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Halfsubgroups

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Abstract: Let G be a group having a partially closed subset S such that S contains the identity element of G and each element in S has an inverse in S. Such subsets of G are called *halfsubgroups of* G. If a halfsubgroup S generates the group G, then S is called a *halfsubgroup generating the group* or *hsgg* in short. In this paper we prove some results on hsggs of a group. Order class of a group are special halfsubgroupoids. Elementary abelian groups are characterized as groups with maximum special halfsubgroupoids. Order class of a group with unity forms a typical halfsubgroup.

Key words: halfsubgroup, hsgg, order class of an element.

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§1. Introduction

According to R.H.Bruck [2] a halfgroupoid is a partially closed set w.r.t. certain operation.

Definition 1.1 *Let $(G,*)$ be a group and S be a subset of G. Let $(S,*)$ be a halfgroupoid (partially closed subset) of G such that*

(i) $e \in S$, e is the identity element of G.

(ii) $a^{-1} \in S$, $\forall\, a \in S$.

Then $(S,*)$ is called a half subgroup of the group G.

Illustration 1.1 Every subgroup of a group G is also halfsubgroup of G but not vise-versa. For example, consider the multiplicative group $G = \{1, -1, i, -i\}$. Then $S = \{1, i, -i\}$ is a halfsubgroup of G which is not a subgroup.

Definition 1.2 *If for a group G there exists a halfsubgroup H without identity such that for all $x, y \in H$, $xy \in H$ whenever $y \neq x^{-1}$ then H is called a special halfsubgroup of G.*

Definition 1.3 *A halfsubgroup $(S,*)$ of a group $(G,*)$ is called a halfsubgroup generating the group (or hsgg in short) if it generates G.*

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It is easy to verify that the union of two hsgg of a group G is again a hsgg of G. In fact, union of any number of hsgg of G is also hsgg of G.

However, the intersection of two hsgg of a group G may not be a hsgg of G.

Theorem 1.1 *If S is a proper hsgg of a group G then $O(S) \geq 3$.*

Proof Let S be a proper hsgg of a group G. Then $S \neq \{e\}$. Let $a \in S, a \neq e$ then $S \neq \{e, a\}$, because if $S = \{e, a\}$ then $a = a^{-1}$ and S can not generate whole G, so S can not be a proper hsgg of G. Thus if $O(G) \leq 3$ then G can not have a proper hsgg. Now if $O(G) \geq 4$ then we can have a proper hsgg $S = \{e, a, b\}$ of G such that $a = a^{-1}$ and $b = b^{-1}$ or $a^{-1} = b$.

As a result there exists an hsgg S such that $O(S) = 3$. Hence we get the result. □

Remark If G is any cyclic group such that $G = (a)$, then $S = \{e, a, a^{-1}\}$ is a minimal hsgg of G.

Definition 1.4 *Let G be a group and S be an hsgg of G. The element $x(\neq e) \in S$ is called a redundant element of S if $S \setminus \{x\}$ is also an hsgg of G.*

An element of S which is not redundant is called an irredundant element.

Definition 1.5 *Let G be a group and S be a hsgg of G such that $a^2 \neq e, \forall a \in S$ and S has no redundant element. Then S is called pure hsgg of G.*

The following results follow trivially.

(1) *Every cyclic group of order ≥ 3 has at least one pure hsgg.*

(2) *A cyclic group of prime order p has $\frac{p-1}{2}$ number of distinct pure hsgg.*

We discuss some Abelian groups in terms of their pure hsgg.

Theorem 1.2 *Every group of prime order can be expressed as the union of its distinct pure hsgg. However, the converse is not true.*

Proof Every group of prime order p is cyclic. Hence the group has $\frac{p-1}{2}$ number of distinct pure hsgg. Each hsgg has two non-identity elements together with an identity element e common in all. Thus G has $2 \cdot \frac{p-1}{2} + 1 = p$ elements. Hence G is the union of all these distinct pure hsggs. □

Theorem 1.3 *If a group G can be written as the union of its distinct pure hsgg then G is a group of odd order.*

Proof It is easy to verify. □

Theorem 1.4 *An elementary Abelian $p-$group, $p > 3$ is a direct product of n cyclic groups each of which is a cyclic $p-$group which is the union of distinct pure hsggs.*

Proof By the definition of elementary Abelian $p-$group,

$$G = C_1 \times C_2 \times \cdots \times C_n$$

where $C_i's$ are cyclic $p-$groups of order p. Now each $C_i = \cup_{r=1}^{(p-1)/2} S_{i_r}$ where $1 \leq i \leq n$ and S_{i_r} are distinct pure hsgg representing each cyclic group C_i of order p given in the above decomposition. Thus G is n times the direct product of union of distinct pure hsgg. □

Theorem 1.5 *Let G be a finite Abelian group of order n. Let $G = C_1 \times C_2 \times \cdots \times C_k$ where each C_i is a cyclic group of order $p_i^{\alpha_i}$. That is $n = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ where p_i are distinct primes and each $\alpha_i > 0$. Then*

$G = \prod_i \{\alpha_i \prod_1^{\alpha_i} \cup_{r=1}^{(p_i-1)/2} S_{i_r})\}$

where $i = 1, \cdots, k$.

Proof The proof follows Theorem 1.4. □

§2. **Order Class**

Definition 2.1 *Let G be a group. A subset O_α of G defined by*

$$O_a = \{b \in G : o(b) = o(a)\}$$

is called an order class of of a.

Definition 2.2 *Let G be a group. Let O_a be an order class of $a \in G$. Then the set of all xy such that $x, y \in O_a$ is called the closure of O_a.It is denoted by $\bar{O}_a$.*

Lemma 2.1 *If G is a finite group and $a \in G$, $a \neq e$ then*

(i) O_a is a halfgroupoid;

(ii) $a^{-1} \in O_a, \forall a \in O_a$.

Proof The proof follows by these definitions of halfgroupoid and order class. □

Notation: We use the notation Θ_a to denote order class of a with unity.

Definition 2.3 *If H is a halfsubgroup of a group G then $H \setminus \{e\}$ is called the halfsubgroupoid of G.*

Every group G has a unique maximum halfsubgroupoid $G \setminus \{e\}$ associated with it.

Definition 2.4 *Let G be a group. Then O_a is a halfsubgroupoid of G. It is called a special halfsubgroupoid of G.*

2.1 **Groups with maximum special halfsubgroupoids**

There exist groups which have only one order class other than $\{e\}$. For such groups closure of the order class of $a(\neq e)$ where $a \in G$, we give below a series of examples of such groups.

Example 2.1 Cyclic groups of prime order without unity such as $Z_5 \setminus \{e\}, Z_7 \setminus \{e\}, \ldots$ are the maximum special halfsubgroupoids.

Example 2.2 All groups with exponent p a prime are such groups.

Example 2.3 All elementary Abelian groups are such groups.

Example 2.4 Extra special groups of order 27 generated by three elements and of order 81 generated by 2 elements are such groups. This has been verified by using GAP ref[3]. The GAP Small Groups Library no. of these groups are [27,3] and [81,12]. These are polycyclic groups of order 27 and order 81 respectively.These are the only groups from the groups of order 100 which which have a single order class other than order class of $\{e\}$.

Example 2.5 The group $GL_3(F, p)$ for odd prime p is such a group.

Example 2.6 George Havas has constructed a biggest 5-group generated by 2 elements. It is of order 5^{34} with exponent 5.

Example 2.7 Dihedral groups of order D_{2p} are such groups.

2.2 Results

Theorem 2.1 *If G has a maximum special halfsubgroupoid then G is a p-group.*

Proof Let a group G has a maximum special halfsubgroupoid. Then every non-identity element of G has same order. If p divides order of G then there exists an element of order p in G. As a result all non-trivial elements of G are of order p. Thus, G is a p-group. □

Now we prove a theorem which gives the characterization of an elementary Abelian groups.

Theorem 2.2 *A group G is elementary Abelian if and only if G has a maximum special halfsubgroupoid.*

Proof Assume G is elementary Abelian, then every element of G is of same order p where p is a prime. Thus the collection of non-identity elements form an order class which is a maximum special halfsubgroupoid. Conversely, If G has a maximum special halfsubgroupoid then by Theorem 2.1 G is a p group and G has a maximum special halfsubgroupoid. Whence G is elementary Abelian. □

Theorem 2.3 *If G be a finite group, $a \in G$ then Θ_a is a halfsubgroup of G.*

Proof The Proof follows Lemma 2.1 and the definition of halfsubgroup of G. □

Definition 2.5 *A halfsubgroup S of a group G is normal in G if and only if $xSx^{-1} \in S, \forall x \in G$.*

Theorem 2.4 *If G is a finite group then Θ_a is a normal halfsubgroup of G.*

Proof If G is abelian then obviously Θ_a is normal in G. If G is non-abelian, then $o(a) = o(xax^{-1}), \forall a \in \Theta_a$. Therefore $xax^{-1} \in \Theta_a$. Hence Θ_a is normal in G. □

Theorem 2.5 *If G is a finite abelian group such that $O(G) = p_1 \cdot p_2 \cdots p_r$ for the primes $p_1, \cdots, p_r$ then G is the direct product of order classes with unity (i.e. halfsubgroup).*

Proof In the decomposition of G every Sylow p_i subgroup is an order class with unity which is also a halfsubgroup. Hence we get the result. □

Corollary 2.1 *Any finite abelian group is a direct product of some order classes with unity (halfsubgroup).*

Theorem 2.6 *Every finite group* G *is the union of halfsubgroups (namely order class with unity)* Θ_a, $a \in G$ *and* $a \neq e$ *which are normal in* G *such that* $\cap_{a \in G} \Theta_a = \{e\}$.

Proof The proof follows from Lemma 2.1 and Theorem 2.4. □

References

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[2] Bruck R. H., *Survey of Binary Systems*, Springer Verlag, Berlin.
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