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On the Basis Number and the Minimum Cycle Bases of the Wreath Product of Two Wheels

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Abstract: A construction of a minimum cycle bases for the wreath product of two wheels is presented. Moreover, the basis numbers for the wreath product of the same classes are investigated.

Key Words: Cycle space, basis number, minimum cycle basis, wreath product.

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§1. Introduction.

Cycle bases of a cycle space have a variety of applications which go back at least as far as Kirchoff's treatise on electrical network [20]. The required bases have been used to give rise to a better understanding and interpretations of the geometric properties of a given graph when MacLane [21] made a connection between the planarity of a graph G and the number of occurrence of edges of G in elements of cycle bases. Recently, the minimum cycle bases are employed in sciences and engineering; for examples, in structural flexibility analysis [19], in chemical structure and in retrieval systems [7] and [9].

In this paper, we investigate the basis number for the wreath product of two wheels and we construct minimum cycle bases for same; also, we give their total length and the length of the longest cycles.

§2. Definitions and Preliminaries

Recall that for a given simple graph $G = (V(G), E(G))$ the set $\mathcal{E}$ of all subsets of $E(G)$ forms an $|E(G)|$-dimensional vector space over Z_2 with vector addition $X \oplus Y = (X\backslash Y) \cup (Y\backslash X)$ and scalar multiplication $1 \cdot X = X$ and $0 \cdot X = \emptyset$ for all $X, Y \in \mathcal{E}$. The *cycle space*, $\mathcal{C}(G)$, of a graph G is the vector subspace of $(\mathcal{E}, \oplus, \cdot)$ spanned by the cycles of G. Note that the non-zero elements of $\mathcal{C}(G)$ are cycles and edge disjoint union of cycles. It is known that the dimension of the cycle space is the *cyclomatic number* or the *first Betti number* $\dim \mathcal{C}(G) = |E(G)| - |V(G)| + r$ where r is the number of components (see [8]).

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A basis $\mathcal{B}$ for $\mathcal{C}(G)$ is *cycle basis* of G. A cycle basis $\mathcal{B}$ of G is called a d-fold if each edge of G occurs in at most d of the cycles in $\mathcal{B}$. *The basis number,* $b(G)$, of G is the least non-negative integer d such that $\mathcal{C}(G)$ has a *d-fold* basis. A *required basis* of $\mathcal{C}(G)$ is a $b(G)$-fold basis. The *length $l(\mathcal{B})$ of a cycle basis* $\mathcal{B}$ is the sum of the lengths of its elements: $l(\mathcal{B}) = \sum_{C\in\mathcal{B}} |C|$. $\lambda(G)$ is defined to be the minimum length of the longest element in an arbitrary cycle basis of G. A *minimum cycle basis* (MCB) is a cycle basis with minimum length. Since the cycle space $\mathcal{C}(G)$ is a matroid in which an element C has weight $|C|$, the greedy algorithm can be used to extract a MCB (see [24]). Chickering, Geiger and Heckerman [6], showed that $\lambda(G)$ is the length of the longest element in a MCB.

Horton [12] presents a polynomial time algorithm that finds a minimum cycle basis in any graph, but the algorithm approach can lead us to miss deeper connections between the structures of graphs and their cycle bases. Therefore, some authors have directly constructed minimum cycle bases and determined the basis number for certain classes of graphs (see [3], [22] and [23]).

Recently, the study of minimum cycle bases and basis numbers of graph products have attracted many authors: Imrich and Stadler [14], Ali and Marougi [2] and Jaradat [16] have each constructed minimum cycle bases and given upper bounds on the basis number of the Cartesian and strong products. Also, Alsardary and Wojciechowski [4] gave an upper bound on the basis number of the Cartesian products of complete graphs. Hammack [10] constructed a minimum cycle basis of the direct product of two bipartite graphs and Jaradat [15] gave an upper bound on the basis number of the same. Most recently, Hammack [11] presented a minimum cycle basis of the direct product of two complete graphs of order greater than 2. Jaradat [16] and Jaradat and Al-Qeyyam [5] investigated basis numbers and constructed minimum cycle bases for certain classes of graphs.

For completeness, we recall the following definitions: Let G and H be two graphs. Then

(1) the Cartesian product $G\Box H$ is the graph whose vertex set is the Cartesian product $V(G)\times V(H)$ and whose edge set is $E(G\Box H) = \{(u_1,v_1)(u_2,v_2)|u_1u_2 \in E(G)$ and $v_1 = v_2,$ or $v_1v_2 \in E(H)$ and $u_1 = u_2\}$.

(2) the lexicographic product $G_1[G_2]$ is the graph with vertex set $V(G)\times V(H)$ and edge set $E(G[H]) = \{(u_1,u_2)(v_1,v_2)|u_1 = v_1$ and $u_2v_2 \in E(H)$ or $u_1v_1 \in E(G)\}$ and the wreath product $G \ltimes H$ is the graph with vertex set $V(G)\times V(H)$ and edge set $E(G\rho H) = \{(u_1,v_1)(u_2,v_2)|u_1 = u_2$ and $v_1v_2 \in H,$ or $u_1u_2 \in G$ and there is $\alpha \in$Aut(H) such that $\alpha(v_1) = v_2\}$ (see [1] and [13]).

The following results will be used frequently in the sequel.

Theorem 2.1(MacLane [21]) *A graph G is planar if and only if $b(G) \leq 2$.*

Lemma 2.2 (Jaradat, et al. [18]) *Let A, B be sets of cycles of a graph G, and suppose that both A and B are linearly independent, and that $E(A)\cap E(B)$ induces a forest in G (we allow the possibility that $E(A)\cap E(B) = \varnothing$). Then $A\cup B$ is linearly independent.*

In this paper, we continue the study initiated in [5] and [17] by investigating the basis number for the wreath products of two wheels W_n and W_m. Moreover, we construct a minimum cycle basis and we give the total lengths and the lengths of longest cycles of the minimum cycle bases of the same.

In the rest of this paper, we let $\{u_1, u_2, \ldots, u_n\}$ be the vertex set of W_n (the star S_n), with $d_{W_n}(u_1) = n-1$ ($d_{S_n}(u_1) = n-1$), and $\{v_1, v_2, \ldots, v_m\}$ be the vertex set W_m (the star S_m), with $d_{W_m}(v_1) = m-1$ ($d_{S_m}(u_1) = m-1$). Wherever they appear a, b, c and l stand for vertices and abc, lab are paths of order 3. Also, $f_B(e)$ stands for the number of elements of B containing the edge e, and $E(B) = \cup_{C\in B} E(C)$ where $B \subseteq \mathcal{C}(G)$.

§3. The Basis Number of $W_n \rho W_m$

In this section, we investigate the basis number of the wreath product of two wheels. Throughout this work we use the notations $\mathcal{V}_{ab}^{(k)}$ and $\mathcal{U}_{lab}^{(k)}$ which were introduced by Jaradat [17] and Al-Qeyyam and Jaradat [5]: For each $k = 1, 2, \ldots, m$,

$$\begin{aligned} \mathcal{V}_{ab}^{(k)} &= \left\{ \mathcal{V}_{ab}^{(k,j)} = (b, v_k)(a, v_j)(a, v_{j+1})(b, v_k) \mid 2 \le j \le m-1 \right\}, \\ \mathcal{U}_{lab}^{(k)} &= \{(l, v_k)(a, v_k)(b, v_k)(l, v_k)\}, \end{aligned}$$

and

$$\mathcal{H}_{ab} = \{(a, v_j)(b, v_i)(a, v_{j+1})(b, v_{i+1})(a, v_j) \mid 2 \le i, j \le m-1\}.$$

Note that $\mathcal{H}_{ab}$ is Schemeichel's 4-fold basis of $\mathcal{C}(ab[N_{m-1}])$ (see Theorem 2.4 in [22]). Moreover, (1) if $e = (a, v_2)(b, v_m)$ or $e = (a, v_m)(b, v_2)$ or $e = (a, v_2)(b, v_2)$ or $e = (a, v_m)(b, v_m)$, then $f_{\mathcal{H}_{ab}}(e) = 1$; (2) if $e = (a, v_2)(b, v_l)$ or $(a, v_j)(b, v_2)$ or $(a, v_m)(b, v_l)$ or $(a, v_j)(b, v_m)$, then $f_{\mathcal{H}_{ab}}(e) \le 2$; and (3) If $e \in E(ab[N_{m-1}])$ and is not of the above forms, then $f_{\mathcal{H}_{ab}}(e) \le 4$.

The following result of Jaradat [17] will be needed in the sequel.

Lemma 3.1 **([17])** *$(\cup_{k=2}^{m} \mathcal{V}_{ab}^{(k)}) \cup (\mathcal{V}_{ba}^{(l)})$ is linearly independent for any $2 \le l \le m$.*

Let

$$\mathcal{D}_{lab} = \mathcal{U}_{lab}^{(m)} \cup \mathcal{H}_{ab} \cup \mathcal{V}_{ba}^{(2)} \cup \mathcal{V}_{ab}^{(2)} \cup \mathcal{U}_{lab}^{(1)}.$$

Lemma 3.2 *$\mathcal{D}_{lab}$ is linearly independent.*

Proof By Schmeichel's Theorems and Lemma 3.1, each of $\mathcal{H}_{ab}, \mathcal{V}_{ba}^{(2)}$ and $\mathcal{V}_{ab}^{(2)}$ is linearly independent. Since $E(\mathcal{U}_{lab}^{(m)}) \cap E(\mathcal{H}_{ab}) = \{(a, v_m)(b, v_m)\}$ which is an edge, $\mathcal{U}_{lab}^{(m)} \cup \mathcal{H}_{ab}$, is linearly independent by Lemma 2.2. By specializing $l = 2$ in Lemma 3.1, we have that $\mathcal{V}_{ba}^{(2)} \cup \mathcal{V}_{ab}^{(2)}$ is linearly independent. Since $E(\mathcal{V}_{ba}^{(2)}) \cup E(\mathcal{V}_{ab}^{(2)}) - \{(a, v_j)(a, v_{j+1}), (b, v_j)(b, v_{j+1}) : 2 \le j \le m-1\}$ is a tree and since any linear combinations of cycles is a cycle or an edge disjoint union of cycles, any linear combination of cycles of $\mathcal{V}_{ba}^{(2)} \cup \mathcal{V}_{ab}^{(2)}$ must contain an edge of the form $(a, v_j)(a, v_{j+1})$

or $(b, v_j)(b, v_{j+1})$ which is not in any cycle of $\mathcal{U}_{lab}^{(m)} \cup \mathcal{H}_{ab}$. Thus, $\mathcal{U}_{lab}^{(m)} \cup \mathcal{H}_{ab} \cup \mathcal{V}_{ba}^{(2)} \cup \mathcal{V}_{ab}^{(2)}$ is linearly independent. Note that $E(\mathcal{U}_{lab}^{(1)}) \cap E(\mathcal{U}_{lab}^{(m)} \cup \mathcal{H}_{ab} \cup \mathcal{V}_{ba}^{(2)} \cup \mathcal{V}_{ab}^{(2)}) = \varnothing$. Therefore, $\mathcal{D}_{lab}$ is linearly independent. $\square$

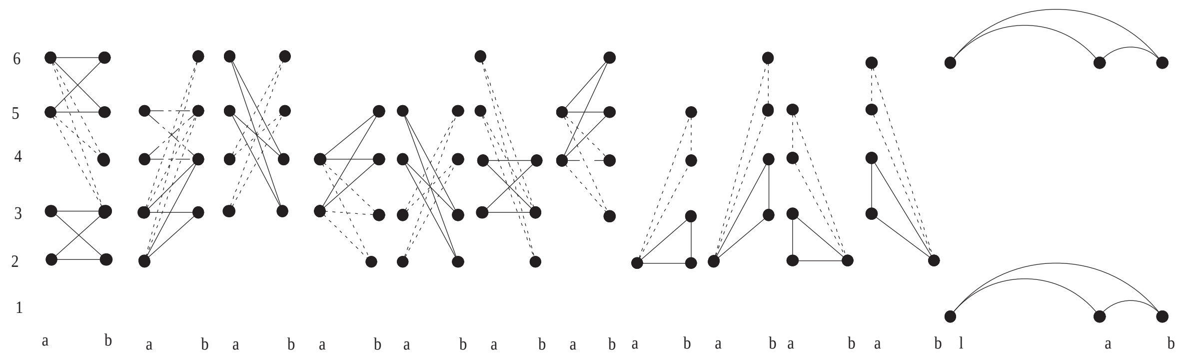

Fig.1 Cycles of $\mathcal{D}_{lab}$ for $m = 6$.

Remark 3.3 Let $e \in E(lab\rho W_m)$. From the definitions of $\mathcal{D}_{lab}$ and by the aid of Figure 2, one can easily see the following:

(1) *If* $e = (a, v_1)(b, v_1)$ *or* $(l, v_1)(a, v_1)$ *or* $(l, v_1)(b, v_1)$ *or* $(l, v_m)(a, v_m)$ *or* $(l, v_m)(b, v_m)$, *then* $f_{\mathcal{D}_{lab}}(e) = 1$.

(2) *If* $e = (a, v_j)(a, v_{j+1})$ *or* $(b, v_j)(b, v_{j+1}), 2 \geq j \geq m-1$, *then* $f_{\mathcal{D}_{lab}}(e) = 1$.

(3) *If* $e = (a, v_2)(b, v_2)$, *then* $f_{\mathcal{D}_{lab}}(e) = 3$.

(4) *If* $e = (a, v_j)(b, v_m)$ *or* $(a, v_m)(b, v_j), 2 \geq j \geq m$, *then* $f_{\mathcal{D}_{lab}}(e) = 2$.

(5) *If* $e = (a, v_j)(b, v_k), 2 \geq j, k \geq m$ *which is not as in (1)-(4), then* $f_{\mathcal{D}_{lab}}(e) \leq 4$.

(6) *If* $e \in E(lab\rho W_m)$ *which is not as in any of (1)-(6), then* $f_{\mathcal{D}_{lab}}(e) = 0$.

The graph $W_n\rho W_m$ is decomposable into $(S_n\rho W_m) \cup C_{n-1}[N_{m-1}] \cup \{(u_j, v_1)(u_{j+1}, v_1) \mid 2 \leq j \ \leq n-1\} \cup \{(u_n, v_1)(u_2, v_1)\}$ where $C_{n-1} = u_2u_3 \ldots u_nu_2$, and N_{m-1} is the null graph with vertex set $V(N_{m-1}) = \{v_2, v_3, \ldots, v_m\}$. Thus, $|E(W_n\rho W_m)| = |E(S_n\rho W_m)| + (n-1)(m-1)^2 + (n-1) = |E(S_n\rho W_m)| + (n-1)(m^2 - 2m + 2)$. Hence,

$$\dim \mathcal{C}(W_n\rho W_m) = \dim \mathcal{C}(S_n\rho W_m) + (n-1)(m^2 - 2m + 2).$$

By Theorem 3.3.2 of [15], we have that

$$\dim \mathcal{C}(S_n\rho W_m) = m^2(n-1) - nm + 2m - 1.$$

Therefore,

$$\dim \mathcal{C}(W_n\rho W_m) = (n-1)(2m^2 - 3m + 2) + (m-1).$$

Lemma 3.4 *$\mathcal{D} = \cup_{i=2}^{n} \mathcal{D}_{u_1 u_i u_{i+1}}$ is linearly independent where $\mathcal{D}_{u_1 u_n u_{n+1}} = \mathcal{D}_{u_1 u_n u_2}$.*

Proof We use the mathematical induction on n. If $n = 2$, then $\mathcal{D} = \mathcal{D}_{u_1 u_2 u_3}$ which is linearly independent by Lemma 3.2. Assume that $n > 2$ and it is true for less than n. Note that $\mathcal{D} = \mathcal{D}_{u_1 u_n u_2} \cup \left(\cup_{i=2}^{n-1} \mathcal{D}_{u_1 u_i u_{i+1}}\right)$. By Lemma 3.2 and the inductive step, each of $\mathcal{D}_{u_1 u_n u_2}$ and $\cup_{i=2}^{n-1} \mathcal{D}_{u_1 u_i u_{i+1}}$ is linearly independent. Note that

$$\begin{aligned} E(\mathcal{D}_{u_1 u_n u_2}) \cap E(\cup_{i=2}^{n-1} \mathcal{D}_{u_1 u_i u_{i+1}}) &= \{(u_1, v_1)(u_n, v_1), (u_1, v_1)(u_2, v_1), (u_1, v_m)(u_n, v_m), \\ &\quad (u_1, v_m)(u_2, v_m)\} \cup \{(u_n, v_j)(u_n, v_{j+1}), (u_2, v_j)(u_2, \\ &\quad v_{j+1}) \mid 2 \le j \le m-1\} \end{aligned}$$

which is an edge set of a forest. Thus, by Lemma 2.2, $\mathcal{D}$ is linearly independent. □

The following set of cycles which were introduced in [17] and [5] will be needed in the coming results:

$$\mathcal{G}_{ab} = \left\{ \mathcal{G}_{ab}^{(j)} = (a, v_1)(a, v_j)(b, v_2)(a, v_{j+1})(a, v_1) \mid 2 \le j \le m-1 \right\},$$

$$\mathcal{W}_{cab} = \{(c, v_1)(c, v_2)(a, v_2)(b, v_m)(b, v_1)(a, v_1)(c, v_1)\},$$

$$\mathcal{E}_{cab} = \left\{ \mathcal{E}_{cab}^{(j)} = (c, v_2)(a, v_j)(b, v_m)(a, v_{j+1})(c, v_2) \mid 2 \le j \le m-1 \right\},$$

$$\mathcal{P}_{a} = \left\{ \mathcal{P}_{a}^{(j)} = (a, v_1)(a, v_j)(a, v_{j+1})(a, v_1) \mid 2 \le j \le m-1 \right\},$$

$$\mathcal{S}_{ab} = \{(a, v_1)(a, v_2)(b, v_2)(b, v_1)(a, v_1)\},$$

and

$$\mathcal{I}_a = \{(a, v_2)(a, v_3) \ldots (a, v_m)(a, v_2)\}.$$

Let

$$\mathcal{F}_{ab} = \mathcal{H}_{ab} \cup \mathcal{G}_{ab} \cup \mathcal{G}_{ba} \cup \mathcal{S}_{ab}$$

and

$$\mathcal{F}_{cab} = \mathcal{E}_{cab} \cup \mathcal{H}_{ca} \cup \mathcal{G}_{ca} \cup \mathcal{W}_{cab}$$

Theorem 3.5 ([5]) *For any star S_n with $n \ge 2$ and wheel W_m with $m \ge 5$, we have that $\mathcal{B}(S_n \rho W_m) = (\cup_{i=2}^{n-1} \mathcal{F}_{u_{i+1} u_1 u_i}) \cup \mathcal{F}_{u_1 u_2} \cup (\cup_{i=1}^{n} \mathcal{P}_{u_i}) \cup (\cup_{i=1}^{n} \mathcal{I}_{u_i})$ is a 4-fold basis of $\mathcal{C}(S_n \rho W_m)$.*

Theorem 3.6 *For any two wheels W_n and W_m with $n \ge 4$ and $m \ge 5$, $b(W_n \rho W_m) \le 4$.*

Proof Define $\mathcal{B}(W_n \rho W_m) = \mathcal{B}(S_n \rho W_m) \cup \mathcal{D}$ where $\mathcal{B}(S_n \rho W_m)$ is as in Theorem 3.5. By Theorem 3.5 and Lemma 3.4, each of $\mathcal{B}(S_n \rho W_m)$ and $\mathcal{D}$ is linearly independent. Note that,

$$E(\mathcal{B}(S_n \rho W_m)) \cap E(\mathcal{D}) = E(S_n \Box \{v_1, v_m\}) \cup E(V(C_{n-1}) \Box P_{m-1})$$

which is an edge set of a forest where $C_{n-1} = u_2u_3 \dots u_nu_2$ and $P_{m-1} = v_2v_3 \dots v_m$. Therefore, by Lemma 2.2, $\mathcal{B}(W_n\rho W_m)$ is linearly independent. Now,

$$|\mathcal{V}_{ba}^{(2)}| = (m-2) \text{ and } |\mathcal{H}_{ab}| = (m-2)^2 \tag{3}$$

and so

$$\begin{aligned} |\mathcal{D}_{u_1u_iu_{i+1}}| &= |\mathcal{D}_{lab}| = |\mathcal{U}_{lab}^{(3)}| + |\mathcal{H}_{ab}| + |\mathcal{V}_{ba}^{(2)}| + |\mathcal{V}_{ab}^{(2)}| + |\mathcal{U}_{lab}^{(1)}| \\ &= 1 + (m-2)^2 + (m-2) + (m-2) + 1 \\ &= (m-2)^2 + 2(m-2) + 2. \end{aligned} \tag{4}$$

By equation (3),

$$\begin{aligned} |\mathcal{D}| &= \sum_{i=2}^{n} |\mathcal{D}_{u_1u_iu_{i+1}}| \\ &= (n-1)((m-2)^2 + 2(m-2) + 2). \end{aligned}$$

Thus,

$$\begin{aligned} |\mathcal{B}(W_n\rho W_m)| &= |\mathcal{B}(S_n\rho W_m)| + |\mathcal{D}| \\ &= m^2(n-1) - nm + 2m - 1 + (n-1)((m-2)^2 + 2(m-2) + 2) \\ &= (n-1)(2m^2 - 3m + 2) + (m-1) \\ &= \dim \mathcal{C}(W_n\rho W_m) \end{aligned}$$

where the last equality followed from (1). Thus $\mathcal{B}(W_n\rho W_m)$ is a basis for $\mathcal{C}(W_n\rho W_m)$. Now,we show that $b(W_n\rho W_m) \leq 4$, for all $n \geq 4$, $m \geq 5$. Let $e \in E(W_n\rho W_m)$. Then we consider the following:

Case a $e \in E(W_n\rho W_m) - E(S_n\Box\{v_1, v_m\}) \cup E(V(C_{n-1})\Box P_{m-1})$ where C_{n-1} and P_{m-1} are as defined above. Then we have the following:

(1) $e = (u_i, v_j)(u_{i+1}, v_k)$ or $(u_i, v_1)(u_{i+1}, v_1)$ with $i \leq n-1$ and $2 \leq j, k \leq m$. Then e occurs only in cycles of $\mathcal{D}_{u_1u_iu_{i+1}}$. And so, by Remark 3.3, $f_{\mathcal{B}(W_n\rho W_m)}(e) = f_{\mathcal{D}_{u_1u_iu_{i+1}}}(e) \leq 4$.

(2) $e = (u_2, v_j)(u_n, v_k)$ or $(u_2, v_1)(u_n, v_1)$ with $2 \leq j, k \leq m$. Then e occurs only in cycles of $\mathcal{D}_{u_1u_nu_2}$. And so, by Remark 3.3, $f_{\mathcal{B}(W_n\rho W_m)}(e) = f_{\mathcal{D}_{u_1u_nu_2}}(e) \leq 4$.

(3) e is not as in (1) or (2). Then e occurs only in cycles of $\mathcal{B}(S_n\rho W_m)$ and so, by Theorem 3.5, $f_{\mathcal{B}(W_n\rho W_m)}(e) \leq f_{\mathcal{B}(S_n\rho W_m)}(e) \leq 4$.

Case b $e \in E(S_n\Box\{v_1, v_m\}) \cup E(V(C_{n-1})\Box P_{m-1})$. Then we have the following:

(1) $e \in E(u_i\Box P_{m-1})$ with $2 \leq i \leq n$. Then e occurs only in $\mathcal{D}_{u_1u_{i-1}u_i}, \mathcal{D}_{u_1u_iu_{i+1}}$ and $\mathcal{B}(S_n\rho W_m)$. Thus, by Remark 3.3 and Theorem 3.5, $f_{\mathcal{B}(W_n\rho W_m)}(e) = f_{\mathcal{D}_{u_1u_{i-1}u_i}}(e) + f_{\mathcal{D}_{u_1u_iu_{i+1}}} + f_{\mathcal{B}(S_n\rho W_m)} \leq 1 + 1 + 2$.

(2) $e = (u_1, v_1)(u_2, v_1)$ or $(u_1, v_m)(u_2, v_m)$. Then e occurs only in cycles of $\mathcal{D}_{u_1u_2u_3}, \mathcal{D}_{u_1u_3u_4}$ and $\mathcal{B}(S_n\rho W_m)$. And so, by Remark 3.3 and Theorem 3.5, $f_{\mathcal{B}(W_n\rho W_m)}(e) = f_{\mathcal{D}_{u_1u_2u_3}}(e) + f_{\mathcal{D}_{u_1u_3u_4}} + f_{\mathcal{B}(S_n\rho W_m)} \leq 1 + 1 + 2$.

(3) $e = (u_1, v_1)(u_i, v_1)$ or $(u_1, v_m)(u_i, v_m)$. Then e occurs only in cycles of $\mathcal{D}_{u_1u_{i-1}u_i}, \mathcal{D}_{u_1u_iu_{i+1}}$ and $\mathcal{B}(S_n\rho W_m)$. And so, by Remark 3.3 and Theorem 3.5, $f_{\mathcal{B}(W_n\rho W_m)}(e) = f_{\mathcal{D}_{u_1u_2u_3}}(e) + f_{\mathcal{D}_{u_1u_3u_4}} + f_{\mathcal{B}(S_n\rho W_m)} \leq 1 + 1 + 2$. □

Corollary 3.7 *For any $n \geq 4$ and $m \geq 6$, we have $3 \leq b(W_n\rho S_m) \leq 4$.*

Proof By Theorem 3.6, it is enough to show that $b(W_n\rho S_m) \geq 3$. Since $S_n\rho S_m$ is a subgraph of $W_n\rho W_m$ and $b(S_n\rho S_m) = 4$ (Theorem 3.2.5 of [17]), $b(W_n\rho S_m) \geq 3$ by MacLane Theorem. □

§4. The Minimum Cycle Basis of $W_n\rho W_m$

In this section, we construct a minimum cycle basis of the wreath product of two wheels. Let

$$\mathcal{X}^*_{lab} = (\cup_{k=2}^m \mathcal{V}^{(k)}_{ab}) \cup \mathcal{V}^{(m)}_{ba} \cup \mathcal{U}^{(1)}_{lab} \cup \mathcal{U}^{(m)}_{lab}$$

Lemma 4.1 *$\mathcal{X}^*_{lab}$ is linearly independent.*

Proof $(\cup_{k=2}^m \mathcal{V}^{(k)}_{ab}) \cup (\mathcal{V}^{(m)}_{ba})$ is a linearly independent set by Lemma 3.1. Since $E((\cup_{k=2}^m \mathcal{V}^{(k)}_{ab}) \cup \mathcal{V}^{(m)}_{ba}) \cap E(\mathcal{U}^{(1)}_{lab}) = \varnothing$, $(\cup_{k=2}^m \mathcal{V}^{(k)}_{ab}) \cup (\mathcal{V}^{(m)}_{ba}) \cup \mathcal{U}^{(1)}_{lab}$ is linearly independent by Lemma 2.2. Similarly, since $E((\cup_{k=2}^m \mathcal{V}^{(k)}_{ab}) \cup (\mathcal{V}^{(m)}_{ba}) \cup \mathcal{U}^{(1)}_{lab}) \cap E(\mathcal{U}^{(3)}_{lab}) = \{(a, v_m)(b, v_m)\}$ which is an edge, we have $\mathcal{X}^*_{lab}$ is linearly independent. □

Lemma 4.2 *$\left(\cup_{i=2}^{n-1} \mathcal{X}^*_{u_1u_iu_{i+1}}\right) \cup \mathcal{X}^*_{u_1u_nu_2}$ is linearly independent.*

Proof We prove that $\cup_{i=2}^{n-1} \mathcal{X}^*_{u_1u_iu_{i+1}}$ is linearly independent using the mathematical induction on n. If $n = 3$, then $\cup_{i=2}^{2} \mathcal{X}^*_{u_1u_iu_{i+1}} = \mathcal{X}^*_{u_1u_2u_3}$ which is linearly independent by Lemma 4.1. Assume that $n \geq 4$ and it is true for less than $n - 1$. Note that $\cup_{i=2}^{n-1} \mathcal{X}^*_{u_1u_iu_{i+1}} = (\cup_{i=2}^{n-2} \mathcal{X}^*_{u_1u_iu_{i+1}}) \cup \mathcal{X}^*_{u_{n-1}u_n}$. Since

$$\begin{aligned} E(\cup_{i=2}^{n-2} \mathcal{X}^*_{u_1u_iu_{i+1}}) \cap E(\mathcal{X}^*_{u_{n-1}u_n}) &= \{(u_1, v_1)(u_{n-1}, v_1), (u_1, v_m)(u_{n-1}, v_m)\} \\ &\quad \cup \{(u_{n-1}, v_j)(u_{n-1}, v_{j+1}) \mid 2 \leq j \leq m-1\} \end{aligned}$$

which is an edge set of a forest, $\cup_{i=2}^{n-1} \mathcal{X}^*_{u_1u_iu_{i+1}}$ is linearly independent by Lemma 2.2. Similarly, Since

$$\begin{aligned} E(\cup_{i=2}^{n-1} \mathcal{X}^*_{u_1u_iu_{i+1}}) \cap E(\mathcal{X}^*_{u_1u_nu_2}) &= \{(u_1, v_1)(u_n, v_1), (u_1, v_m)(u_n, v_m), (u_1, v_1)(u_2, v_1), \\ &\quad (u_1, v_m)(u_2, \ v_m)\} \\ &\quad \cup \{(u_n, v_j)(u_n, v_{j+1}), (u_2, v_j)(u_2, v_{j+1}) \mid 2 \leq j \leq m-1\} \end{aligned}$$

which is an edge set of a forest, $\left(\cup_{i=2}^{n-1}\mathcal{X}^*_{u_1u_iu_{i+1}}\right)\cup\mathcal{X}^*_{u_1u_nu_2}$ is linearly independent. $\square$

Throughout the following results, $B_{a\square W_m}$ stands for the cycle basis of $a\square W_m$ which consists of 3-cycles.

Lemma 4.3 *$\mathcal{B}^*(W_n\rho W_m) = \mathcal{B}^*(S_n\rho W_m)\cup(\cup_{i=2}^{n-1}\mathcal{X}^*_{u_1u_iu_{i+1}})\cup\mathcal{X}^*_{u_1u_nu_2}$ is a cycle basis of $\mathcal{C}(W_n\rho W_m)$ where $\mathcal{B}^*(S_n\rho W_m) = (\cup_{i=2}^{n}\cup_{j=2}^{m}\mathcal{V}^{(j)}_{u_1u_i})\cup(\cup_{i=2}^{n}\mathcal{V}^{(m)}_{u_iu_1})\cup(\cup_{i=1}^{n}B_{u_i\square W_m})\cup(\cup_{i=2}^{n}\mathcal{S}_{u_1u_i})$.*

Proof $\mathcal{B}^*(S_n\rho W_m)$ is linearly independent by Lemma 4.3.2 of [5]. Since $E(\mathcal{B}^*(S_n\rho W_m))\cap E(\left(\cup_{i=2}^{n}\mathcal{X}^*_{u_1u_iu_{i+1}}\right)\cup\mathcal{X}^*_{u_1u_nu_2}) = E(S_n\square\{v_1,v_m\})\cup E(V(P_{n-1})\square P_{m-1})$, which is an edge set of a forest, as a result $\mathcal{B}^*(W_n\rho W_m)$ is linearly independent by Lemma 2.2 where $P_{n-1}=u_2u_3\cdots u_n$ and $P_{m-1}=v_2v_3\ldots v_m$. Now,

$$\begin{aligned}|\mathcal{X}^*_{u_1u_iu_{i+1}}| &= |\mathcal{X}^*_{lab}|\\ &= \sum_{k=2}^{m}|V^{(k)}_{ab}|+|V^{(m)}_{ab}|+2\\ &= \sum_{k=2}^{m}(m-2)+(m-2)+2\\ &= m(m-2)+2.\end{aligned}$$

Thus,

$$\begin{aligned}|\mathcal{B}^*(W_n\rho W_m)| &= |\mathcal{B}^*(S_n\rho W_m)|+|(\cup_{i=2}^{n}\mathcal{X}^*_{u_1u_iu_{i+1}})|\\ &= m^2(n-1)-mn+2m-1+\sum_{i=2}^{n}(m(m-2)+2)\\ &= m^2(n-1)-mn+2m-1+(n-1)(m(m-2)+2)\\ &= (n-1)(2m^2-3m+2)+(m-1)\\ &= \dim\mathcal{C}(W_n\rho W_m).\end{aligned}$$

Therefore, $\mathcal{B}^*(W_n\rho W_m)$ is a cycle basis for $W_n\rho W_m$. $\square$

Theorem 4.4 *$\mathcal{B}^*(W_n\rho W_m)$ is minimum cycle basis of $\mathcal{C}(S_n\rho W_m)$ for each $n,m\geq 5$.*

Proof Let $P^*=\cup_{i=1}^{n}B_{u_i\square W_m}$. Since $B_{u_i\square W_m}$ is a basis for $\mathcal{C}(u_i\square W_m)$ for each $1\leq i\leq n$ and since $E(u_i\square W_m)\cap E(u_i\square W_m)=\varnothing$ for any $i\neq j$, we have P^* is a cycle basis for the subgraph $\cup_{i=1}^{n}(u_i\square W_m)$. Let $Q^*=\mathcal{B}^*(W_n\rho W_m)-(P^*\cup(\cup_{i=2}^{n}\mathcal{S}_{u_1u_i}))$ and $(W_n\rho W_m)^-=(W_n\rho W_m)-\cup_{i=1}^{n}(E((u_i\square S_m)\cup\{(u_i,v_2)(u_i,v_m)\}))$. Note that $(W_n\rho W_m)^-$ consists of two components with $V((W_n\rho W_m)^-)=V(W_n\rho W_m)$. Also,

$$\begin{aligned}|E((W_n\rho W_m)^-)| &= |E(W_n\rho W_m)|-\sum_{i=1}^{n}(|E(u_i\square S_m)+1)\\ &= |E(W_n\rho W_m)|-nm.\end{aligned}$$

Thus,

$$\begin{aligned}\dim\mathcal{C}((W_n\rho W_m)^-) &= |E(W_n\rho W_m)|-nm-mn+2\\ &= \dim\mathcal{C}(W_n\rho W_m)-mn+1\end{aligned}$$

Now,

$$|B_{a\Box W_m}| = m-1$$

Hence,

$$\begin{aligned}|Q^*| &= |\mathcal{B}^*(W_n\rho W_m)| - |P^*| - |\cup_{i=2}^{n}\mathcal{S}_{u_1u_i}| \\ &= \dim\mathcal{C}(W_n\rho W_m) - n(m-1) - (n-1) \\ &= \dim\mathcal{C}(W_n\rho W_m) - mn + 1 \\ &= \dim\mathcal{C}((W_n\rho W_m)^-).\end{aligned}$$

Therefore, Q^* is a basis for $(W_n\rho W_m)^-$. Now, we show that $L = \mathcal{B}^*(W_n\rho W_m) - (\cup_{i=2}^{n}\mathcal{S}_{u_1u_i})$ is the largest linearly independent subset of $W_n\rho W_m$ containing L and consisting of 3-cycles. Suppose that $\{C\}\cup L$ is linearly independent where C is a 3-cycle of $W_n\rho W_m$. Then we have the following three cases:

Case 1: $E(C) \subseteq E(\cup_{i=1}^{n} u_i\Box W_m)$. Then $C \in P^*$ because the cycles of P^* is the only 3-cycles of $\cup_{i=1}^{n}(u_i\Box W_m)$. This is a contradiction.

Case 2: $E(C) \subseteq E((W_n\rho W_m)^-)$. Then C can be written as a linear combination of Q^* because Q^* is a basis for $(W_n\rho W_m)^-$. This is a contradiction.

Case 3: $E(C)$ neither a subset of $E(\cup_{i=1}^{n} u_i\Box W_m)$ nor of $E((W_n\rho W_m)^-)$. Thus, C contains at least one edge which does not belong to $\cup_{i=1}^{n} u_i\Box W_m$ and at least one edge which does not belong to $(W_n\rho W_m)^-$. Note that

$$E((W_n\rho W_m)^-) \cap E(\cup_{i=1}^{n} u_i\Box W_m) = \cup_{i=1}^{n}\left(u_i\Box v_2v_3\ldots v_m\right).$$

Thus, C must contains at least one edge of $(\cup_{i=1}^{n} u_i\Box W_m) - (\cup_{i=1}^{n} u_i\Box v_2v_3\ldots v_m)$ and at least one edge of $(W_n\rho W_m)^- - (\cup_{i=1}^{n} u_i\Box v_2v_3\ldots v_m)$. To this end, we have two subcases:

Subcase 3a: $(u_i,v_2)(u_i,v_m) \in E(C)$ for some i. Then $C = (u_i,v_2)(u_i,v_m)(u_k\ ,v_s)(u_i,v_2)$ where $u_iu_k \in E(W_n)$ and $2 \le s \le m$. Thus, C can be written as a linear combination of 3-cycle as follows:

$$\begin{aligned}C \;=\; & \left(\oplus_{j=2}^{m-1}(u_i,v_j)(u_i,v_{j+1})(u_i,v_1)(u_i,v_j)\right) \oplus (u_i,v_2)(u_i,v_m)(u_i,v_1)(u_i,v_2) \\ & \oplus_{j=2}^{m-1}(u_i,v_j)(u_i,v_{j+1})(u_k,v_s)(u_i,v_j).\end{aligned}$$

Note that each of $(u_i,v_j)(u_i,v_{j+1})(u_i,v_1)(u_i,v_j)$ and $(u_i,v_2)(u_i,v_m)(u_i,v_1)(u_i,\ v_2)$ belongs to P^*. Also, $(u_i,v_j)(u_i,v_{j+1})(u_k,v_s)(u_i,v_2)$ is a linear combinations of $(\cup_{l=2}^{m}\mathcal{V}_{u_iu_k}^{(l)}) \cup (\mathcal{V}_{u_iu_k}^{(m)})$ because $(u_i,v_j)(u_i,v_{j+1})(u_k,v_s)(u_i,v_2) \subseteq u_iu_k[v_2v_3\ldots v_m]$ and $(\cup_{l=2}^{m}\mathcal{V}_{u_iu_k}^{(l)}) \cup (\mathcal{V}_{u_iu_k}^{(m)})$ is a basis for $u_iu_k[v_2v_3\ldots v_m]$. Thus, C is a linear combinations of L. That is a contradiction.

Subcase 3b: $(u_i,v_2)(u_i,v_m) \notin E(C)$ for each i. Then C contains at least one edge of $\cup_{i=1}^{n}E(u_i\Box S_m)$ and one edge of $(W_n\rho W_m)^-$. Therefore, by the construction of $W_n\rho W_m$, C must contains at least two edges of $\cup_{i=1}^{n}(u_i\Box W_m)$ and two other edges of $(W_n\rho W_m)^-$. This is a contradiction.

Since the cycle space is a matroid and each cycle of $\cup_{i=2}^{n}\mathcal{S}_{u_1 u_i}$ is of length 4. Then $\mathcal{B}^*(W_n\rho W_m)$ is a minimum cycle basis for $W_n\rho W_m$. ■

Corollary 3.5 $l(W_n\rho W_m) = 3((n-1)(2m^2-3m+1)+(m-1))+4(n-1)$, *and* $\lambda(W_n\rho W_m) = 4$.

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