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Flexibility of Embeddings of a Halin Graph on the Projective Plane

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Abstract: A basic problem in graph embedding theory is to determine distinct embeddings of planar graphs on higher surfaces. Tutte's work on graph connectivity shows that wheels or wheel-like configurations plays a key role in 3-connected graphs. In this paper we investigate the flexibility of a Halin grap on N_1, the projective plane, and show that embeddings of a Halin graph on N_1 is determined by making either a twist or a 3-patchment of a vertex in a wheel. Further more, as applications, we give a correspondence between a Halin graph and its embeddings on the projective plane. Based on this, the numbers of some types of such embeddings are determined.

Key Words: Halin graph, embedding, face-width.

AMS(2000): 05C30, 05C45.

§1. Introduction

Throughout this paper we consider simple connected labeled graphs and their embeddings on surfaces. Terms and notations not defined may be found in [1,3] and [11].

A surface is a compact closed 2-manifold. An(A) orientable (non-orientable) surface of genus g is the sphere with g *handles* (or *crosscaps*) which is denoted by S_g (or N_g). A *map* M or *embedding* on $S_g (or N_g)$ is a graph drawn on the surface so that each vertex is a point on the surface, each edge {x,y}, $x \neq y$, is a simple open curve whose endpoints are x and y, each loop incident to a vertex x is a simple closed curve containing x, no edge contains a vertex to which it is not incident, and each connected region of the complement of the graph in the surface is homeomorphic to a disc and is called a *face*. It is clear that maps(or embeddings) here are combinatorial. A map or An embedding is called *strong* if the boundaries of all the facial walks are simply cycles. A curve (or circuit) C on a surface Σ is called *non-contractible* (or *essential*) if none of the regions of $\Sigma - C$ is homeomorphic to an open disc; otherwise it is called *contractible* (or *trivial*). Let T be a tree without subdivisions of edges and embedded in the plane with its one-valent vertices being $v_1, v_2, ..., v_m$ under the rotation of T. A *leaf* is an edge incident to a vertex of valence 1. If new edges $(v_i, v_{i+1}) (i = 1, 2, ..., m)$ are added to $E(T)$, the edge-set of T, then T together with the cycle $(v_1, v_2, ..., v_m)$ forms a planar map

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called *Halin graph*. This cycle is defined as *leaf-cycle* (i.e., the boundary of the outer face) and is denoted as ∂f_r. In convenience, we always let T to denote the tree which orients a Halin graph. It is clear that Halin graphs are generalized wheels on the plane. Tutte showed in his book[11] that a 3-connected graph are obtained from the wheels by a series of *edge* or *vertex splitting* operations. Further, Vitray[12] pointed out that wheels play a key role in embeddings of a 3-connected graph since in that case the *local structures* (i.e., neighbour of a vertex) may be viewed as a wheel.

A major subject about planar graphs is to determine all of their distinct embeddings on a non-planar surface. This theory has been developed and deepened by people such as R.Vitray[12], N.Robertson and R.Vitray[7], B.Mohar and N.Robertson[4,6], and C.Thomassen[8] etc. Recently, Mohar et al[5] showed the existence of upper bounds for the distinct embeddings of a 3-connected graph in general orientable surfaces. In this paper we investigate the embeddings of a Halin map on N_1 and show that strong embeddings of a Halin graph on N_1 is determined by making 3-patchments on inner vertices of a wheel and present a correspondence between a Halin graph and its (strong) embeddings in the projective plane. Based on this, the number of such embeddings is determined.

Let $\mathcal{H}, \mathcal{H}_p$ be the set of all the Halin graphs and their embeddings on the projective plane, respectively. Then we have the following result:

Theorem A. *For a map $M \in \mathcal{H}$ with $s(M)$ edges, there are*

$$\sum_{v\in V(M)} \binom{d(v)}{2} - s(M)$$

maps in $\mathcal{H}_p$ corresponding to M whose face-width are all 1.

Here, the concept of *face-width* of an embedding is defined in the next section. The readers may also see[12] for a reference. Based on Theorem 1 one may calculate the number of such embeddings on N_1. For instance, there are exactly 6 such embeddings of W_4 in N_1 as depicted in Fig.1.

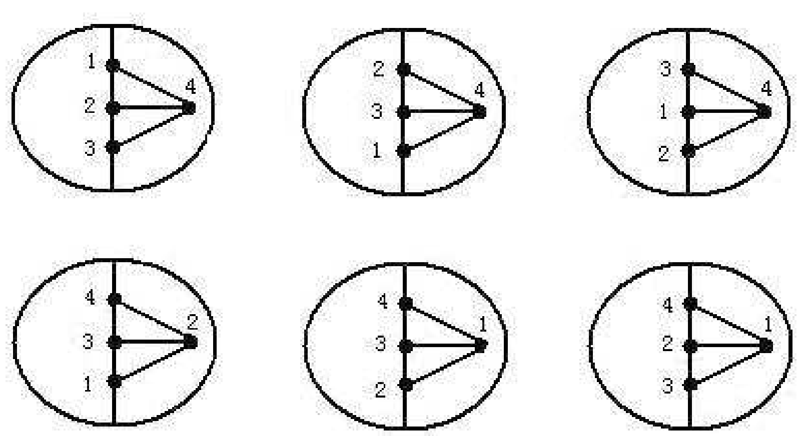

Fig.1 Six face-width-1 embeddings of W_4 in N_1

In the case of strong embeddings or maps (i.e., the boundary of each facial walk is a cycle), the following result shows that any strong embedding of a Halin graph in N_1 is in fact determined by a corresponding strong embedding of a wheel.

Theorem B. Let M be a Halin graph. Then its strong embeddings are determined by strong embeddings of wheels.

As applications of Theorem B, we have

Theorem C. *For a Halin graph G, there are*

$$\sum_{v \in V - \partial f_r} \left(2^{d(v)-1} - d(v)\right).$$

elements in $\mathcal{H}_p$ corresponding to G whose face-width are all 2.

Based on the formula presented in Theorem C, one may calculate the number of strong embeddings of a Halin graph on N_1. The following Fig.2 shows a Halin graph and its strong embeddings in N_1.

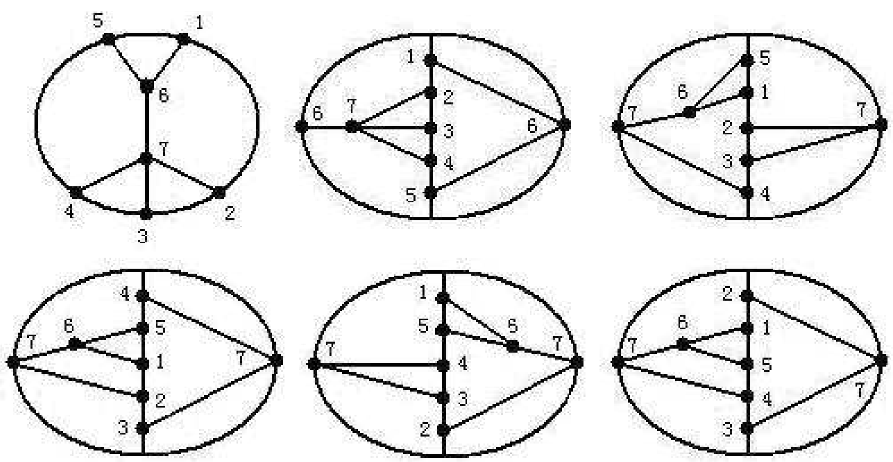

Fig.2 A Halin graph with five distinct strong embeddings in N_1

§2. Some Preliminary Works

In this section we shall give some lemmas on graph embeddings before proving of our main results.

Lemma 1 *A planar Halin graph is* 3*-connected and has at least two facial walks which are* 3*-gons.*

Proof. Let G be a planar Halin graph oriented by a tree, say $T(G)$. One may easily see that G is 3-connected. In fact, for any two distinct vertices x and y not on the leaf-cycle and with their valencies not less than 3, there are two paths in $T(G)$ connecting two leaves for each of them. Those four paths are pairwise inner disjoint. It is easy for one to see that those paths together with a pair of segments (which are determined by the four leaves) of the leaf-cycle form a pair of inner disjoint paths connecting x and y. If we consider the unique path from x to y in $T(G)$, then there are three inner disjoint paths joining x and y in G. Since the same property holds for other locations of x and y, G is 3-connected by *Menger's Theorem*. As for the existence of 3-gons, one may find at least two such triangles along the longest paths in $T(G)$. □

A fundamental result on topological graph theory by H.Whitney[13] states that any 3-connected graph has at most one planar embedding, i.e.,

Lemma 2 *There is only one way to embed a* 3*-connected planar graph in the plane.*

W.Tutte[10] obtained Whitney's uniqueness result from a combinatorial view of facial walks-*induced non-separating cycle* (for a reference, one may see[8]), i.e.,

Lemma 3 *Every facial walk of a* 3*-connected planar graph is an induced non-separating cycle.*

Later, C.Thomassen[9] generalized the above two results to *LEW-embeddings* (a concept by J.Hutchison[2]) in general surfaces and found that such embeddings share many properties with planar graphs.

Based on Lemmas 1, 2, and 3, we have the following

Lemma 4 *If a Halin graph is embedded in a non-planar surface* Σ*, then every facial walk of it (viewed as a planar map) is either a contractible cycle (hence also a facial walk) or a non-contractible cycle (or essential as some people called it) of* Σ.

When a planar graph G is embedded in a non-planar surface Σ, then some very important properties will appear. For instance, R.Vitray[12] found (late proved by N.Robertson et al[8] and C.Thomassen[9] independently) that the *face-width* $\rho_\Sigma(G)$ of G on Σ is at most 2, where $\rho_\Sigma(G)$ is defined as

$$\rho_\Sigma(G) = \min\{|\, C \cap V(G) \,|\; |C \text{ is an noncontractible curve of } \Sigma\}$$

In view of intuition, face-width is a measure of how densely a graph is embedded in a given surface. The above property says that every embedding, if possible, of a planar graph on non-planar surfaces is relatively sparse. A basic problem is how to determine the face-width of an embedding or to find that what operations performed on the graphs may not change its representativity. It is easy for one to check that the following result (which is depicted in Fig.3) presents such operations.

Lemma 5 *Let* G *be a graph embedded in a surface. Then the* $\triangle - Y$ *and* $\triangle - I$ *operations defined below do not change the face-width.*

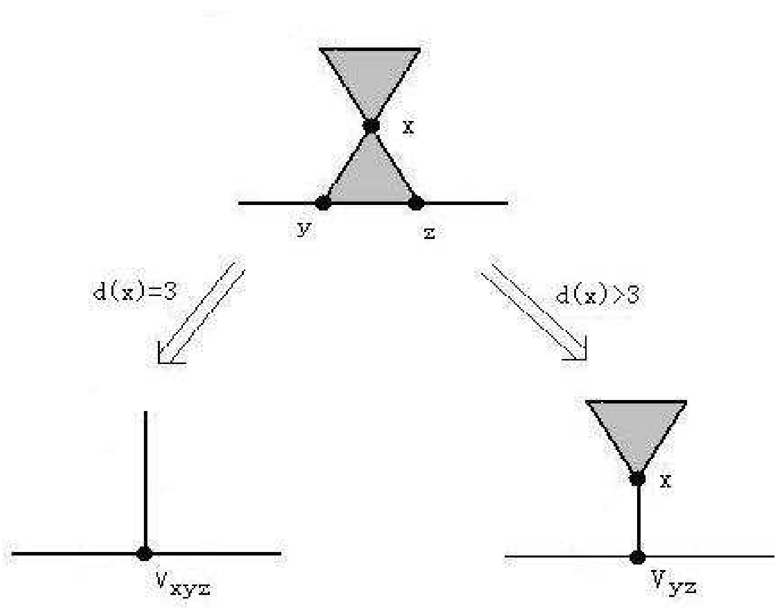

Fig.3 Popping a planar triangle into a vertex or an edge, $d(y) = d(z) = 3$

§3. Projective Planar Maps

In this section we shall prove Theorems A, B and C.

According to Lemma 3, the leaf-cycles of those in $\mathcal{H}_p$ are either facial walks or non-contractible cycles on N_1. Thus, $\mathcal{H}_p$ may be partitioned into two parts as

$$\mathcal{H}_p = \mathcal{H}_p(1) + \mathcal{H}_p(2),$$

where

$$\mathcal{H}_p(1) = \{M \mid \rho_{\mathrm{N}_1}(\mathrm{G(M)}) = 1\}; \quad (1)$$

$$\mathcal{H}_p(2) = \{M \mid \rho_{\mathrm{N}_1}(\mathrm{G(M)}) = 2\} \quad (2)$$

and $G(M)$ is the *underline graph* of $M(M)$.

Proof of Theorem A Let M be a map in $\mathcal{H}_p(1)$. Then it is determined by making a twist at a vertex of a Halin graph. On the other hand, by making a twist at each pair of corners around every vertex of a Halin graph G will induce an embedding of G on N_1. One may see that for each element in $\mathcal{H}_p(1)$ no more than one such twists are permitted since otherwise by reversing the specific twists (which will change a facial walk into two whose boundaries are simple cycles) we may see that a 3-connected planar Halin graph will have at least two distinct embeddings in the plane and hence contradicts Lemma 2 or 3. This completes the proof. □

We now concentrate on the structures of the maps in $\mathcal{H}_p(2)$.

Lemma 6 *Let M be a map in $\mathcal{H}_p(2)$. Then the leaf-cycle of M is non-contractible.*

Proof: It is easy to see its validity for smaller maps. Suppose it holds for those having fewer than n edges. Let $M \in \mathcal{H}_p(2)$ be a counter example with n edges. Then its leaf-cycle is contractible. Under this case we will show that its face-width is 1. By the definition of M, there exists a Halin graph G such that M is an embedding of it in N_1. Notice that both of the them share the same leaf-cycle (and consequently the same outer facial walk). By Lemmas 1 and 3, G has a 3-cycle, say (x, y, z), which is either a 3-gon or non-contractible in M. If (x, y, z) is non-contractible in M and the leaf-cycle is on the only one side of it, then we have $\rho_{N_1}(G(M)) = 1$ since at least two vertices of $\{x, y, z\}$ are on the leaf-cycle and trivalent and the two edges not on it are incident to them are on the same side of the 3-cycle. If (x, y, z) is non-contractible with edges of the leaf-cycle lying on the both sides of it, the leaf-cycle of M is not a simple cycle (i.e., containing vertices repeated more than twice), a contradiction as required. Next, we consider the case that the 3-cycle (x, y, z) is a 3-gon of M. In this case the face-width is 2 by performing operations in Lemma 5 and the Induction hypothesis says that the leaf-cycle is non-contractible. This contradiction completes the proof. □

Let M be a map in $\mathcal{H}_p(2)$. Then by Lemma 6 its leaf-cycle is non-contractible and all the *leaves* are distributed alternatively on the both sides of the leaf-cycle since otherwise we will have its face-width 1. Thus, leaves together with their 1-valent vertices are classified into two groups lying on the "both sides" of the leaf-cycles. One may see that this is not accurate since on N_1 every non-contractible cycle has only one side. But this description will not ruin our proofs. By a *foot* we mean a maximal group of leaves together with the 1-valent vertices which appear to the same side of the leaf-cycle consecutively. Further, we have

Lemma 7 *The feet on ∂f_r (the boundary of the leaf-cycle) must appear alternatively (i.e., there exists three feet B_1, B_2 and B_3 such that their appearing order is B_1, B_2, B_3, where B_1 and B_3 are on the same side of ∂f_r and B_2 on the other side (the right hand side of Fig.4 presents a case of this structure).*

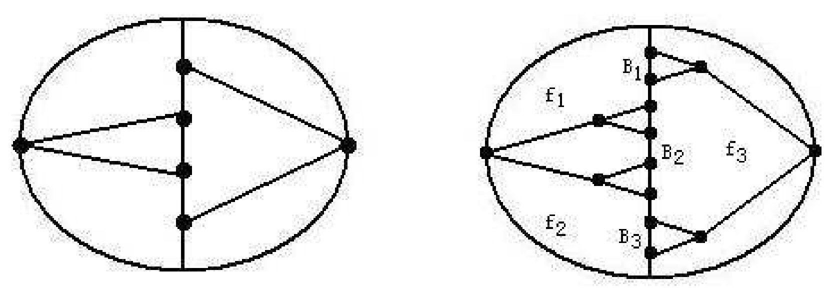

Fig.4 An embedding of W_5 which will induce
a strong embedding of a Halin map

Proof This follows from the fact that maps in $\mathcal{H}_p(2)$ have face-width 2. □

We say that a group of leaves will *induce a subtree* of T if there is a vertex v in T such that those leaves consist of all the leaves of a subtree of T rooted at v. By the definition of Halin graphs one may see the following

Lemma 8 *Let M be a map in $\mathcal{H}_p(2)$ which has a structure in Lemma 7. Then the foot B_2 will induce a subtree of T*

Proof Let the two ends of B_i be x_i and $y_i (1 \leq i \leq 3)$ and f be the face on the opposite of side of B_2. Let f_1 and f_2 be, respectively, the faces on the other side of the edges (y_1, x_2) and (y_2, x_3). Then one may see the following fact from $\rho_{N_1}(G(M)) = 2$ and the definition of Halin graphs.

Fact 1 $f_1 \neq f_2$.

Our next proofs are divided into two cases.

Case 1 $\partial f_1 \bigcap \partial f_2 \neq \emptyset$.

One may choose a vertex u on the common boundary of f_1 and f_2 such that the path from y_2 to u in T is shortest. Then by the 3-connectness of $G(M)$ and the definition of a Halin graph there is an unique path, say P, connecting u and a vertex v on ∂f. Choose v such that P is as short as possible. Then we have

Fact 2 $\mid V(P) \mid \leq 2$.

Since otherwise there will exist an internal vertex w on P. By the definition of a Halin map there is a path Q (in T) connecting w and a vertex w_1 on ∂f_r. It is clear that $w_1 \notin V(B_i)$. If we view B_i as a vertex v_{B_i} for $1 \leq i \leq 3$, then the set $\{v, w, w_1, v_{B_1}, v_{B_2}, v_{B_3}\}$ will guarantee the existence of a subgraph of $G(M)$ which is a subdivision of $K_{3,3}$, a Kuratowski graph.

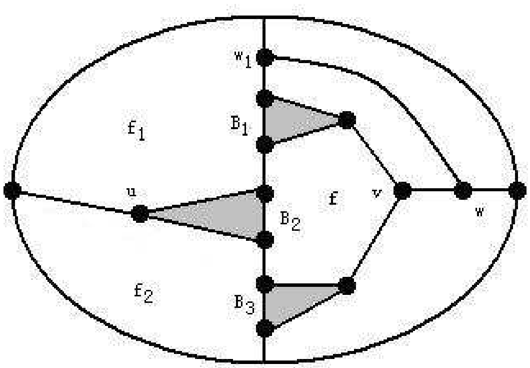

Fig.5

This is shown in Fig.5, contradicts to that Lemma 8.

Case 2 $\partial f_1 \bigcap \partial f_2 = \emptyset$.

Then there are two paths, say P and Q, from ∂f_1 and ∂f_2 to ∂f, respectively. We may choose P and Q such that they are from x_2 and y_2 to ∂f respectively and $V(P)\bigcap\partial f_{=}\{v\}$, $V(Q)\bigcap\partial f=\{u\}$. If $u\neq v$, then there would be a subgraph which

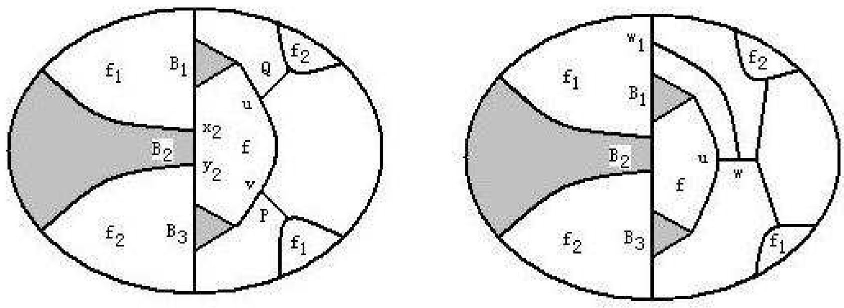

Fig.6

is a subdivision of $K_{3,3}$ (in fact it is induced by the set $\{u,v,x_2,y_2,v_{B_1},v_{B_3}\}$ as depicted in the left side of Fig.6). So, $u=v$. If $P\bigcap Q$ is a path with length≥ 1, then the length of $P\bigcap Q$ is 1. Since otherwise we may choose an internal vertex w (as we did previously) which may lead to a path from w to ∂f and hence will imply the existence of a non-planar subgraph of $G(M)$ (which is determined by the vertex-set $\{u,w,w_1,v_{B_1},v_{B_2},v_{B_3}\}$ as shown in the right side of Fig.6). This contradiction shows that $P\bigcap Q$ is a path with no more than two vertices. Combining all the possible situations in the two cases completes the proof. □.

Lemma 9 *Let M be an embedding of a Halin map with representativity 2. Let $B_i(1\leq i\leq 5)$ be five feet appearing alternatively on the two sides of ∂f_r. Then the two trees induced by B_2 and B_4 are rooted at the same vertex of T.*

Proof. Let f_i be the face on the opposite side of B_i and T_i be the subtree induced by B_i for $1\leq i\leq 5$. Then by Lemma 8 the tree T_4 (which corresponds to B_4) is rooted at some vertex u in ∂f_4. Let P be a path from u to B_5 along ∂f_4 and Q be a path from B_4 to u. Then the cycle $C=QuPx_5y_5$ is non-contractible. Similarly, choose Q_1 be a path from B_2 to a vertex v on ∂f_2 such that T_2 is rooted at v. Let P_1 be path from v to x_3 on ∂f_2. The cycle $C'=Q_1vP_1x_3y_2$ is also non-contractible. Notice that any pair of non-contractible cycles (curves) on N_1 will intersect at a vertex, we conclude that C and C' will intersect at a vertex w on the path P. If $u\neq v$, then as we have discussed before, there is a non-planar subgraph of $G(M)$. This contradiction shows that $u=v$. It follows from Lemma 8 that the vertex w is also on the boundary of f_2. This ends the proof. □

Proof of Theorem B Let M be a Halin Map and M' an embedding of it on N_1 with $\rho_{N_1}(G(M))=2$. Let B_i be the feet of M' and induces a subtree T_i for $1\leq i\leq s$. Then by Lemma 6 the leaf-cycle ∂f_r is non-contractible and all the feet are lying on the two sides of ∂f_r alternatively by Lemma 8. Lemmas 6-9 show that all the subtrees T_i are rooted at some vertex v of T. Let $d(v)=m$. For each $T_i(1\leq i\leq s)$, its root-vertex is v and edges incident to v is $e_{l_1},e_{l_2},...,e_{l_i}$. One may view T_i together B_i as a claw K_{1,l_i} whose edges are correspondent to $e_{l_1}, e_{l_2}, ..., e_{l_i}$. Then one may get a bigger claw $K_{1,m}$ which together with ∂f_r forms a wheel W_{m+1} whose underlying graph is planar, where $m = \sum_1^s l_i$. This procedure is shown in Fig.4 where the case of $m = 4$ is shown. It is clear that W_{m+1} is strongly embedded in N_1. Since this procedure is reversible, the theorem follows. □

Proof of Theorem C Let M be a Halin graph with T and ∂f as its orienting tree and leaf cycle. Then by Theorem B its strong embeddings are completely determined by performing 3-patchments on its inner vertices (i.e., those not on ∂f). So, taking an inner vertex, say $v \in V - \partial f$, and considering the number of ways of performing 3-patchments at v. Let $d(v) = m$. Then the corresponding 3-patchments is induced by those of W_{m+1}, the wheel with m spokes. So, we only need to restrain our procedure on the strong embeddings of W_{m+1} on N_1 . Notice that in the case of $m \geq 4$, there is only one leaf-cycle for W_{m+1} which is non-contractible. We can determine all the possible strong embeddings of W_{m+1} this way: We first draw the leaf-cycle $(1, 2, ..., m)$ into N_1 such that the leaf-cycle is non-contractible and then consider the ways of choosing alternating feet on $(1, 2, ..., m)$ as described in Lemma 7. It is clear that the number of alternating feet must be an odd number. Let the number of leaves in feet B_i is x_i. Then the total number of ways of choosing alternating feet is equal to the number of ways of assigning $2k + 1$ groups of consecutive vertices on a m-vertex-cycle. This correspondence is shown in Fig.7, where a 3-patchment on the center of a wheel will produce 7 alternating feet on the leaf-cycle.

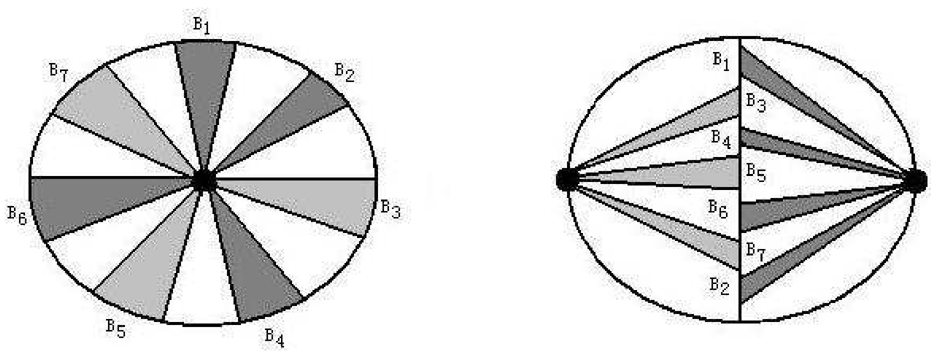

Fig.7 Generating a strong embedding in N_1 by performing a 3-patchment at the center of W_{m+1}

Let $f(m, k)$ be the number of ways of grouping k sets of consecutive vertices. Then it is clear that $f(m, k)$ satisfies the following recursive relation:

$$\begin{cases} f(m,k) = f(m-1,k-1) + f(m-1,k), \quad m \geq k \geq 2; \\ f(m,m) = f(0,0) = 1. \end{cases}$$

Since the combinatorial number $\binom{m}{k}$ also satisfies the above relation, we have that $f(m, k) = \binom{m}{k}$. Hence, Theorem C follows from the case of $f(m, 2k + 1)$. □

Final Remark By using the same procedure used in our proof of Theorem B, one may find that a Halin graph has no strong embeddings in orientable surface other than the sphere. This seems resulted from the fact that the face-distance (i.e., the shortest length of face-chain connecting two faces) is not greater than 2. With the increase of genera, the possibility of strong embedding is decrease. Hence, we think that the sphere and the projective plane are the only two possible surfaces on which a Halin graph may be strongly embedded.

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