_1, v_2, \ldots, v_{p-1}, v_p\}$ and let $E(W_p) = \{v_i v_{i+1} : 1 \leq i \leq p-2\} \bigcup \{v_1 v_{p-1}\} \bigcup \{v_p v_i : 1 \leq i \leq p-1\}$. Let

$$\psi = \begin{cases} \{(v_{i+1}, v_i, v_p, v_{\frac{p-1}{2}+i}, v_{\frac{p+1}{2}+i}) : 1 \leq i \leq \frac{p-3}{2}\} \bigcup \{(v_{\frac{p+1}{2}}, v_{\frac{p-1}{2}}, v_p, v_{p-1}, v_1)\}, \text{if p is odd,} \\ \{(v_{i+1}, v_i, v_p, v_{\frac{p-2}{2}+i}, v_{\frac{p}{2}+i}) : 1 \leq i \leq \frac{p-2}{2}\} \bigcup \{(v_p, v_{p-1}, v_1)\}, \text{ if p iseven,} \end{cases}$$

then ψ is a EPPD with $|\psi| = \lfloor \frac{p}{2} \rfloor$ and hence $\pi_P(W_p) \leq \lfloor \frac{p}{2} \rfloor$. Since every odd degree vertex of W_p is an end vertex of a path in any path cover of W_p, we have $\pi_P(W_p) \geq \lfloor \frac{p}{2} \rfloor$. Then $\pi_P(W_p) = \lfloor \frac{p}{2} \rfloor$. □

Theorem 2.20 *For a 3-regular graph G, $\pi_P(G) = \frac{p}{2}$.*

Proof It follows from Theorem 1.3 that every 3-regular graph is P_4 decomposable and hence $\pi_P(G) \leq \frac{q}{3} = \frac{p}{2}$. Further, since every vertex of G is of odd degree, they are the end vertices of paths in any path cover of G. So, we have $\pi_P(G) \geq \frac{p}{2}$. Thus $\pi_P(G) = \frac{p}{2}$. □

Theorem 2.21 *For any $n \geq 1$, $\pi_P(K_{2n}) = n$.*

Proof From Theorems 1.4 and 2.6, it follows that $\pi_P(K_{2n}) \leq n$. Further, since every vertex of K_{2n} is of odd degree, they are the end vertices of paths in any path cover of K_{2n}. So, we have $\pi_P(K_{2n}) \geq n$ and hence $\pi_P(K_{2n}) = n$. □

Theorem 2.22 *For any $n \geq 1$,*

$$\pi_P(K_{2n+1}) = \begin{cases} n+1 & if\ n\ is\ even, \\ n+2 & if\ n\ is\ odd. \end{cases}$$

Proof Let $V(K_{2n+1}) = \{v_1, v_2, \cdots, v_{2n+1}\}$.

Case (i) n is even.

Consider paths following:

$P_1 = (v_{2n+1}, v_3, v_{2n}, v_4, v_{2n-1}, \ldots, v_n, v_{n+3}, v_{n+1}, v_{n+2}, v_1, v_2)$,

$P_2 = (v_2, v_4, v_{2n+1}, v_5, v_{2n}, \cdots, v_{n+1}, v_{n+4}, v_{n+2}, v_{n+3}, v_1, v_3)$,

$P_3 = (v_3, v_5, v_2, v_6, v_{2n+1}, \cdots, v_{n+2}, v_{n+5}, v_{n+3}, v_{n+4}, v_1, v_4)$,

$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$,

$P_n = (v_n, v_{n+2}, v_{n-1}, v_{n+3}, v_{n-2}, \cdots, v_{2n-1}, v_2, v_{2n}, v_{2n+1}, v_1, v_{n+1})$,

$P_{n+1} = (v_{2n+1}, v_2, v_3, v_4, v_5, cldots, v_{n-1}, v_n, v_{n+1})$.

The paths P_i $(1 \leq i \leq n)$ can be obtained from n hamiltonian cycles of K_{2n+1} by removing an edge from each cycle and the path P_{n+1} is obtained by joining the removed edges. It follows that the lengths of $P_i, 1 \leq i \leq n$ are $2n$ and the length of P_{n+1} is n, so that $\psi = \{P_1, P_2, \ldots, P_n, P_{n+1}\}$ is an EPPD and hence $\pi_{_P}(K_{2n+1}) \leq |\psi| = n+1$. From Theorems 1.4 and 2.6, it follows that $\pi_p(K_{2n+1}) \geq n+1$ and hence $\pi_P(K_{2n+1}) = n+1$.

Case (ii) n is odd.

Consider the hamilton cycles of K_{2n+1}

$C_1 = (v_1, v_2, v_{2n+1}, v_3, v_{2n}, v_4, v_{2n-1}, \ldots, v_n, v_{n+3}, v_{n+1}, v_{n+2}, v_1)$,
$C_2 = (v_1, v_3, v_2, v_4, v_{2n+1}, v_5, v_{2n}, \ldots, v_{n+1}, v_{n+4}, v_{n+2}, v_{n+3}, v_1)$,
$C_3 = (v_1, v_4, v_3, v_5, v_2, v_6, v_{2n+1}, \ldots, v_{n+2}, v_{n+5}, v_{n+3}, v_{n+4}, v_1)$,
$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$,
$C_{\frac{n-1}{2}} = (v_1, v_{\frac{n+1}{2}}, v_{\frac{n-1}{2}}, v_{\frac{n+3}{2}}, v_{\frac{n-3}{2}}, \ldots, v_{\frac{3n-3}{2}}, v_{\frac{3n+3}{2}}, v_{\frac{3n-1}{2}}, v_{\frac{3n+1}{2}}, v_1)$,
$C_{\frac{n+1}{2}} = (v_1, v_{\frac{n+3}{2}}, v_{\frac{n+1}{2}}, v_{\frac{n+5}{2}}, v_{\frac{n-1}{2}}, \ldots, v_{\frac{3n-1}{2}}, v_{\frac{3n+5}{2}}, v_{\frac{3n+1}{2}}, v_{\frac{3n+3}{2}}, v_1)$.
$C_{\frac{n+3}{2}} = (v_1, v_{\frac{n+5}{2}}, v_{\frac{n+3}{2}}, v_{\frac{n+7}{2}}, v_{\frac{n+1}{2}}, \ldots, v_{\frac{3n+1}{2}}, v_{\frac{3n+7}{2}}, v_{\frac{3n+3}{2}}, v_{\frac{3n+5}{2}}, v_1)$,
$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$,
$C_{n-1} = (v_1, v_n, v_{n-1}, v_{n+1}, v_{n-2}, v_{n+2}, v_{n-3}, \cdots, v_{2n-2}, v_{2n+1}, v_{2n-1}, v_{2n}, v_1)$,
$C_n = (v_1, v_{n+1}, v_n, v_{n+2}, v_{n-1}, v_{n+3}, v_{n-2}, \ldots, v_{2n-1}, v_2, v_{2n}, v_{2n+1}, v_1)$.

We will construct the following paths from the above hamilton cycles. Let
$P_1 = C_1 - (v_1, v_2, v_{2n+1})$,
$P_2 = C_2 - (v_{2n+1}, v_5, v_{2n})$,
$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$,
$P_{\frac{n-1}{2}} = C_{\frac{n-1}{2}} - (v_{\frac{3n+7}{2}}, v_{\frac{3n-5}{2}}, v_{\frac{3n+5}{2}})$,
$P_{\frac{n+1}{2}} = C_{\frac{n+1}{2}} - (v_{\frac{3n+5}{2}}, v_{\frac{3n+1}{2}}, v_{\frac{3n+3}{2}})$,
$P_{\frac{n+3}{2}} = C_{\frac{n+3}{2}} - (v_{\frac{3n-1}{2}}, v_{\frac{3n+7}{2}}, v_{\frac{3n+1}{2}})$,
$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$,
$P_{n-1} = C_{n-1} - (v_{n+2}, v_{n-3}, v_{n+3})$,
$P_n = C_n - (v_{n+1}, v_n, v_{n+2})$,
$P_{n+1} = (v_1, v_2, v_{2n+1}, v_5, v_{2n}, \ldots, v_{\frac{3n+7}{2}}, v_{\frac{3n-5}{2}}, v_{\frac{3n+5}{2}}, v_{\frac{3n+1}{2}})$,
$P_{n+2} = (v_{\frac{3n+3}{2}}, v_{\frac{3n+1}{2}}, v_{\frac{3n+7}{2}}, v_{\frac{3n-1}{2}}, \ldots, v_{n+3}, v_{n-3}, v_{n+2}, v_n, v_{n+1})$.

The paths P_i $(1 \le i \le n)$ can be obtained from n hamiltonian cycles of K_{2n+1} by removing two adjacent edges from each cycle and the paths P_{n+1} and P_{n+2} are obtained by joining the removed edges. It follows that the lengths of $P_i, 1 \le i \le n$ are $2n-1$ and the lengths of P_{n+1} and P_{n+2} are n, so that $\psi = \{P_1, P_2, \cdots, P_n, P_{n+1}\}$ is an OPPD and hence $\pi_P(K_{2n+1}) \le |\psi| = n+2$. From Theorems 1.4 and T2.6 , it follows that $\pi_P(K_{2n+1}) \ge n+1$. Now , since n is odd, $q = n(2n+1)$ is odd. Thus $\pi_P(K_{2n+1})$ is odd, so that $\pi_P(K_{2n+1}) \ge n+2$ and hence $\pi_P(K_{2n+1}) = n+2$. □

We now proceed to obtain upper bounds for π_p involving circumference of a graph and characterize graphs attaining the bounds.

Theorem 2.23 *For a graph G, $\pi_P(G) \le q-c+3$, where c is the circumference of G. Further, equality holds if and only if G is an odd cycle.*

Proof Let C be a longest cycle of length c. Let c be even. Then the path of length $c-1$, together with the remaining edges form an OPPD and hence $\pi_P(G) \le q-(c-1)+1 = q-c+2$. Let c be odd. Then path of length $p-2$, together with the remaining edges form an OPPD and hence $\pi_P(G) \le q-(c-2)+1 = q-c+3$. Thus from both the cases, it follows that $\pi_P(G) \le q-c+3$.

Suppose G is a graph with $\pi_P(G) = q-c+3$. Let $C = (v_1, v_2, \ldots, v_c, v_1)$ be a longest cycle in G. If c is even, then as in the first paragraph of the proof, $\pi_P(G) \le q-c+2$ and so c is odd.

Now, we claim that C has no chords. Suppose it is not. Let $e = v_1v_i$ be a chord in C. Let $P_1 = (v_1, v_2, \ldots, v_{c-1})$ and $P_2 = (v_{c-1}, v_c, v_1, v_i)$. Since c is odd, $\psi = \{P_1, P_2\} \bigcup S$ where S is the set of edges of G not covered by P_1, P_2 is an OPPD of G such that $|\psi| < q - c + 3$, which is a contradiction. Thus C has no chords.

Next, we claim that $V(G) = V(C)$. Suppose there exists a vertex v not on C adjacent to a vertex of C, say v_1. Let $P_1 = (v_1, v_2, \ldots, v_{c-1})$ and $P_2 = (v_{c-1}, v_c, v_1, v)$. Since c is odd, $\psi = \{P_1, P_2\} \bigcup S$ where S is the set of edges of G not covered by P_1, P_2 is an OPPD of G such that $|\psi| < q - c + 3$, which is a contradiction. Then it follows that $V(G) = V(C)$. Thus G is an odd cycle.

The converse is obvious. □

Theorem 2.24 *For a graph G, $\pi_P(G) = q - c + 2$ if and only if G is either an even cycle or a $\theta-$graph of odd size or a kite with tail length 1 of odd size.*

Proof Clearly, the result is true for $p = 3, 4$ and 5. So we assume that $p \geq 6$.

Suppose $\pi_P(G) = q - c + 2$. Let $C = (v_1, v_2, \ldots, v_c, v_1)$ be a longest cycle in G.

Claim 1 c is even.

Suppose c is odd. Since the value of π_P for an odd cycle is $q - c + 3$, it follows that $G \neq C$. Hence C has a chord, say $e = v_1v_i$ (Fig.3).

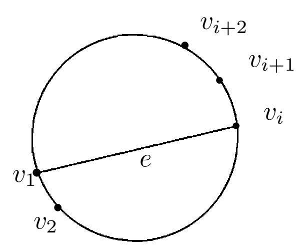

Fig. 3

Let $P_1 = (v_1, v_i, v_{i+1}, v_{i+2})$ and $P_2 = (v_{i+2}, v_{i+3}, \cdots, v_c, v_1, v_2, \ldots, v_i)$. Then $\psi = \{P_1, P_2\} \bigcup S$ where S is the set of edges of G not covered by P_1, P_2 is an OPPD of G such that $|\psi| < q - c + 2$, which is a contradiction.

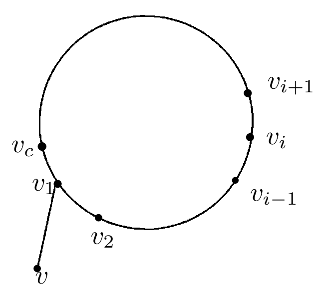

Fig. 4

Hence there is a vertex v not on C adjacent to a vertex of C, say v_1 (Fig.4). Let $P_1 = (v, v_1, v_2, v_3)$ and $P_2 = (v_3, v_4, \ldots, v_{c-1}, v_c, v_1)$. Since c is odd, $\psi = \{P_1, P_2\} \bigcup S$ where S is the set of edges of G not covered by P_1, P_2 is an OPPD of G such that $|\psi| < q - c + 2$, which is a contradiction. Thus c is even.

Case (i) $V(G) = V(C)$

We now prove that C has at most one chord.

Claim 2 No two chords of C are adjacent.

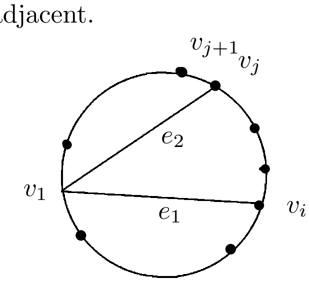

Fig.5

Suppose there exists two adjacent chords $e_1 = v_1v_i$ and $e_2 = v_1v_j$ $(1 < i < j)$ in C (Fig 5). Let P_1 be the (v_j, v_{j+1})-section of C containing v_1 and let $P_2 = (v_{j+1}, v_j, v_1, v_i)$. From Claim 1, it follows that $\psi = \{P_1, P_2\} \bigcup S$ where S is the set of edges of G not covered by P_1, P_2 is an OPPD of G such that $|\psi| < q - c + 2$, which is a contradiction. Thus no two chords of C are adjacent.

Next we define some sections of cycle.

A section C' of length greater than 1 of a cycle C is said to be of type 1 if the end vertices of C' are adjacent and no internal vertex of C' is an end vertex of a chord of C.

A section C' of a cycle C is said to be of type 2 if the end vertices of C' are the end vertices of two different chords of C and no internal vertex of C' is an end vertex of a chord of C.

Claim 3 The type 2 sections of C formed by any two nonadjacent chords are of even length.

Let e_1 and e_2 be two nonadjacent chords of C. Then the choices of e_1 and e_2 are as in the following figure (Fig.6).

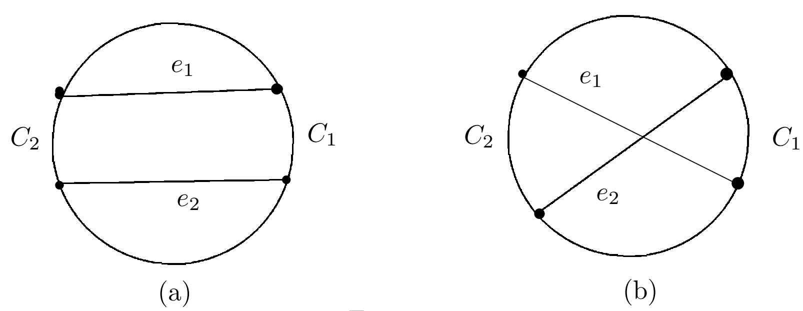

Fig. 6

Let C_1 and C_2 be the sections of C. We now claim that the section C_1 is of even length. Suppose not. Now, let $P_1 = e_1 \circ C_1 \circ e_2$ and $P_2 = C - C_1$. Then it follows from Claim 1 that

$\psi = \{P_1, P_2\} \bigcup S$ where S is the set of edges of G not covered by P_1, P_2 is an OPPD of G such that $|\psi| < q - c + 2$, which is a contradiction. Hence the section C_1 is of even length. Similarly, we can prove that the section C_2 is of even length.

Claim 4 Type 1 sections of C formed by three mutually disjoint chords are of even length.

Let e_1,e_2 and e_3 be three mutually disjoint chords of C. Let C_1 be a type 1 section of C formed by e_1. We now claim that C_1 is of even length. Suppose not. Then by claim 1, $C - C_1$ is of odd length. Now, since there are exactly six sections of either type 1 or type 2 formed by e_1, e_2 and e_3 in C and since $C - C_1$ is of odd length, it follows from claim 3 that there is a type 1 section C_2 of odd length formed either by e_2 or e_3, say e_2. then the chord e_3 is as in the following Fig. 7.

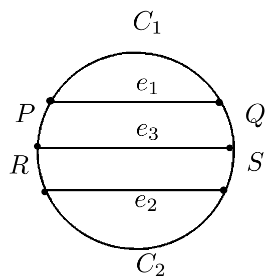

Fig.7

Let P, Q, R and S denote the remaining type 2 sections of C as in Fig. 7. Then it follows from claim 3 that the sections P, Q, R and S are of even length. Now, let $P_1 = e_2 \circ S \circ e_3 \circ P \circ e_1$, $P_2 = C_1 \circ Q$, $P_3 = C_2 \circ R$. Then $\psi = \{P_1, P_2, P_3\} \bigcup S$ where S is the set of edges of G not covered by P_1, P_2 and P_3 is an OPPD of G such that $|\psi| < q - c + 2$, which is a contradiction. Thus the type 1 sections of C formed by three mutually disjoint chords are of even length.

Claim 5 C has at most one chord.

Suppose C has exactly two chords, say e_1 and e_2. Then by Claim 2 the choices of e_1 and e_2 are as in Fig. 6. Also, there are exactly 4 sections of type 1 or type 2 in C, say C_1, C_2, C_3 and C_4. Suppose e_1 and e_2 are as in Fig. 6(b). Then the sections C_1, C_2, C_3 and C_4 are of type 2 and hence it follows from Claim 3 that each is of even length.

Now, let $P_1 = e_1 \circ C_2 \circ e_2$ and $P_2 = C_3 \circ C_1 \circ C_4$. Then $\psi = \{P_1, P_2\}$ is an EPPD of G and hence $\pi_P(G) = 2 < q - c + 2$, which is a contradiction.

Suppose e_1 and e_2 are as in Fig. 6(a). Then the sections C_1 and C_3 are of type 1 and the sections C_2 and C_4 are of type 2 and hence it follows from Claims 3 4 that each is of even length.

Now, let $P_1 = e_1 \circ C_2 \circ e_2$ and $P_2 = C_3 \circ C_1 \circ C_4$. Then $\psi = \{P_1, P_2\}$ is an EPPD of G and hence $\pi_P(G) = 2 < q - c + 2$, which is a contradiction.

Thus C does not have exactly two chords.

Suppose C has 3 chords, say e_1, e_2 and e_3. Then by Claim 2 the choices of e_1, e_2 and e_3 are as in Fig. 8.

Also, there are exactly 6 sections of types 1 or 2 in C, say C_1, C_2, C_3, C_4, C_5 and C_6. By Claim 3 and 4, any section of C formed by the chords is of even length and so C_1, C_2, C_3, C_4, C_5

and C_6 are of even length.

If e_1, e_2 and e_3 are as in Fig 8(a), let $P_1 = C_1 \circ C_6 \circ e_2$, $P_2 = C_5 \circ e_1$ and $P_3 = e_3 \circ C_2 \circ C_3 \circ C_4$. If e_1, e_2 and e_3 are as in Fig. 8(b), let $P_1 = C_1 \circ C_6 \circ e_2$, $P_2 = C_4 \circ C_3 \circ C_2 \circ e_3$ and $P_3 = C_5 \circ e_1$.

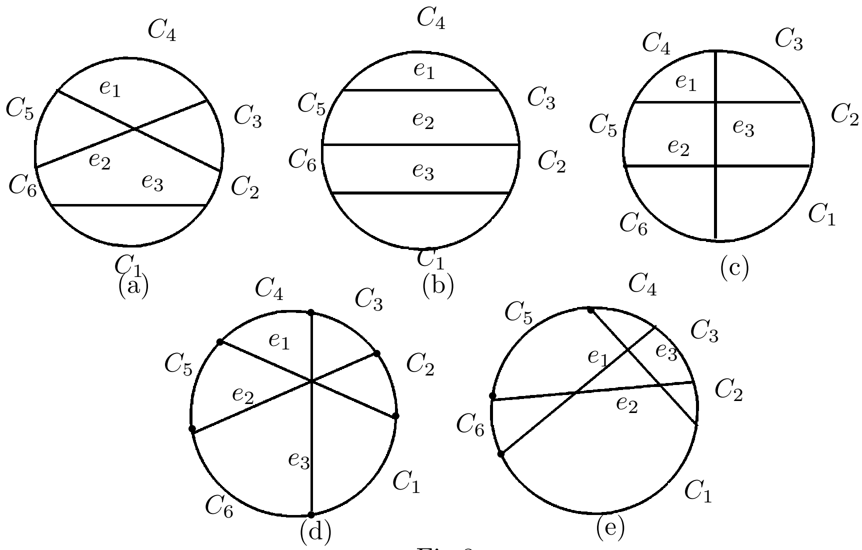

Fig.8

If e_1, e_2 and e_3 are as in Fig. 8(c), let $P_1 = C_1 \circ e_3 \circ C_4$, $P_2 = C_3 \circ C_2 \circ e_2$ and $P_3 = C_6 \circ C_5 \circ e_1$. If e_1, e_2 and e_3 are as in Fig.8(d), let $P_1 = C_6 \circ C_1 \circ e_1$, $P_2 = C_2 \circ C_3 \circ e_3$ and $P_3 = C_4 \circ C_5 \circ e_2$. If e_1, e_2 and e_3 are as in Fig.8(e), let $P_1 = e_1 \circ C_1 \circ C_2$, $P_2 = C_4 \circ C_3 \circ e_2$ and $P_3 = C_6 \circ C_5 \circ e_3$. Then $\psi = \{P_1, P_2, P_3\} \bigcup S$ where S is the set of edges of G not covered by P_1, P_2 and P_3 is an OPPD of G such that $|\psi| < q - c + 2$, which is a contradiction. Thus by Claims 1 and 5, G is either an even cycle or a $\theta-$graph of odd size.

Case (ii) $V(G) \neq V(C)$

Let $C = (v_1, v_2, \ldots, v_c, v_1)$ be a longest cycle of length c in G.

Claim 6 Every vertex not on C is a pendant vertex.

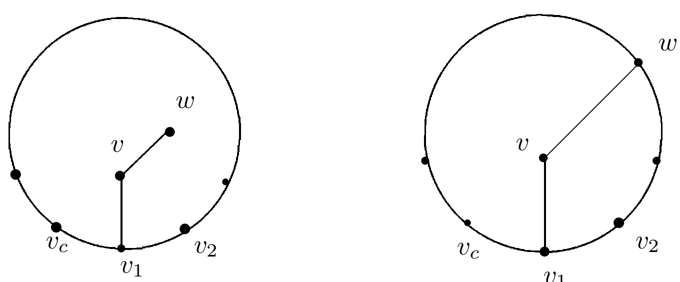

Fig.9

Suppose there exists a vertex v with $degv \geq 2$, not on C adjacent to a vertex of C, say v_1. Let w be a vertex which is adjacent to v. Note that w may be either on C or not on C (Fig 9) . Let $P_1 = (v_1, v_2, \ldots, v_c)$ and $P_2 = (v_c, v_1, v, w)$. Then $\psi = \{P_1, P_2\} \bigcup S$, where S is the set of edges of G not covered by P_1, P_2 is an OPPD of G such that $|\psi| < q - c + 2$ which is a contradiction. Thus every vertex not on C is a pendant vertex.

Claim 7 The cycle C has no chord.

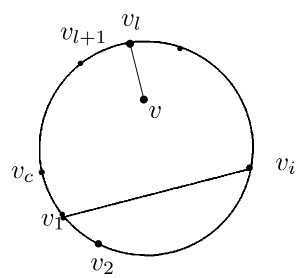

Fig.10

Suppose C has a chord, say v_1v_i (Fig 10). Let v be a pendant vertex not on C, which is adjacent to some vertex, say v_l on C. Suppose v_l is different from v_1 and v_i. If (v_1, v_l)- section is odd, then let $P_1 = (v, v_l, v_{l+1}, \cdots, v_c, v_1, v_i)$ and $P_2 = (v_1, v_2, \cdots, v_i, v_{i+1}, \cdots, v_l)$ and if that section is even, then let $P_1 = (v_l, v_{l+1}, \ldots, v_c, v_1, v_i)$ and $P_2 = (v_1, v_2, \ldots, v_i, v_{i+1}, \ldots, v_l, v)$.

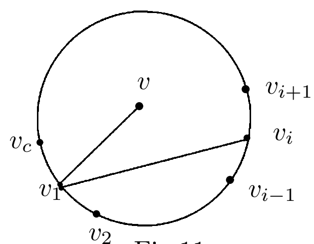

Fig.11

Suppose v_l is either v_1 or v_i. Without loss of generality, let $v_l = v_1$(Fig 11). Let $P_1 = (v, v_1, v_i, v_{i+1})$ and $P_2 = (v_{i+1}, v_{i+2}, \ldots, v_c, v_1, v_2, \ldots, v_{i-1}, v_i)$. Then $\psi = \{P_1, P_2\} \bigcup S$, where S is the set of edges of G not covered by P_1, P_2 is an OPPD of G such that $|\psi| < q - c + 2$ which is a contradiction. Hence the cycle C has no chord. Thus G is a unicyclic graph.

Claim 8 Every vertex on C has degree less than or equal to 3.

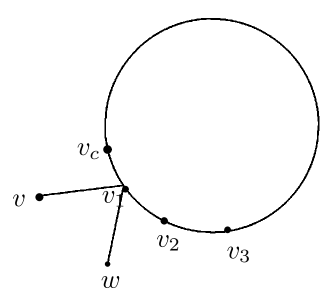

Fig.12

Suppose there is a vertex, say v_1 on C with degree of $v \geq 4$ (Fig 12). From Claims 6 and 7, it follows that there are two pendant vertices, say v and w not on C which are adjacent to v_1. Let $P_1 = (w, v_1, v_2, v_3)$ and $P_2 = (v_3, v_4, \ldots, v_c, v_1, v)$. Since c is even, $\psi = \{P_1, P_2\} \bigcup S$, where S is the set of edges of G not covered by P_1, P_2 is an OPPD of G such that $|\psi| < q - c + 2$ which is a contradiction. Thus every vertex on C has degree less than or equal to 3.

Claim 9 Exactly one vertex on C has degree 3.

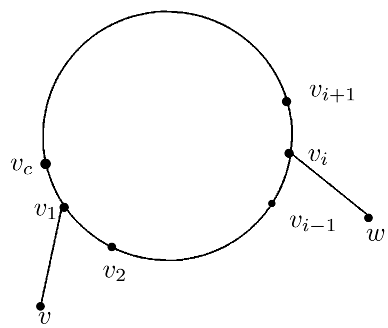

Fig.13

Suppose there are two vertices on C have degree 3, say v_1 and v_i (Fig. 13). By claim 6, there are two pendant vertices v and w not on C, adjacent to v_1 and v_i respectively. If the length of (v_1, v_i)- section not containing v_c is odd, then let $P_1 = (v, v_1, v_2, \ldots, v_{i-1}, v_i, w)$ and $P_2 = (v_i, v_{i+1}, \ldots, v_c, v_1)$ and if that section is even, then let $P_1 = (v, v_1, v_2, \ldots, v_{i-1}, v_i)$ and $P_2 = (w, v_i, v_{i+1}, \ldots, v_c, v_1)$. Since c is even, $\psi = \{P_1, P_2\} \bigcup S$, where S is the set of edges of G not covered by P_1, P_2 is an OPPD of G such that $|\psi| < q - c + 2$ which is a contradiction. Thus exactly one vertex on C has degree 3. Thus G is a kite with tail length 1 of odd size.

The converse is obvious. □

Remark 2.25 In the Theorem 2.24, for the case $V(G) = V(C)$, we have $c = p$ and the condition becomes $\pi_P(G) = q - p + 2$.

We conclude this paper by posing the following problems for further investigation.

(*i*) *For a tree T of even size, prove that $\pi(T) = \pi_P(T)$.*

(*ii*) *If G is a unicyclic graph, find $\pi_P(G)$.*

(*iii*) *For a graph G of even size, prove that $\pi(G) \leq \pi_P(G) \leq \pi(G) + 1$.*

References

[1] B. D. Acharya and E. Sampathkumar, Graphoidal covers and graphoidal covering number of a graph, *Indian J. pure appl. Math.,* **18** (10)(1987), 882 - 890.

[2] S. Arumugam and I. Sahul Hamid, Simple path covers in graphs, *International Journal of Mathematical Combinatorics,* **3** (2008), 94 - 104.

[3] S. Arumugam and I. Sahul Hamid, Simple Acyclic graphoidal covers in a graph, *Australasian Journal of Combinatorics,* **37** (2007), 243 - 255.

[4] S. Arumugam and J. Suresh Suseela, Acyclic graphoidal covers and path partitions in a graph, *Discrete Math.,* **190**(1998), 67 - 77.

[5] G. Chartrand and L. Lesniak, *Graphs and Digraphs,* Fourth Edition, CRC Press, Boca Raton, 2004.

[6] F. Harary,*Graph Theory,* Addison-Wesley, Reading, Mass, 1972.

[7] F. Harary and A. J. Schwenk, Evolution of the path number of a graph, covering and packing in graphs II, *Graph Theory and Computing,* Ed. R. C. Road, Academic Press, New York, (1972), 39 - 45.

[8] K. Nagarajan, A. Nagarajan and S. Somasundram, 2-graphoidal Path Covers *International Journal of Applied Mathematics,* **21**(4) (2008), 615 - 628.

[9] B. Peroche, The path number of some multipartite graphs, *Annals of Discrete Math.,* **9**(1982), 193 - 197.

[10] R. G. Stanton, D. D. Cowan and L. O. James, Some results on path numbers, *Proc. Louisiana Conf. on Combinatorics, Graph Theory and computing,* (1970), 112 - 135.