International J.Math. Combin. Vol.1 (2008), 16-35

Curvature Equations on Combinatorial Manifolds with Applications to Theoretical Physics

Linfan Mao

(Chinese Academy of Mathematics and System Science, Beijing 100080, P.R.China)

E-mail: maolinfan@163.com

Abstract Curvature equations are very important in theoretical physics for describing various classical fields, particularly for gravitational field by Einstein. For applying Smarandache multi-spaces to *parallel universes*, the conception of combinatorial manifolds was introduced under a combinatorial speculation for mathematical sciences in [9], which are similar to manifolds in the local but different in the global. Similarly, we introduce curvatures on combinatorial manifolds and find their structural equations in this paper. These Einstein's equations for a gravitational field are established again by the choice of a combinatorial Riemannian manifold as its spacetime and some multi-space solutions for these new equations are also gotten by applying the *projective principle* on multi-spaces in this paper.

Key Words: curvature, combinatorial manifold, combinatorially Euclidean space, equations of gravitational field, multi-space solution.

AMS(2000): 51M15, 53B15, 53B40, 57N16, 83C05, 83F05.

§1. Introduction

As an efficiently mathematical tool used by Einstein in his general relativity, tensor analysis mainly dealt with transformations on manifolds had gotten considerable developments by both mathematicians and physicists in last century. Among all of these, much concerns were concentrated on an important tensor called curvature tensor for understanding the behavior of curved spaces. For example, the famous Einstein's gravitational field equations

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = -8\pi G T_{\mu\nu}$$

are consisted of curvature tensors and energy-momentum tensors of the curved space.

Notice that all curved spaces considered in classical fields are homogenous. Achievements of physics had shown that the multiple behavior of the cosmos in last century, enables the model of parallel universe for the cosmos born([14]). Then *can we construct a new mathematical theory, or generalized manifolds usable for this multiple, non-homogenous physics appeared in 21st century*? The answer is YES in logic at least. That is the *Smarandache multi-space theory*, see [7] for details.

For applying Smarandache multi-spaces to *parallel universes*, combinatorial manifolds were

[1]Received October 25, 2007. Accepted December 12, 2007

introduced endowed with a topological or differential structure under a combinatorial speculation for mathematical sciences in [9], i.e. *mathematics can be reconstructed from or turned into combinatorization*([8]), which are similar to manifolds in the local but different in the global. Whence, geometries on combinatorial manifolds are nothing but these *Smarandache* geometries([12]-[13]).

Now we introduce the conception of combinatorial manifolds in the following. For an integer $s \geq 1$, let $n_1, n_2, \cdots, n_s$ be an integer sequence with $0 < n_1 < n_2 < \cdots < n_s$. Choose s open unit balls $B_1^{n_1}, B_2^{n_2}, \cdots, B_s^{n_s}$ with $\bigcap\limits_{i=1}^{s} B_i^{n_i} \neq \emptyset$ in $\mathbf{R}^n$, where $n = n_1 + n_2 + \cdots n_s$. A *unit open combinatorial ball of degree* s is a union

$$\widetilde{B}(n_1, n_2, \cdots, n_s) = \bigcup_{i=1}^{s} B_i^{n_i}.$$

A combinatorial manifold $\widetilde{M}$ is defined in the next.

Definition 1.1 *For a given integer sequence $n_1, n_2, \cdots, n_m, m \geq 1$ with $0 < n_1 < n_2 < \cdots < n_m$, a combinatorial manifold $\widetilde{M}$ is a Hausdorff space such that for any point $p \in \widetilde{M}$, there is a local chart (U_p, φ_p) of p, i.e., an open neighborhood U_p of p in $\widetilde{M}$ and a homoeomorphism $\varphi_p : U_p \rightarrow \widetilde{B}(n_1(p), n_2(p), \cdots, n_{s(p)}(p))$ with $\{n_1(p), n_2(p), \cdots, n_{s(p)}(p)\} \subseteq \{n_1, n_2, \cdots, n_m\}$ and $\bigcup\limits_{p \in \widetilde{M}} \{n_1(p), n_2(p), \cdots, n_{s(p)}(p)\} = \{n_1, n_2, \cdots, n_m\}$, denoted by $\widetilde{M}(n_1, n_2, \cdots, n_m)$ or $\widetilde{M}$ on the context and*

$$\widetilde{\mathcal{A}} = \{(U_p, \varphi_p) | p \in \widetilde{M}(n_1, n_2, \cdots, n_m))\}$$

an atlas on $\widetilde{M}(n_1, n_2, \cdots, n_m)$. The maximum value of $s(p)$ and the dimension $\widehat{s}(p)$ of $\bigcap\limits_{i=1}^{s(p)} B_i^{n_i}$ are called the dimension and the intersectional dimensional of $\widetilde{M}(n_1, n_2, \cdots, n_m)$ at the point p, denoted by $d(p)$ and $\widehat{d}(p)$, respectively.

A combinatorial manifold $\widetilde{M}$ is called *finite* if it is just combined by finite manifolds without one manifold contained in the union of others, is called *smooth* if it is finite endowed with a C^∞ differential structure. For a smoothly combinatorial manifold $\widetilde{M}$ and a point $p \in \widetilde{M}$, it has been shown in [9] that $\dim T_p\widetilde{M}(n_1, n_2, \cdots, n_m) = \widehat{s}(p) + \sum\limits_{i=1}^{s(p)} (n_i - \widehat{s}(p))$ and $\dim T_p^*\widetilde{M}(n_1, n_2, \cdots, n_m) = \widehat{s}(p) + \sum\limits_{i=1}^{s(p)} (n_i - \widehat{s}(p))$ with a basis

$$\{\frac{\partial}{\partial x^{hj}}|_p | 1 \leq j \leq \widehat{s}(p)\} \bigcup (\bigcup_{i=1}^{s(p)} \bigcup_{j=\widehat{s}(p)+1}^{n_i} \{\frac{\partial}{\partial x^{ij}}|_p \mid 1 \leq j \leq s\})$$

or

$$\{dx^{hj}|_p|\} 1 \leq j \leq \widehat{s}(p)\} \bigcup (\bigcup_{i=1}^{s(p)} \bigcup_{j=\widehat{s}(p)+1}^{n_i} \{dx^{ij}|_p \mid 1 \leq j \leq s\}$$

for a given integer $h, 1 \leq h \leq s(p)$.

Definition 1.2 *A connection $\widetilde{D}$ on a smoothly combinatorial manifold $\widetilde{M}$ is a mapping $\widetilde{D}$: $\mathscr{X}(\widetilde{M}) \times T^r_s\widetilde{M} \to T^r_s\widetilde{M}$ on tensors of $\widetilde{M}$ with $\widetilde{D}_X\tau = \widetilde{D}(X,\tau)$ such that for $\forall X, Y \in \mathscr{X}\widetilde{M}$, $\tau, \pi \in T^r_s(\widetilde{M})$,$\lambda \in \mathbf{R}$ and $f \in C^\infty(\widetilde{M})$,*

(1) $\widetilde{D}_{X+fY}\tau = \widetilde{D}_X\tau + f\widetilde{D}_Y\tau$; and $\widetilde{D}_X(\tau + \lambda\pi) = \widetilde{D}_X\tau + \lambda\widetilde{D}_X\pi$;

(2) $\widetilde{D}_X(\tau \otimes \pi) = \widetilde{D}_X\tau \otimes \pi + \sigma \otimes \widetilde{D}_X\pi$;

(3) *for any contraction C on $T^r_s(\widetilde{M})$, $\widetilde{D}_X(C(\tau)) = C(\widetilde{D}_X\tau)$.*

A combinatorially connection space is a 2-tuple $(\widetilde{M}, \widetilde{D})$ consisting of a smoothly combinatorial manifold $\widetilde{M}$ with a connection $\widetilde{D}$ and a torsion tensor $\widetilde{T} : \mathscr{X}(\widetilde{M}) \times \mathscr{X}(\widetilde{M}) \to \mathscr{X}(\widetilde{M})$ on $(\widetilde{M}, \widetilde{D})$ is defined by $\widetilde{T}(X, Y) = \widetilde{D}_XY - \widetilde{D}_YX - [X, Y]$ for $\forall X, Y \in \mathscr{X}(\widetilde{M})$. If $\widetilde{T}|_U(X, Y) \equiv 0$ in a local chart $(U, [\varphi])$, then $\widetilde{D}$ is called torsion-free on $(U, [\varphi])$.

Similar to that of Riemannian geometry, metrics on a smoothly combinatorial manifold and the combinatorially Riemannian geometry are defined in next definition.

Definition 1.3 *Let $\widetilde{M}$ be a smoothly combinatorial manifold and $g \in A^2(\widetilde{M}) = \bigcup_{p\in\widetilde{M}} T^0_2(p, \widetilde{M})$. If g is symmetrical and positive, then $\widetilde{M}$ is called a combinatorially Riemannian manifold, denoted by $(\widetilde{M}, g)$. In this case, if there is a connection $\widetilde{D}$ on $(\widetilde{M}, g)$ with equality following hold*

$$Z(g(X, Y)) = g(\widetilde{D}_Z, Y) + g(X, \widetilde{D}_ZY)$$

then $\widetilde{M}$ is called a combinatorially Riemannian geometry, denoted by $(\widetilde{M}, g, \widetilde{D})$.

It has been showed that there exists a unique connection $\widetilde{D}$ on $(\widetilde{M}, g)$ such that $(\widetilde{M}, g, \widetilde{D})$ is a combinatorially Riemannian geometry in [9].

We all known that curvature equations are very important in theoretical physics for describing various classical fields, particularly for gravitational field by Einstein. The main purpose of this paper is to establish curvature tensors with equations on combinatorial manifolds and apply them to describe the gravitational field. For this objective, we introduce the conception of curvatures on combinatorial manifolds and establish symmetrical relations for curvature tensors, particularly for combinatorially Riemannian manifolds in the next two sections. Structural equations of curvature tensors on combinatorial manifolds are also established. These generalized Einstein's equations of gravitational field on combinatorially Riemannian manifolds are constructed in Section 4. By applying the *projective principle* on multi-spaces, multi-space solutions for these new equations are gotten in Section 5.

Terminologies and notations used in this paper are standard and can be found in [1], [4] for those of manifolds [9] − [11] for combinatorial manifolds and [6] − [7] for graphs, respectively.

§2. Curvatures on Combinatorially Connection Spaces

As a first step for introducing curvatures on combinatorial manifolds, we define combinatorially curvature operators on smoothly combinatorial manifolds in the next.

Definition 2.1 *Let $(\widetilde{M},\widetilde{D})$ be a combinatorially connection space. For $\forall X,Y\in\mathscr{X}(\widetilde{M})$, a combinatorially curvature operator $\widetilde{\mathcal{R}}(X,Y):\mathscr{X}(\widetilde{M})\to\mathscr{X}(\widetilde{M})$ is defined by*

$$\widetilde{\mathcal{R}}(X,Y)Z=\widetilde{D}_X\widetilde{D}_YZ-\widetilde{D}_Y\widetilde{D}_XZ-\widetilde{D}_{[X,Y]}Z$$

for $\forall Z\in\mathscr{X}(\widetilde{M})$.

For a given combinatorially connection space $(\widetilde{M},\widetilde{D})$, we know properties following on combinatorially curvature operators similar to those of the Riemannian geometry.

Theorem 2.1 *Let $(\widetilde{M},\widetilde{D})$ be a combinatorially connection space. Then for $\forall X,Y,Z\in\mathscr{X}(\widetilde{M})$, $\forall f\in C^\infty(\widetilde{M})$,*

(1) $\widetilde{\mathcal{R}}(X,Y)=-\widetilde{\mathcal{R}}(Y,X)$;

(2) $\widetilde{\mathcal{R}}(fX,Y)=\widetilde{\mathcal{R}}(X,fY)=f\widetilde{\mathcal{R}}(X,Y)$;

(3) $\widetilde{\mathcal{R}}(X,Y)(fZ)=f\widetilde{\mathcal{R}}(X,Y)Z$.

Proof For $\forall X,Y,Z\in\mathscr{X}(\widetilde{M})$, we know that $\widetilde{\mathcal{R}}(X,Y)Z=-\widetilde{\mathcal{R}}(Y,X)Z$ by definition. Whence, $\widetilde{\mathcal{R}}(X,Y)=-\widetilde{\mathcal{R}}(Y,X)$.

Now since

$$\begin{aligned}
\widetilde{\mathcal{R}}(fX,Y)Z &= \widetilde{D}_{fX}\widetilde{D}_YZ-\widetilde{D}_Y\widetilde{D}_{fX}Z-\widetilde{D}_{[fX,Y]}Z\\
&= f\widetilde{D}_X\widetilde{D}_YZ-\widetilde{D}_Y(f\widetilde{D}_XZ)-\widetilde{D}_{f[X,Y]-Y(f)X}Z\\
&= f\widetilde{D}_X\widetilde{D}_YZ-Y(f)\widetilde{D}_XZ-f\widetilde{D}_Y\widetilde{D}_XZ\\
&- f\widetilde{D}_{[X,Y]}Z+Y(f)\widetilde{D}_XZ\\
&= f\widetilde{\mathcal{R}}(X,Y)Z,
\end{aligned}$$

we get that $\widetilde{\mathcal{R}}(fX,Y)=f\widetilde{\mathcal{R}}(X,Y)$. Applying the quality (1), we find that

$$\widetilde{\mathcal{R}}(X,fY)=-\widetilde{\mathcal{R}}(fY,X)=-f\widetilde{\mathcal{R}}(Y,X)=f\widetilde{\mathcal{R}}(X,Y).$$

This establishes (2). Now calculation shows that

$$\begin{aligned}
\widetilde{\mathcal{R}}(X,Y)(fZ) &= \widetilde{D}_X\widetilde{D}_Y(fZ)-\widetilde{D}_Y\widetilde{D}_X(fZ)-\widetilde{D}_{[X,Y]}(fZ)\\
&= \widetilde{D}_X(Y(f)Z+f\widetilde{D}_YZ)-\widetilde{D}_Y(X(f)Z+f\widetilde{D}_XZ)\\
&- ([X,Y](f))Z-f\widetilde{D}_{[X,Y]}Z\\
&= X(Y(f))Z+Y(f)\widetilde{D}_XZ+X(f)\widetilde{D}_YZ\\
&+ f\widetilde{D}_X\widetilde{D}_YZ-Y(X(f))Z-X(f)\widetilde{D}_YZ-Y(f)\widetilde{D}_XZ\\
&- f\widetilde{D}_Y\widetilde{D}_XZ-([X,Y](f))Z-f\widetilde{D}_{[X,Y]}Z\\
&= f\widetilde{\mathcal{R}}(X,Y)Z.
\end{aligned}$$

Whence, we know that

$$\widetilde{\mathcal{R}}(X,Y)(fZ)=f\widetilde{\mathcal{R}}(X,Y)Z.$$

□

Theorem 2.2 *Let $(\widetilde{M},\widetilde{D})$ be a combinatorially connection space. If the torsion tensor $\widetilde{T}\equiv 0$ on $\widetilde{D}$, then the first and second Bianchi equalities following hold.*

$$\widetilde{\mathcal{R}}(X,Y)Z+\widetilde{\mathcal{R}}(Y,Z)X+\widetilde{\mathcal{R}}(Z,X)Y=0$$

and

$$(\widetilde{D}_X\widetilde{R})(Y,Z)W+(\widetilde{D}_Y\widetilde{R})(Z,X)W+(\widetilde{D}_Z\widetilde{R})(X,Y)W=0.$$

Proof Notice that $\widetilde{T}\equiv 0$ is equal to $\widetilde{D}_XY-\widetilde{D}_YX=[X,Y]$ for $\forall X,Y\in\mathscr{X}(\widetilde{M})$. Thereafter, we know that

$$\begin{aligned}
&\widetilde{\mathcal{R}}(X,Y)Z+\widetilde{\mathcal{R}}(Y,Z)X+\widetilde{\mathcal{R}}(Z,X)Y\\
=\ &\widetilde{D}_X\widetilde{D}_YZ-\widetilde{D}_Y\widetilde{D}_XZ-\widetilde{D}_{[X,Y]}Z+\widetilde{D}_Y\widetilde{D}_ZX-\widetilde{D}_Z\widetilde{D}_YX\\
-\ &\widetilde{D}_{[Y,Z]}X+\widetilde{D}_Z\widetilde{D}_XY-\widetilde{D}_X\widetilde{D}_ZY-\widetilde{D}_{[Z,X]}Y\\
=\ &\widetilde{D}_X(\widetilde{D}_YZ-\widetilde{D}_ZY)-\widetilde{D}_{[Y,Z]}X+\widetilde{D}_Y(\widetilde{D}_ZX-\widetilde{D}_XZ)\\
-\ &\widetilde{D}_{[Z,X]}Y+\widetilde{D}_Z(\widetilde{D}_XY-\widetilde{D}_YX)-\widetilde{D}_{[X,Y]}Z\\
=\ &\widetilde{D}_X[Y,Z]-\widetilde{D}_{[Y,Z]}X+\widetilde{D}_Y[Z,X]-\widetilde{D}_{[Z,X]}Y\\
+\ &\widetilde{D}_Z[X,Y]-\widetilde{D}_{[X,Y]}Z\\
=\ &[X,[Y,Z]]+[Y,[Z,X]]+[Z,[X,Y]].
\end{aligned}$$

By the Jacobi equality $[X,[Y,Z]]+[Y,[Z,X]]+[Z,[X,Y]]=0$, we get that

$$\widetilde{\mathcal{R}}(X,Y)Z+\widetilde{\mathcal{R}}(Y,Z)X+\widetilde{\mathcal{R}}(Z,X)Y=0.$$

By definition, we know that

$$\begin{aligned}
&(\widetilde{D}_X\widetilde{R})(Y,Z)W=\\
&\widetilde{D}_X\widetilde{R}(Y,Z)W-\widetilde{R}(\widetilde{D}_XY,Z)W-\widetilde{R}(Y,\widetilde{D}_XZ)W-\widetilde{R}(Y,Z)\widetilde{D}_XW\\
&=\widetilde{D}_X\widetilde{D}_Y\widetilde{D}_ZW-\widetilde{D}_X\widetilde{D}_Z\widetilde{D}_YW-\widetilde{D}_X\widetilde{D}_{[Y,Z]}W-\widetilde{D}_{\widetilde{D}_XY}\widetilde{D}_ZW\\
&+\widetilde{D}_Z\widetilde{D}_{\widetilde{D}_XY}W+\widetilde{D}_{[\widetilde{D}_XY,Z]}W-\widetilde{D}_Y\widetilde{D}_{\widetilde{D}_XZ}W+\widetilde{D}_{\widetilde{D}_XZ}\widetilde{D}_YW\\
&+\widetilde{D}_{[Y,\widetilde{D}_XZ]}W-\widetilde{D}_Y\widetilde{D}_Z\widetilde{D}_XW+\widetilde{D}_Z\widetilde{D}_Y\widetilde{D}_XW+\widetilde{D}_{[Y,Z]}\widetilde{D}_XW.
\end{aligned}$$

Let

$$A^W(X,Y,Z)=\widetilde{D}_X\widetilde{D}_Y\widetilde{D}_ZW-\widetilde{D}_X\widetilde{D}_Z\widetilde{D}_YW-\widetilde{D}_Y\widetilde{D}_Z\widetilde{D}_XW+\widetilde{D}_Z\widetilde{D}_Y\widetilde{D}_XW,$$

$$B^W(X,Y,Z)=-\widetilde{D}_X\widetilde{D}_{\widetilde{D}_YZ}W+\widetilde{D}_X\widetilde{D}_{\widetilde{D}_ZY}W+\widetilde{D}_Z\widetilde{D}_{\widetilde{D}_XY}W-\widetilde{D}_Y\widetilde{D}_{\widetilde{D}_XZ}W,$$

$$C^W(X,Y,Z) = -\widetilde{D}_{\widetilde{D}_XY}\widetilde{D}_ZW + \widetilde{D}_{\widetilde{D}_XZ}\widetilde{D}_YW + \widetilde{D}_{\widetilde{D}_YZ}\widetilde{D}_XW - \widetilde{D}_{\widetilde{D}_ZY}\widetilde{D}_XW$$

and

$$D^W(X,Y,Z) = \widetilde{D}_{[\widetilde{D}_XY,Z]}W - \widetilde{D}_{[\widetilde{D}_XZ,Y]}W.$$

Applying the equality $\widetilde{D}_XY - \widetilde{D}_YX = [X,Y]$, we find that

$$(\widetilde{D}_X\widetilde{R})(Y,Z)W = A^W(X,Y,Z) + B^W(X,Y,Z) + C^W(X,Y,Z) + D^W(X,Y,Z).$$

We can check immediately that

$$A^W(X,Y,Z) + A^W(Y,Z,X) + A^W(Z,X,Y) = 0,$$
$$B^W(X,Y,Z) + B^W(Y,Z,X) + B^W(Z,X,Y) = 0,$$
$$C^W(X,Y,Z) + C^W(Y,Z,X) + C^W(Z,X,Y) = 0$$

and

$$D^W(X,Y,Z) + D^W(Y,Z,X) + D^W(Z,X,Y)$$
$$= \widetilde{D}_{[X,[Y,Z]]+[Y,[Z,X]]+[Z,[X,Y]]}W = \widetilde{D}_0W = 0$$

by the Jacobi equality $[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0$. Therefore, we get finally that

$$(\widetilde{D}_X\widetilde{R})(Y,Z)W + (\widetilde{D}_Y\widetilde{R})(Z,X)W + (\widetilde{D}_Z\widetilde{R})(X,Y)W$$
$$= A^W(X,Y,Z) + B^W(X,Y,Z) + C^W(X,Y,Z) + D^W(X,Y,Z)$$
$$+A^W(Y,Z,X) + B^W(Y,Z,X) + C^W(Y,Z,X) + D^W(Y,Z,X)$$
$$+A^W(Z,X,Y) + B^W(Z,X,Y) + C^W(Z,X,Y) + D^W(Z,X,Y) = 0.$$

This completes the proof. □

According to Theorem 2.1, the curvature operator $\widetilde{\mathcal{R}}(X,Y) : \mathscr{X}(\widetilde{M}) \to \mathscr{X}(\widetilde{M})$ is a tensor of type $(1,1)$. By applying this operator, we can define a curvature tensor in the next definition.

Definition 2.2 *Let $(\widetilde{M},\widetilde{D})$ be a combinatorially connection space. For $\forall X,Y,Z \in \mathscr{X}(\widetilde{M})$, a linear multi-mapping $\widetilde{\mathcal{R}} : \mathscr{X}(\widetilde{M}) \times \mathscr{X}(\widetilde{M}) \times \mathscr{X}(\widetilde{M}) \to \mathscr{X}(\widetilde{M})$ determined by*

$$\widetilde{\mathcal{R}}(Z,X,Y) = \widetilde{\mathcal{R}}(X,Y)Z$$

is said a curvature tensor of type $(1,3)$ on $(\widetilde{M},\widetilde{D})$.

Let $(\widetilde{M},\widetilde{D})$ be a combinatorially connection space and

$$\{e_{ij}|1 \le i \le s(p), 1 \le j \le n_i \text{ and } e_{i_1j} = e_{i_2j} \text{ for } 1 \le i_1, i_2 \le s(p) \text{ if } 1 \le j \le \widehat{s}(p)\}$$

a local frame with a dual

$$\{\omega^{ij}|1 \leq i \leq s(p), 1 \leq j \leq n_i \text{ and } \omega^{i_1 j} = \omega^{i_2 j} \text{ for } 1 \leq i_1, i_2 \leq s(p) \text{ if } 1 \leq j \leq \widehat{s}(p)\},$$

abbreviated to $\{e_{ij}\}$ and $\{\omega^{ij}\}$ at a point $p \in \widetilde{M}$, where $\widetilde{M} = \widetilde{M}(n_1, n_2, \cdots, n_m)$. Then there exist smooth functions $\Gamma^{\sigma\varsigma}_{(\mu\nu)(\kappa\lambda)} \in C^\infty(\widetilde{M})$ such that

$$\widetilde{D}_{e_{\kappa\lambda}} e_{\mu\nu} = \Gamma^{\sigma\varsigma}_{(\mu\nu)(\kappa\lambda)} e_{\sigma\varsigma}$$

called connection coefficients in the local frame $\{e_{ij}\}$. Define

$$\omega^{\sigma\varsigma}_{\mu\nu} = \Gamma^{\sigma\varsigma}_{(\mu\nu)(\kappa\lambda)} \omega^{\kappa\lambda}.$$

We get that

$$\widetilde{D} e_{\kappa\lambda} = \omega^{\sigma\varsigma}_{\mu\nu} e_{\sigma\varsigma}.$$

Theorem 2.3 *Let $(\widetilde{M}, \widetilde{D})$ be a combinatorially connection space and $\{e_{ij}\}$ a local frame with a dual $\{\omega^{ij}\}$ at a point $p \in \widetilde{M}$. Then*

$$\widetilde{d}\omega^{\mu\nu} - \omega^{\kappa\lambda} \wedge \omega^{\mu\nu}_{\kappa\lambda} = \frac{1}{2} \widetilde{T}^{\mu\nu}_{(\kappa\lambda)(\sigma\varsigma)} \omega^{\kappa\lambda} \wedge \omega^{\sigma\varsigma},$$

where $\widetilde{T}^{\mu\nu}_{(\kappa\lambda)(\sigma\varsigma)}$ is a component of the torsion tensor $\widetilde{T}$ in the frame $\{e_{ij}\}$, i.e., $\widetilde{T}^{\mu\nu}_{(\kappa\lambda)(\sigma\varsigma)} = \omega^{\mu\nu}(\widetilde{T}(e_{\kappa\lambda}, e_{\sigma\varsigma}))$ and

$$\widetilde{d}\omega^{\kappa\lambda}_{\mu\nu} - \omega^{\sigma\varsigma}_{\mu\nu} \wedge \omega^{\kappa\lambda}_{\sigma\varsigma} = \frac{1}{2} \widetilde{R}^{\kappa\lambda}_{(\mu\nu)(\sigma\varsigma)(\eta\theta)} \omega^{\sigma\varsigma} \wedge \omega^{\eta\theta}$$

with $\widetilde{R}^{\kappa\lambda}_{(\mu\nu)(\sigma\varsigma)(\sigma\varsigma)} e_{\kappa\lambda} = \widetilde{R}(e_{\sigma\varsigma}, e_{\eta\theta}) e_{\mu\nu}$.

Proof By definition, for any given $e_{\sigma\varsigma}, e_{\eta\theta}$ we know that (see Theorem 3.6 in [9])

$$\begin{aligned}
(\widetilde{d}\omega^{\mu\nu} - \omega^{\kappa\lambda} \wedge \omega^{\mu\nu}_{\kappa\lambda})(e_{\sigma\varsigma}, e_{\eta\theta}) &= e_{\sigma\varsigma}(\omega^{\mu\nu}(e_{\eta\theta})) - e_{\eta\theta}(\omega^{\mu\nu}(e_{\sigma\varsigma})) - \omega^{\mu\nu}([e_{\sigma\varsigma}, e_{\eta\theta}]) \\
&- \omega^{\kappa\lambda}(e_{\sigma\varsigma})\omega^{\mu\nu}_{\kappa\lambda}(e_{\eta\theta}) + \omega^{\kappa\lambda}(e_{\eta\theta})\omega^{\mu\nu}_{\kappa\lambda}(e_{\sigma\varsigma}) \\
&= -\omega^{\mu\nu}_{\sigma\varsigma}(e_{\eta\theta}) + \omega^{\mu\nu}_{\eta\theta}(e_{\sigma\varsigma}) - \omega^{\mu\nu}([e_{\sigma\varsigma}, e_{\eta\theta}]) \\
&= -\Gamma^{\mu\nu}_{(\sigma\varsigma)(\eta\theta)} + \Gamma^{\mu\nu}_{(\eta\theta)(\sigma\varsigma)} - \omega^{\mu\nu}([e_{\sigma\varsigma}, e_{\eta\theta}]) \\
&= \omega^{\mu\nu}(\widetilde{D}_{e_{\sigma\varsigma}} e_{\eta\theta} - \widetilde{D}_{e_{\eta\theta}} e_{\sigma\varsigma} - [e_{\sigma\varsigma}, e_{\eta\theta}]) \\
&= \omega^{\mu\nu}(\widetilde{T}(e_{\sigma\varsigma}, e_{\eta\theta})) = \widetilde{T}^{\mu\nu}_{(\sigma\varsigma)(\eta\theta)}.
\end{aligned}$$

Whence,

$$\widetilde{d}\omega^{\mu\nu} - \omega^{\kappa\lambda} \wedge \omega^{\mu\nu}_{\kappa\lambda} = \frac{1}{2} \widetilde{T}^{\mu\nu}_{(\kappa\lambda)(\sigma\varsigma)} \omega^{\kappa\lambda} \wedge \omega^{\sigma\varsigma}.$$

Now since

$$\begin{aligned}
&(\widetilde{d}\omega_{\mu\nu}^{\kappa\lambda} - \omega_{\mu\nu}^{\vartheta\iota} \wedge \omega_{\vartheta\iota}^{\kappa\lambda})(e_{\sigma\varsigma}, e_{\eta\theta}) \\
&= e_{\sigma\varsigma}(\omega_{\mu\nu}^{\kappa\lambda}(e_{\eta\theta})) - e_{\eta\theta}(\omega_{\mu\nu}^{\kappa\lambda}(e_{\sigma\varsigma})) - \omega_{\mu\nu}^{\kappa\lambda}([e_{\sigma\varsigma}, e_{\eta\theta}]) \\
&-\omega_{\mu\nu}^{\vartheta\iota}(e_{\sigma\varsigma})\omega_{\vartheta\iota}^{\kappa\lambda}(e_{\eta\theta}) + \omega_{\mu\nu}^{\vartheta\iota}(e_{\eta\theta})\omega_{\vartheta\iota}^{\kappa\lambda}(e_{\sigma\varsigma}) \\
&= e_{\sigma\varsigma}(\Gamma_{(\mu\nu)(\eta\theta)}^{\kappa\lambda}) - e_{\eta\theta}(\Gamma_{(\mu\nu)(\sigma\varsigma)}^{\kappa\lambda}) - \omega^{\vartheta\iota}([e_{\sigma\varsigma}, e_{\eta\theta}])\Gamma_{(\mu\nu)(\vartheta\iota)}^{\kappa\lambda} \\
&-\Gamma_{(\mu\nu)(\sigma\varsigma)}^{\vartheta\iota}\Gamma_{(\vartheta\iota)(\eta\theta)}^{\kappa\lambda} + \Gamma_{(\mu\nu)(\eta\theta)}^{\vartheta\iota}\Gamma_{(\vartheta\iota)(\sigma\varsigma)}^{\kappa\lambda}
\end{aligned}$$

and

$$\begin{aligned}
\widetilde{R}(e_{\sigma\varsigma}, e_{\eta\theta})e_{\mu\nu} &= \widetilde{D}_{e_{\sigma\varsigma}}\widetilde{D}_{e_{\eta\theta}}e_{\mu\nu} - \widetilde{D}_{e_{\eta\theta}}\widetilde{D}_{e_{\sigma\varsigma}}e_{\mu\nu} - \widetilde{D}_{[e_{\sigma\varsigma}, e_{\eta\theta}]}e_{\mu\nu} \\
&= \widetilde{D}_{e_{\sigma\varsigma}}(\Gamma_{(\mu\nu)(\eta\theta)}^{\kappa\lambda}e_{\kappa\lambda}) - \widetilde{D}_{e_{\eta\theta}}(\Gamma_{(\mu\nu)(\sigma\varsigma)}^{\kappa\lambda}e_{\kappa\lambda}) - \omega^{\vartheta\iota}([e_{\sigma\varsigma}, e_{\eta\theta}])\Gamma_{(\mu\nu)(\vartheta\iota)}^{\kappa\lambda}e_{\kappa\lambda} \\
&= (e_{\sigma\varsigma}(\Gamma_{(\mu\nu)(\eta\theta)}^{\kappa\lambda}) - e_{\eta\theta}(\Gamma_{(\mu\nu)(\sigma\varsigma)}^{\kappa\lambda}) + \Gamma_{(\mu\nu)(\eta\theta)}^{\vartheta\iota}\Gamma_{(\vartheta\iota)(\sigma\varsigma)}^{\kappa\lambda} \\
&- \Gamma_{(\mu\nu)(\sigma\varsigma)}^{\vartheta\iota}\Gamma_{(\vartheta\iota)(\eta\theta)}^{\kappa\lambda} - \omega^{\vartheta\iota}([e_{\sigma\varsigma}, e_{\eta\theta}])\Gamma_{(\mu\nu)(\vartheta\iota)}^{\kappa\lambda})e_{\kappa\lambda} \\
&= (\widetilde{d}\omega_{\mu\nu}^{\kappa\lambda} - \omega_{\mu\nu}^{\vartheta\iota} \wedge \omega_{\vartheta\iota}^{\kappa\lambda})(e_{\sigma\varsigma}, e_{\eta\theta})e_{\kappa\lambda}.
\end{aligned}$$

Therefore, we get that

$$(\widetilde{d}\omega_{\mu\nu}^{\kappa\lambda} - \omega_{\mu\nu}^{\vartheta\iota} \wedge \omega_{\vartheta\iota}^{\kappa\lambda})(e_{\sigma\varsigma}, e_{\eta\theta}) = \widetilde{R}_{(\mu\nu)(\sigma\varsigma)(\eta\theta)}^{\kappa\lambda},$$

that is,

$$\widetilde{d}\omega_{\mu\nu}^{\kappa\lambda} - \omega_{\mu\nu}^{\sigma\varsigma} \wedge \omega_{\sigma\varsigma}^{\kappa\lambda} = \frac{1}{2}\widetilde{R}_{(\mu\nu)(\sigma\varsigma)(\eta\theta)}^{\kappa\lambda}\omega^{\sigma\varsigma} \wedge \omega^{\eta\theta}.$$

□

Definition 2.3 *Let $(\widetilde{M}, \widetilde{D})$ be a combinatorially connection space. Differential 2-forms $\Omega^{\mu\nu} = \widetilde{d}\omega^{\mu\nu} - \omega^{\mu\nu} \wedge \omega_{\kappa\lambda}^{\mu\nu}$, $\Omega_{\mu\nu}^{\kappa\lambda} = \widetilde{d}\omega_{\mu\nu}^{\kappa\lambda} - \omega_{\mu\nu}^{\sigma\varsigma} \wedge \omega_{\sigma\varsigma}^{\kappa\lambda}$ and equations*

$$\widetilde{d}\omega^{\mu\nu} = \omega^{\kappa\lambda} \wedge \omega_{\kappa\lambda}^{\mu\nu} + \Omega^{\mu\nu}, \quad \widetilde{d}\omega_{\mu\nu}^{\kappa\lambda} = \omega_{\mu\nu}^{\sigma\varsigma} \wedge \omega_{\sigma\varsigma}^{\kappa\lambda} + \Omega_{\mu\nu}^{\kappa\lambda}$$

are called torsion forms, curvature forms and structural equations in a local frame $\{e_{ij}\}$ of $(\widetilde{M}, \widetilde{D})$, respectively.

By Theorem 2.3 and Definition 2.3, we get local forms for torsion tensor and curvature tensor in a local frame following.

Corollary 2.1 *Let $(\widetilde{M}, \widetilde{D})$ be a combinatorially connection space and $\{e_{ij}\}$ a local frame with a dual $\{\omega^{ij}\}$ at a point $p \in \widetilde{M}$. Then*

$$\widetilde{T} = \Omega^{\mu\nu} \otimes e_{\mu\nu} \quad \textit{and} \quad \widetilde{R} = \omega^{\mu\nu} \otimes e_{\kappa\lambda} \otimes \Omega_{\mu\nu}^{\kappa\lambda},$$

i.e., for $\forall X, Y \in \mathscr{X}(\widetilde{M})$,

$$\widetilde{T}(X,Y) = \Omega^{\mu\nu}(X,Y)e_{\mu\nu} \quad \textit{and} \quad \widetilde{R}(X,Y) = \Omega_{\mu\nu}^{\kappa\lambda}(X,Y)\omega^{\mu\nu} \otimes e_{\mu\nu}.$$

Theorem 2.4 *Let $(\widetilde{M},\widetilde{D})$ be a combinatorially connection space and $\{e_{ij}\}$ a local frame with a dual $\{\omega^{ij}\}$ at a point $p\in\widetilde{M}$. Then*

$$\widetilde{d}\Omega^{\mu\nu}=\omega^{\kappa\lambda}\wedge\Omega^{\mu\nu}_{\kappa\lambda}-\Omega^{\kappa\lambda}\wedge\omega^{\mu\nu}_{\kappa\lambda}\quad and\quad \widetilde{d}\Omega^{\kappa\lambda}_{\mu\nu}=\omega^{\sigma\varsigma}_{\mu\nu}\wedge\Omega^{\kappa\lambda}_{\sigma\varsigma}-\Omega^{\sigma\varsigma}_{\mu\nu}\wedge\omega^{\kappa\lambda}_{\sigma\varsigma}.$$

Proof Notice that $\widetilde{d}^2=0$. Differentiating the equality $\Omega^{\mu\nu}=\widetilde{d}\omega^{\mu\nu}-\omega^{\mu\nu}\wedge\omega^{\mu\nu}_{\kappa\lambda}$ on both sides, we get that

$$\begin{aligned}\widetilde{d}\Omega^{\mu\nu} &= -\widetilde{d}\omega^{\mu\nu}\wedge\omega^{\mu\nu}_{\kappa\lambda}+\omega^{\mu\nu}\wedge\widetilde{d}\omega^{\mu\nu}_{\kappa\lambda}\\ &= -(\Omega^{\kappa\lambda}+\omega^{\sigma\varsigma}\wedge\omega^{\kappa\lambda}_{\sigma\varsigma})\wedge\omega^{\mu\nu}_{\kappa\lambda}+\omega^{\kappa\lambda}\wedge(\Omega^{\mu\nu}_{\kappa\lambda}+\omega^{\sigma\varsigma}_{\kappa\lambda}\wedge\omega^{\mu\nu}_{\sigma\varsigma})\\ &= \omega^{\kappa\lambda}\wedge\Omega^{\mu\nu}_{\kappa\lambda}-\Omega^{\kappa\lambda}\wedge\omega^{\mu\nu}_{\kappa\lambda}.\end{aligned}$$

Similarly, differentiating the equality $\Omega^{\kappa\lambda}_{\mu\nu}=\widetilde{d}\omega^{\kappa\lambda}_{\mu\nu}-\omega^{\sigma\varsigma}_{\mu\nu}\wedge\omega^{\kappa\lambda}_{\sigma\varsigma}$ on both sides, we can also find that

$$\widetilde{d}\Omega^{\kappa\lambda}_{\mu\nu}=\omega^{\sigma\varsigma}_{\mu\nu}\wedge\Omega^{\kappa\lambda}_{\sigma\varsigma}-\Omega^{\sigma\varsigma}_{\mu\nu}\wedge\omega^{\kappa\lambda}_{\sigma\varsigma}.$$

□

Corollary 2.2 *Let (M,D) be an affine connection space and $\{e_i\}$ a local frame with a dual $\{\omega^i\}$ at a point $p\in M$. Then*

$$d\Omega^i=\omega^j\wedge\Omega^i_j-\Omega^j\wedge\omega^i_j\quad and\quad d\Omega^j_i=\omega^k_i\wedge\Omega^j_k-\Omega^k_i\wedge\Omega^j_k.$$

According to Theorems 2.1−2.4 there is a type $(1,3)$ tensor $\widetilde{\mathcal{R}}_p:T_p\widetilde{M}\times T_p\widetilde{M}\times T_p\widetilde{M}\to T_p\widetilde{M}$ determined by $\widetilde{\mathcal{R}}(w,u,v)=\widetilde{\mathcal{R}}(u,v)w$ for $\forall u,v,w\in T_p\widetilde{M}$ at each point $p\in\widetilde{M}$. Particularly, we get its a concrete local form in the standard basis $\{\frac{\partial}{\partial x^{\mu\nu}}\}$.

Theorem 2.5 *Let $(\widetilde{M},\widetilde{D})$ be a combinatorially connection space. Then for $\forall p\in\widetilde{M}$ with a local chart $(U_p;[\varphi_p])$,*

$$\widetilde{\mathcal{R}}=\widetilde{\mathcal{R}}^{\eta\theta}_{(\sigma\varsigma)(\mu\nu)(\kappa\lambda)}dx^{\sigma\varsigma}\otimes\frac{\partial}{\partial x^{\eta\theta}}\otimes dx^{\mu\nu}\otimes dx^{\kappa\lambda}$$

with

$$\widetilde{\mathcal{R}}^{\eta\theta}_{(\sigma\varsigma)(\mu\nu)(\kappa\lambda)}=\frac{\partial\Gamma^{\eta\theta}_{(\sigma\varsigma)(\kappa\lambda)}}{\partial x^{\mu\nu}}-\frac{\partial\Gamma^{\eta\theta}_{(\sigma\varsigma)(\mu\nu)}}{\partial x^{\kappa\lambda}}+\Gamma^{\vartheta\iota}_{(\sigma\varsigma)(\kappa\lambda)}\Gamma^{\eta\theta}_{(\vartheta\iota)(\mu\nu)}-\Gamma^{\vartheta\iota}_{(\sigma\varsigma)(\mu\nu)}\Gamma^{\eta\theta}_{(\vartheta\iota)(\kappa\lambda)})\frac{\partial}{\partial x^{\vartheta\iota}},$$

where, $\Gamma^{\sigma\varsigma}_{(\mu\nu)(\kappa\lambda)}\in C^\infty(U_p)$ is determined by

$$\widetilde{D}_{\frac{\partial}{\partial x^{\mu\nu}}}\frac{\partial}{\partial x^{\kappa\lambda}}=\Gamma^{\sigma\varsigma}_{(\kappa\lambda)(\mu\nu)}\frac{\partial}{\partial x^{\sigma\varsigma}}.$$

Proof We only need to prove that for integers $\mu,\nu,\kappa,\lambda,\sigma,\varsigma,\iota$ and θ,

$$\widetilde{\mathcal{R}}(\frac{\partial}{\partial x^{\mu\nu}}, \frac{\partial}{\partial x^{\kappa\lambda}})\frac{\partial}{\partial x^{\sigma\varsigma}} = \widetilde{\mathcal{R}}^{\eta\theta}_{(\sigma\varsigma)(\mu\nu)(\kappa\lambda)}\frac{\partial}{\partial x^{\eta\theta}}$$

at the local chart $(U_p; [\varphi_p])$. In fact, by definition we get that

$$\begin{aligned}
&\widetilde{\mathcal{R}}(\frac{\partial}{\partial x^{\mu\nu}}, \frac{\partial}{\partial x^{\kappa\lambda}})\frac{\partial}{\partial x^{\sigma\varsigma}} \\
&= \widetilde{D}_{\frac{\partial}{\partial x^{\mu\nu}}}\widetilde{D}_{\frac{\partial}{\partial x^{\kappa\lambda}}}\frac{\partial}{\partial x^{\sigma\varsigma}} - \widetilde{D}_{\frac{\partial}{\partial x^{\kappa\lambda}}}\widetilde{D}_{\frac{\partial}{\partial x^{\mu\nu}}}\frac{\partial}{\partial x^{\sigma\varsigma}} - \widetilde{D}_{[\frac{\partial}{\partial x^{\mu\nu}}, \frac{\partial}{\partial x^{\kappa\lambda}}]}\frac{\partial}{\partial x^{\sigma\varsigma}} \\
&= \widetilde{D}_{\frac{\partial}{\partial x^{\mu\nu}}}(\Gamma^{\eta\theta}_{(\sigma\varsigma)(\kappa\lambda)}\frac{\partial}{\partial x^{\eta\theta}}) - \widetilde{D}_{\frac{\partial}{\partial x^{\kappa\lambda}}}(\Gamma^{\eta\theta}_{(\sigma\varsigma)(\mu\nu)}\frac{\partial}{\partial x^{\eta\theta}}) \\
&= \frac{\partial\Gamma^{\eta\theta}_{(\sigma\varsigma)(\kappa\lambda)}}{\partial x^{\mu\nu}}\frac{\partial}{\partial x^{\eta\theta}} + \Gamma^{\eta\theta}_{(\sigma\varsigma)(\kappa\lambda)}\widetilde{D}_{\frac{\partial}{\partial x^{\mu\nu}}}\frac{\partial}{\partial x^{\eta\theta}} - \frac{\partial\Gamma^{\eta\theta}_{(\sigma\varsigma)(\mu\nu)}}{\partial x^{\kappa\lambda}}\frac{\partial}{\partial x^{\eta\theta}} - \Gamma^{\eta\theta}_{(\sigma\varsigma)(\mu\nu)}\widetilde{D}_{\frac{\partial}{\partial x^{\kappa\lambda}}}\frac{\partial}{\partial x^{\eta\theta}} \\
&= (\frac{\partial\Gamma^{\eta\theta}_{(\sigma\varsigma)(\kappa\lambda)}}{\partial x^{\mu\nu}} - \frac{\partial\Gamma^{\eta\theta}_{(\sigma\varsigma)(\mu\nu)}}{\partial x^{\kappa\lambda}})\frac{\partial}{\partial x^{\eta\theta}} + \Gamma^{\eta\theta}_{(\sigma\varsigma)(\kappa\lambda)}\Gamma^{\vartheta\iota}_{(\eta\theta)(\mu\nu)}\frac{\partial}{\partial x^{\vartheta\iota}} - \Gamma^{\eta\theta}_{(\sigma\varsigma)(\mu\nu)}\Gamma^{\vartheta\iota}_{(\eta\theta)(\kappa\lambda)}\frac{\partial}{\partial x^{\vartheta\iota}} \\
&= (\frac{\partial\Gamma^{\eta\theta}_{(\sigma\varsigma)(\kappa\lambda)}}{\partial x^{\mu\nu}} - \frac{\partial\Gamma^{\eta\theta}_{(\sigma\varsigma)(\mu\nu)}}{\partial x^{\kappa\lambda}} + \Gamma^{\vartheta\iota}_{(\sigma\varsigma)(\kappa\lambda)}\Gamma^{\eta\theta}_{(\vartheta\iota)(\mu\nu)} - \Gamma^{\vartheta\iota}_{(\sigma\varsigma)(\mu\nu)}\Gamma^{\eta\theta}_{(\vartheta\iota)(\kappa\lambda)})\frac{\partial}{\partial x^{\vartheta\iota}} \\
&= \widetilde{\mathcal{R}}^{\eta\theta}_{(\sigma\varsigma)(\mu\nu)(\kappa\lambda)}\frac{\partial}{\partial x^{\eta\theta}}.
\end{aligned}$$

This completes the proof. □

For the curvature tensor $\widetilde{\mathcal{R}}^{\eta\theta}_{(\sigma\varsigma)(\mu\nu)(\kappa\lambda)}$, we can also get these *Bianchi identities* in the next result.

Theorem 2.6 *Let $(\widetilde{M}, \widetilde{D})$ be a combinatorially connection space. Then for $\forall p \in \widetilde{M}$ with a local chart $(U_p, [\varphi_p])$, if $\widetilde{T} \equiv 0$, then*

$$\widetilde{R}^{\mu\nu}_{(\kappa\lambda)(\sigma\varsigma)(\eta\theta)} + \widetilde{R}^{\mu\nu}_{(\sigma\varsigma)(\eta\theta)(\kappa\lambda)} + \widetilde{R}^{\mu\nu}_{(\eta\theta)(\kappa\lambda)(\sigma\varsigma)} = 0$$

and

$$\widetilde{D}_{\vartheta\iota}\widetilde{R}^{\kappa\lambda}_{(\mu\nu)(\sigma\varsigma)(\eta\theta)} + \widetilde{D}_{\sigma\varsigma}\widetilde{R}^{\kappa\lambda}_{(\mu\nu)(\eta\theta)(\vartheta\iota)} + \widetilde{D}_{\eta\theta}\widetilde{R}^{\kappa\lambda}_{(\mu\nu)(\vartheta\iota)(\sigma\varsigma)} = 0,$$

where,

$$\widetilde{D}_{\vartheta\iota}\widetilde{R}^{\kappa\lambda}_{(\mu\nu)(\sigma\varsigma)(\eta\theta)} = \widetilde{D}_{\frac{\partial}{\partial x^{\vartheta\iota}}}\widetilde{R}^{\kappa\lambda}_{(\mu\nu)(\sigma\varsigma)(\eta\theta)}.$$

Proof By definition of the curvature tensor $\widetilde{\mathcal{R}}^{\eta\theta}_{(\sigma\varsigma)(\mu\nu)(\kappa\lambda)}$, we know that

$$\begin{aligned}
&\widetilde{R}^{\mu\nu}_{(\kappa\lambda)(\sigma\varsigma)(\eta\theta)} + \widetilde{R}^{\mu\nu}_{(\sigma\varsigma)(\eta\theta)(\kappa\lambda)} + \widetilde{R}^{\mu\nu}_{(\eta\theta)(\kappa\lambda)(\sigma\varsigma)} \\
&= \widetilde{R}(\frac{\partial}{\partial x^{\sigma\varsigma}}, \frac{\partial}{\partial x^{\eta\theta}})\frac{\partial}{\partial x^{\kappa\lambda}} + \widetilde{R}(\frac{\partial}{\partial x^{\eta\theta}}, \frac{\partial}{\partial x^{\kappa\lambda}})\frac{\partial}{\partial x^{\sigma\varsigma}} + \widetilde{R}(\frac{\partial}{\partial x^{\kappa\lambda}}, \frac{\partial}{\partial x^{\sigma\varsigma}})\frac{\partial}{\partial x^{\eta\theta}} \\
&= 0
\end{aligned}$$

with

$$X = \frac{\partial}{\partial x^{\sigma\varsigma}}, \quad Y = \frac{\partial}{\partial x^{\eta\theta}} \text{ and } Z = \frac{\partial}{\partial x^{\kappa\lambda}}. \quad \natural$$

in the first Bianchi equality and

$$\begin{aligned}
&\widetilde{D}_{\vartheta\iota}\widetilde{R}^{\kappa\lambda}_{(\mu\nu)(\sigma\varsigma)(\eta\theta)} + \widetilde{D}_{\sigma\varsigma}\widetilde{R}^{\kappa\lambda}_{(\mu\nu)(\eta\theta)(\vartheta\iota)} + \widetilde{D}_{\eta\theta}\widetilde{R}^{\kappa\lambda}_{(\mu\nu)(\vartheta\iota)(\sigma\varsigma)} \\
&= \widetilde{D}_{\vartheta\iota}\widetilde{R}(\frac{\partial}{\partial x^{\sigma\varsigma}}, \frac{\partial}{\partial x^{\eta\theta}})\frac{\partial}{\partial x^{\kappa\lambda}} + \widetilde{D}_{\sigma\varsigma}\widetilde{R}(\frac{\partial}{\partial x^{\eta\theta}}, \frac{\partial}{\partial x^{\vartheta\iota}})\frac{\partial}{\partial x^{\kappa\lambda}} + \widetilde{D}_{\eta\theta}\widetilde{R}(\frac{\partial}{\partial x^{\vartheta\iota}}, \frac{\partial}{\partial x^{\sigma\varsigma}})\frac{\partial}{\partial x^{\kappa\lambda}} \\
&= 0.
\end{aligned}$$

with

$$X = \frac{\partial}{\partial x^{\vartheta\iota}}, \ Y = \frac{\partial}{\partial x^{\sigma\varsigma}}, \ Z = \frac{\partial}{\partial x^{\eta\theta}}, \ W = \frac{\partial}{\partial x^{\kappa\lambda}}$$

in the second Bianchi equality of Theorem 2.2. □

§3. Curvatures on Combinatorially Riemannian Manifolds

Now we turn our attention to combinatorially Riemannian manifolds and characterize curvature tensors on combinatorial manifolds further.

Definition 3.1 *Let $(\widetilde{M}, g, \widetilde{D})$ be a combinatorially Riemannian manifold. A combinatorially Riemannian curvature tensor*

$$\widetilde{R} : \mathscr{X}(\widetilde{M}) \times \mathscr{X}(\widetilde{M}) \times \mathscr{X}(\widetilde{M}) \times \mathscr{X}(\widetilde{M}) \to C^{\infty}(\widetilde{M})$$

of type $(0, 4)$ is defined by

$$\widetilde{R}(X, Y, Z, W) = g(\widetilde{R}(Z, W)X, Y)$$

for $\forall X, Y, Z, W \in \mathscr{X}(\widetilde{M})$.

Then we find symmetrical relations of $\widetilde{R}(X, Y, Z, W)$ following.

Theorem 3.1 *Let $\widetilde{R} : \mathscr{X}(\widetilde{M}) \times \mathscr{X}(\widetilde{M}) \times \mathscr{X}(\widetilde{M}) \times \mathscr{X}(\widetilde{M}) \to C^{\infty}(\widetilde{M})$ be a combinatorially Riemannian curvature tensor. Then for $\forall X, Y, Z, W \in \mathscr{X}(\widetilde{M})$,*

(1) $\widetilde{R}(X, Y, Z, W) + \widetilde{R}(Z, Y, W, X) + \widetilde{R}(W, Y, X, Z) = 0$.

(2) $\widetilde{R}(X, Y, Z, W) = -\widetilde{R}(Y, X, Z, W)$ *and* $\widetilde{R}(X, Y, Z, W) = -\widetilde{R}(X, Y, W, Z)$.

(3) $\widetilde{R}(X, Y, Z, W) = \widetilde{R}(Z, W, X, Y)$.

Proof For the equality (1), calculation shows that

$$\begin{aligned}
&\widetilde{R}(X, Y, Z, W) + \widetilde{R}(Z, Y, W, X) + \widetilde{R}(W, Y, X, Z) \\
&= g(\widetilde{R}(Z, W)X, Y) + g(\widetilde{R}(W, X)Z, Y) + g(\widetilde{R}(X, Z)W, Y) \\
&= g(\widetilde{R}(Z, W)X + \widetilde{R}(W, X)Z + \widetilde{R}(X, Z)W, Y) = 0
\end{aligned}$$

by definition and Theorem 2.1(4).

For (2), by definition and Theorem 2.1(1), we know that

$$\begin{aligned}\widetilde{R}(X,Y,Z,W) &= g(\widetilde{R}(Z,W)X,Y) = g(-\widetilde{R}(W,Z)X,Y)\\ &= -g(\widetilde{R}(W,Z)X,Y) = -\widetilde{R}(X,Y,W,Z).\end{aligned}$$

Now since $\widetilde{D}$ is a combinatorially Riemannian connection, we know that ([9])

$$Z(g(X,Y)) = g(\widetilde{D}_Z X,Y) + g(X,\widetilde{D}_Z Y).$$

Therefore, we find that

$$\begin{aligned}g(\widetilde{D}_Z\widetilde{D}_W X,Y) &= Z(g(\widetilde{D}_W X,Y)) - g(\widetilde{D}_W X,\widetilde{D}_Z Y)\\ &= Z(W(g(X,Y))) - Z(g(X,\widetilde{D}_W Y))\\ &- W(g(X,\widetilde{D}_Z Y)) + g(X,\widetilde{D}_W\widetilde{D}_Z Y).\end{aligned}$$

Similarly, we have that

$$\begin{aligned}g(\widetilde{D}_W\widetilde{D}_Z X,Y) &= W(Z(g(X,Y))) - W(g(X,\widetilde{D}_Z Y))\\ &- Z(g(X,\widetilde{D}_W Y)) + g(X,\widetilde{D}_Z\widetilde{D}_W Y).\end{aligned}$$

Notice that

$$g(\widetilde{D}_{[Z,W]},Y) = [Z,W]g(X,Y) - g(X,\widetilde{D}_{[Z,W]}Y).$$

By definition, we get that

$$\begin{aligned}\widetilde{R}(X,Y,Z,W) &= g(\widetilde{D}_Z\widetilde{D}_W X - \widetilde{D}_W\widetilde{D}_Z X - \widetilde{D}_{[Z,W]}X,Y)\\ &= g(\widetilde{D}_Z\widetilde{D}_W X,Y) - g(\widetilde{D}_W\widetilde{D}_Z X,Y) - g(\widetilde{D}_{[Z,W]}X,Y)\\ &= Z(W(g(X,Y))) - Z(g(X,\widetilde{D}_W Y)) - W(g(X,\widetilde{D}_Z Y))\\ &+ g(X,\widetilde{D}_W\widetilde{D}_Z Y) - W(Z(g(X,Y))) + W(g(X,\widetilde{D}_Z Y))\\ &+ Z(g(X,\widetilde{D}_W Y)) - g(X,\widetilde{D}_Z\widetilde{D}_W Y) - [Z,W]g(X,Y)\\ &- g(X,\widetilde{D}_{[Z,W]}Y)\\ &= Z(W(g(X,Y))) - W(Z(g(X,Y))) + g(X,\widetilde{D}_W\widetilde{D}_Z Y)\\ &- g(X,\widetilde{D}_Z\widetilde{D}_W Y) - [Z,W]g(X,Y) - g(X,\widetilde{D}_{[Z,W]}Y)\\ &= g(X,\widetilde{D}_W\widetilde{D}_Z Y - \widetilde{D}_Z\widetilde{D}_W Y + \widetilde{D}_{[Z,W]}Y)\\ &= -g(X,\widetilde{R}(Z,W)Y) = -\widetilde{R}(Y,X,Z,W).\end{aligned}$$

Applying the equality (1), we know that

$$\widetilde{R}(X,Y,Z,W)+\widetilde{R}(Z,Y,W,X)+\widetilde{R}(W,Y,X,Z)=0, \quad (3.1)$$

$$\widetilde{R}(Y,Z,W,X)+\widetilde{R}(W,Z,X,Y)+\widetilde{R}(X,Z,Y,W)=0. \quad (3.2)$$

Then $(3.1)+(3.2)$ shows that

$$\begin{aligned}\widetilde{R}(X,Y,Z,W) &+ \widetilde{R}(W,Y,X,Z)\\ &+ \widetilde{R}(W,Z,X,Y)+\widetilde{R}(X,Z,Y,W)=0\end{aligned}$$

by applying (2). We also know that

$$\begin{aligned}\widetilde{R}(W,Y,X,Z)-\widetilde{R}(X,Z,Y,W) &= -(\widetilde{R}(Z,Y,W,X)-\widetilde{R}(W,X,Z,Y))\\ &= \widetilde{R}(X,Y,Z,W)-\widetilde{R}(Z,W,X,Y).\end{aligned}$$

This enables us getting the equality (3)

$$\widetilde{R}(X,Y,Z,W)=\widetilde{R}(Z,W,X,Y).$$

□

Applying Theorems 2.2, 2.3 and 3.1, we also get the next result.

Theorem 3.2 *Let* $(\widetilde{M},g,\widetilde{D})$ *be a combinatorially Riemannian manifold and* $\Omega_{(\mu\nu)(\kappa\lambda)} = \Omega_{\mu\nu}^{\sigma\varsigma}g_{(\sigma\varsigma)(\kappa\lambda)}$. *Then*

(1) $\Omega_{(\mu\nu)(\kappa\lambda)}=\frac{1}{2}\widetilde{R}_{(\mu\nu)(\kappa\lambda)(\sigma\varsigma)(\eta\theta)}\omega^{\sigma\varsigma}\wedge\omega^{\eta\theta}$;

(2) $\Omega_{(\mu\nu)(\kappa\lambda)}+\Omega_{(\kappa\lambda)(\mu\nu)}=0$;

(3) $\omega^{\mu\nu}\wedge\Omega_{(\mu\nu)(\kappa\lambda)}=0$;

(4) $\widetilde{d}\Omega_{(\mu\nu)(\kappa\lambda)}=\omega_{\mu\nu}^{\sigma\varsigma}\wedge\Omega_{(\sigma\varsigma)(\kappa\lambda)}-\omega_{\kappa\lambda}^{\sigma\varsigma}\wedge\Omega_{(\sigma\varsigma)(\mu\nu)}$.

Proof Notice that $\widetilde{T}\equiv 0$ in a combinatorially Riemannian manifold $(\widetilde{M},g,\widetilde{D})$. We find that

$$\Omega_{\mu\nu}^{\kappa\lambda}=\frac{1}{2}\widetilde{R}_{(\mu\nu)(\sigma\varsigma)(\eta\theta)}^{\kappa\lambda}\omega^{\sigma\varsigma}\wedge\omega^{\eta\theta}$$

by Theorem 2.2. By definition, we know that

$$\begin{aligned}\Omega_{(\mu\nu)(\kappa\lambda)} &= \Omega_{\mu\nu}^{\sigma\varsigma}g_{(\sigma\varsigma)(\kappa\lambda)}\\ &= \frac{1}{2}\widetilde{R}_{(\mu\nu)(\eta\theta)(\vartheta\iota)}^{\sigma\varsigma}g_{(\sigma\varsigma)(\kappa\lambda)}\omega^{\eta\theta}\wedge\omega^{\vartheta\iota}=\frac{1}{2}\widetilde{R}_{(\mu\nu)(\kappa\lambda)(\sigma\varsigma)(\eta\theta)}\omega^{\sigma\varsigma}\wedge\omega^{\eta\theta}.\end{aligned}$$

Whence, we get the equality (1). For (2), applying Theorem 3.1(2), we find that

$$\Omega_{(\mu\nu)(\kappa\lambda)}+\Omega_{(\kappa\lambda)(\mu\nu)}=\frac{1}{2}(\widetilde{R}_{(\mu\nu)(\kappa\lambda)(\sigma\varsigma)(\eta\theta)}+\widetilde{R}_{(\kappa\lambda)(\mu\nu)(\sigma\varsigma)(\eta\theta)})\omega^{\sigma\varsigma}\wedge\omega^{\eta\theta}=0.$$

By Corollary 2.1, a connection $\widetilde{D}$ is torsion-free only if $\Omega^{\mu\nu} \equiv 0$. This fact enables us to get these equalities (3) and (4) by Theorem 2.3. □

For any point $p \in \widetilde{M}$ with a local chart $(U_p, [\varphi_p])$, we can also find a local form of $\widetilde{R}$ in the next result.

Theorem 3.3 *Let $\widetilde{R} : \mathscr{X}(\widetilde{M}) \times \mathscr{X}(\widetilde{M}) \times \mathscr{X}(\widetilde{M}) \times \mathscr{X}(\widetilde{M}) \to C^\infty(\widetilde{M})$ be a combinatorially Riemannian curvature tensor. Then for $\forall p \in \widetilde{M}$ with a local chart $(U_p; [\varphi_p])$,*

$$\widetilde{R} = \widetilde{R}_{(\sigma\varsigma)(\eta\theta)(\mu\nu)(\kappa\lambda)} dx^{\sigma\varsigma} \otimes dx^{\eta\theta} \otimes dx^{\mu\nu} \otimes dx^{\kappa\lambda}$$

with

$$\begin{aligned}\widetilde{R}_{(\sigma\varsigma)(\eta\theta)(\mu\nu)(\kappa\lambda)} &= \frac{1}{2}\Big(\frac{\partial^2 g_{(\mu\nu)(\sigma\varsigma)}}{\partial x^{\kappa\lambda}\partial x^{\eta\theta}} + \frac{\partial^2 g_{(\kappa\lambda)(\eta\theta)}}{\partial x^{\mu\nu\nu}\partial x^{\sigma\varsigma}} - \frac{\partial^2 g_{(\mu\nu)(\eta\theta)}}{\partial x^{\kappa\lambda}\partial x^{\sigma\varsigma}} - \frac{\partial^2 g_{(\kappa\lambda)(\sigma\varsigma)}}{\partial x^{\mu\nu}\partial x^{\eta\theta}}\Big) \\ &+ \Gamma^{\vartheta\iota}_{(\mu\nu)(\sigma\varsigma)}\Gamma^{\xi o}_{(\kappa\lambda)(\eta\theta)} g_{(\xi o)(\vartheta\iota)} - \Gamma^{\xi o}_{(\mu\nu)(\eta\theta)}\Gamma_{(\kappa\lambda)(\sigma\varsigma)^{\vartheta\iota}} g_{(\xi o)(\vartheta\iota)},\end{aligned}$$

where $g_{(\mu\nu)(\kappa\lambda)} = g(\frac{\partial}{\partial x^{\mu\nu}}, \frac{\partial}{\partial x^{\kappa\lambda}})$.

Proof Notice that

$$\begin{aligned}\widetilde{R}_{(\sigma\varsigma)(\eta\theta)(\mu\nu)(\kappa\lambda)} &= \widetilde{R}\Big(\frac{\partial}{\partial x^{\sigma\varsigma}}, \frac{\partial}{\partial x^{\eta\theta}}, \frac{\partial}{\partial x^{\mu\nu}}, \frac{\partial}{\partial x^{\kappa\lambda}}\Big) = \widetilde{R}\Big(\frac{\partial}{\partial x^{\mu\nu}}, \frac{\partial}{\partial x^{\kappa\lambda}}, \frac{\partial}{\partial x^{\sigma\varsigma}}, \frac{\partial}{\partial x^{\eta\theta}}\Big) \\ &= g\Big(\widetilde{R}\Big(\frac{\partial}{\partial x^{\sigma\varsigma}}, \frac{\partial}{\partial x^{\eta\theta}}\Big)\frac{\partial}{\partial x^{\mu\nu}}, \frac{\partial}{\partial x^{\kappa\lambda}}\Big) = \widetilde{R}^{\vartheta\iota}_{(\mu\nu)(\sigma\varsigma)(\eta\theta)} g_{(\vartheta\iota)(\kappa\lambda)}\end{aligned}$$

By definition and Theorem 3.1(3). Now we have know that (eqn.(3.5) in [9])

$$\frac{\partial g_{(\mu\nu)(\kappa\lambda)}}{\partial x^{\sigma\varsigma}} = \Gamma^{\eta\theta}_{(\mu\nu)(\sigma\varsigma)} g_{(\eta\theta)(\kappa\lambda)} + \Gamma^{\eta\theta}_{(\kappa\lambda)(\sigma\varsigma)} g_{(\mu\nu)(\eta\theta)}.$$

Applying Theorem 2.4, we get that

$$\begin{aligned}
&\widetilde{R}_{(\sigma\varsigma)(\eta\theta)(\mu\nu)(\kappa\lambda)}\\
&=(\frac{\partial\Gamma^{\vartheta\iota}_{(\sigma\varsigma)(\kappa\lambda)}}{\partial x^{\mu\nu}}-\frac{\partial\Gamma^{\vartheta\iota}_{(\sigma\varsigma)(\mu\nu)}}{\partial x^{\kappa\lambda}}+\Gamma^{\xi o}_{(\sigma\varsigma)(\kappa\lambda)}\Gamma^{\vartheta\iota}_{(\xi o)(\mu\nu)}-\Gamma^{\xi o}_{(\sigma\varsigma)(\mu\nu)}\Gamma^{\vartheta\iota}_{(\xi o)(\kappa\lambda)})g_{(\vartheta\iota)(\eta\theta)}\\
&=\frac{\partial}{\partial x^{\mu\nu}}(\Gamma^{\vartheta\iota}_{(\sigma\varsigma)(\kappa\lambda)}g_{(\vartheta\iota)(\eta\theta)})-\Gamma^{\vartheta\iota}_{(\sigma\varsigma)(\kappa\lambda)}\frac{\partial g_{(\vartheta\iota)(\eta\theta)}}{\partial x^{\mu\nu}}-\frac{\partial}{\partial x^{\kappa\lambda}}(\Gamma^{\vartheta\iota}_{(\sigma\varsigma)(\mu\nu)}g_{(\vartheta\iota)(\eta\theta)})\\
&+\Gamma^{\vartheta\iota}_{(\sigma\varsigma)(\mu\nu)}\frac{\partial g_{(\vartheta\iota)(\eta\theta)}}{\partial x^{\kappa\lambda}}+\Gamma^{\xi o}_{(\sigma\varsigma)(\kappa\lambda)}\Gamma^{\vartheta\iota}_{(\xi o)(\mu\nu)}g_{(\vartheta\iota)(\kappa\lambda)}-\Gamma^{\xi o}_{(\sigma\varsigma)(\mu\nu)}\Gamma^{\vartheta\iota}_{(\xi o)(\kappa\lambda)}g_{(\vartheta\iota)(\eta\theta)}\\
&=\frac{\partial}{\partial x^{\mu\nu}}(\Gamma^{\vartheta\iota}_{(\sigma\varsigma)(\kappa\lambda)}g_{(\vartheta\iota)(\eta\theta)})-\frac{\partial}{\partial x^{\kappa\lambda}}(\Gamma^{\vartheta\iota}_{(\sigma\varsigma)(\mu\nu)}g_{(\vartheta\iota)(\eta\theta)})\\
&+\Gamma^{\vartheta\iota}_{(\sigma\varsigma)(\mu\nu)}(\Gamma^{\xi o}_{(\vartheta\iota)(\kappa\lambda)}g_{(\xi o)(\eta\theta)}+\Gamma^{\xi o}_{(\eta\theta)(\kappa\lambda)}g_{(\vartheta\iota)(\xi o)})+\Gamma^{\xi o}_{(\sigma\varsigma)(\kappa\lambda)}\Gamma^{\vartheta\iota}_{(\xi o)(\mu\nu)}g_{(\vartheta\iota)(\kappa\lambda)}\\
&-\Gamma^{\vartheta\iota}_{(\sigma\varsigma)(\kappa\lambda)}(\Gamma^{\xi o}_{(\vartheta\iota)(\mu\nu)}g_{(\xi o)(\eta\theta)}+\Gamma^{\xi o}_{(\eta\theta)(\mu\nu)}g_{(\vartheta\iota)(\xi o)})-\Gamma^{\xi o}_{(\sigma\varsigma)(\mu\nu)}\Gamma^{\vartheta\iota}_{(\xi o)(\kappa\lambda)})g_{(\vartheta\iota)(\eta\theta)}\\
&=\frac{1}{2}\frac{\partial}{\partial x^{\mu\nu}}(\frac{\partial g_{(\sigma\varsigma)(\eta\theta)}}{\partial x^{\kappa\lambda}}+\frac{\partial g_{(\kappa\lambda)(\eta\theta)}}{\partial x^{\sigma\varsigma}}-\frac{\partial g_{(\sigma\varsigma)(\kappa\lambda)}}{\partial x^{\eta\theta}})+\Gamma^{\xi o}_{(\sigma\varsigma)(\kappa\lambda)}\Gamma^{\vartheta\iota}_{(\xi o)(\mu\nu)}g_{(\vartheta\iota)(\kappa\lambda)}\\
&-\frac{1}{2}\frac{\partial}{\partial x^{\kappa\lambda}}(\frac{\partial g_{(\sigma\varsigma)(\eta\theta)}}{\partial x^{\mu\nu}}+\frac{\partial g_{(\mu\nu)(\eta\theta)}}{\partial x^{\sigma\varsigma}}-\frac{\partial g_{(\sigma\varsigma)(\mu\nu)}}{\partial x^{\eta\theta}})-\Gamma^{\xi o}_{(\sigma\varsigma)(\mu\nu)}\Gamma^{\vartheta\iota}_{(\xi o)(\kappa\lambda)})g_{(\vartheta\iota)(\eta\theta)}\\
&=\frac{1}{2}(\frac{\partial^2 g_{(\mu\nu)(\sigma\varsigma)}}{\partial x^{\kappa\lambda}\partial x^{\eta\theta}}+\frac{\partial^2 g_{(\kappa\lambda)(\eta\theta)}}{\partial x^{\mu\nu}\partial x^{\sigma\varsigma}}-\frac{\partial^2 g_{(\mu\nu)(\eta\theta)}}{\partial x^{\kappa\lambda}\partial x^{\sigma\varsigma}}-\frac{\partial^2 g_{(\kappa\lambda)(\sigma\varsigma)}}{\partial x^{\mu\nu}\partial x^{\eta\theta}})\\
&+\Gamma^{\xi o}_{(\sigma\varsigma)(\kappa\lambda)}\Gamma^{\vartheta\iota}_{(\xi o)(\mu\nu)}g_{(\vartheta\iota)(\kappa\lambda)}-\Gamma^{\xi o}_{(\sigma\varsigma)(\mu\nu)}\Gamma^{\vartheta\iota}_{(\xi o)(\kappa\lambda)})g_{(\vartheta\iota)(\eta\theta)}.
\end{aligned}$$

This completes the proof. □

Combining Theorems 2.5, 3.1 and 3.3, we have the following consequence.

Corollary 3.1 *Let $\widetilde{R}_{(\mu\nu)(\kappa\lambda)(\sigma\varsigma)(\eta\theta)}$ be a component of a combinatorially Riemannian curvature tensor $\widetilde{R}$ in a local chart $(U,[\varphi])$ of a combinatorially Riemannian manifold $(\widetilde{R},g,\widetilde{D})$. Then*

(1) $\widetilde{R}_{(\mu\nu)(\kappa\lambda)(\sigma\varsigma)(\eta\theta)}=-\widetilde{R}_{(\kappa\lambda)(\mu\nu)(\sigma\varsigma)(\eta\theta)}=-\widetilde{R}_{(\mu\nu)(\kappa\lambda)(\eta\theta)(\sigma\varsigma)}$;

(2) $\widetilde{R}_{(\mu\nu)(\kappa\lambda)(\sigma\varsigma)(\eta\theta)}=\widetilde{R}_{(\sigma\varsigma)(\eta\theta)(\mu\nu)(\kappa\lambda)}$;

(3) $\widetilde{R}_{(\mu\nu)(\kappa\lambda)(\sigma\varsigma)(\eta\theta)}+\widetilde{R}_{(\eta\theta)(\kappa\lambda)(\mu\nu)(\sigma\varsigma)}+\widetilde{R}_{(\sigma\varsigma)(\kappa\lambda)(\eta\theta)(\mu\nu)}=0$;

(4) $\widetilde{D}_{\vartheta\iota}\widetilde{R}_{(\mu\nu)(\kappa\lambda)(\sigma\varsigma)(\eta\theta)}+\widetilde{D}_{\sigma\varsigma}\widetilde{R}_{(\mu\nu)(\kappa\lambda)(\eta\theta)(\vartheta\iota)}+\widetilde{D}_{\eta\theta}\widetilde{R}_{(\mu\nu)(\kappa\lambda)(\vartheta\iota)(\sigma\varsigma)}=0$.

§4. Einstein's Gravitational Equations on Combinatorial Manifolds

Application of results in last two sections enables us to establish these Einstein' gravitational filed equations on combinatorially Riemannian manifolds in this section and find their multi-space solutions in next section under a *projective principle* on the behavior of particles in multi-spaces.

Let $(\widetilde{M},g,\widetilde{D})$ be a combinatorially Riemannian manifold. A type $(0,2)$ tensor $\mathscr{E}:\mathscr{X}(\widetilde{M})\times\mathscr{X}(\widetilde{M})\rightarrow C^{\infty}(\widetilde{M})$ with

$$\mathscr{E}=\mathscr{E}_{(\mu\nu)(\kappa\lambda)}dx^{\mu\nu}\otimes dx^{\kappa\lambda}\quad(4.1)$$

is called an *energy-momentum tensor* if it satisfies the conservation laws $\widetilde{D}(\mathscr{E})=0$, i.e., for any indexes κ,λ, $1\leq\kappa\leq m$, $1\leq\lambda\leq n_{\kappa}$,

$$\frac{\partial \mathscr{E}_{(\mu\nu)(\kappa\lambda)}}{\partial x^{\kappa\lambda}} - \Gamma^{\sigma\varsigma}_{(\mu\nu)(\kappa\lambda)}\mathscr{E}_{(\sigma\varsigma)(\kappa\lambda)} - \Gamma^{\sigma\varsigma}_{(\kappa\lambda)(\kappa\lambda)}\mathscr{E}_{(\mu\nu)(\sigma\varsigma)} = 0 \quad (4.2)$$

in a local chart $(U_p, [\varphi_p])$ for any point $p \in \widetilde{M}$. Define the *Ricci tensor* $\widetilde{R}_{(\mu\nu)(\kappa\lambda)}$, *Rocci scalar tensor* $\mathbf{R}$ and *Einstein tensor* $\mathscr{G}_{(\mu\nu)(\kappa\lambda)}$ respectively by

$$\widetilde{R}_{(\mu\nu)(\kappa\lambda)} = \widetilde{R}^{\sigma\varsigma}_{(\mu\nu)(\sigma\varsigma)(\kappa\lambda)}, \quad \mathbf{R} = g^{(\mu\nu)(\kappa\lambda)}\widetilde{R}_{(\mu\nu)(\kappa\lambda)} \quad (4.3)$$

and

$$\mathscr{G}_{(\mu\nu)(\kappa\lambda)} = \widetilde{R}_{(\mu\nu)(\kappa\lambda)} - \frac{1}{2}g_{(\mu\nu)(\kappa\lambda)}\mathbf{R}. \quad (4.4)$$

Then we get results following hold by Theorems 2.4, 2.5 and 3.1.

$$\widetilde{R}_{(\mu\nu)(\kappa\lambda)} = \widetilde{R}_{(\kappa\lambda)(\mu\nu)}, \quad (4.5)$$

$$\widetilde{R}_{(\mu\nu)(\kappa\lambda)} == \frac{\partial \Gamma^{\sigma\varsigma}_{(\mu\nu)(\kappa\lambda)}}{\partial x^{\sigma\varsigma}} - \frac{\partial \Gamma^{\sigma\varsigma}_{(\mu\nu)(\sigma\varsigma)}}{\partial x^{\kappa\lambda}} + \Gamma^{\vartheta\iota}_{(\mu\nu)(\kappa\lambda)}\Gamma^{\sigma\varsigma}_{(\vartheta\iota)(\sigma\varsigma)} - \Gamma^{\vartheta\iota}_{(\mu\nu)(\sigma\varsigma)}\Gamma^{\sigma\varsigma}_{(\vartheta\iota)(\kappa\lambda)}. \quad (4.6)$$

and

$$\frac{\partial \mathscr{G}_{(\mu\nu)(\kappa\lambda)}}{\partial x^{\kappa\lambda}} - \Gamma^{\sigma\varsigma}_{(\mu\nu)(\kappa\lambda)}\mathscr{G}_{(\sigma\varsigma)(\kappa\lambda)} - \Gamma^{\sigma\varsigma}_{(\kappa\lambda)(\kappa\lambda)}\mathscr{G}_{(\mu\nu)(\sigma\varsigma)} = 0. \quad (4.7)$$

i.e., $\widetilde{D}(\mathscr{G}) = 0$. *Einstein's* principle of general relativity says that *a law of physics should take a same form in any reference system*, which claims that a right form for a physics law should be presented by tensors in mathematics. For a multi-spacetime, we conclude that *Einstein's* principle of general relativity is still true, if we take the multi-spacetime being a combinatorially Riemannian manifold. Whence, a physics law should be also presented by tensor equations in the multi-spacetime case.

Just as the establishing of *Einstein's* gravitational equations in the classical case, these equations should satisfy two conditions following.

(C1) *They should be* $(0, 2)$ *type tensor equations related to the energy-momentum tensor* $\mathscr{E}$ *linearly;*

(C2) *Their forms should be the same as in a classical gravitational field.*

By these two conditions, *Einstein's* gravitational equations in a multi-spacetime should be taken the following form

$$\mathscr{G} = c\mathscr{E}$$

with c a constant. Now since these equations should take the same form in the classical case, i.e.,

$$\mathscr{G}_{ij} = -8\pi G\mathscr{E}_{ij}$$

for $1 \le i, j \le n$ at a point p in a manifold of $\widetilde{M}$ not contained in the others. Whence, it must be $c = -8\pi G$ for c being a constant. This enables us finding these *Einstein's* gravitational equations in a multi-spacetime to be

$$\widetilde{\mathcal{R}}_{(\mu\nu)(\kappa\lambda)} - \frac{1}{2}\mathbf{R}g_{(\mu\nu)(\kappa\lambda)} = -8\pi G\mathscr{E}_{(\mu\nu)(\kappa\lambda)}. \quad (4.8)$$

Certainly, we can also add a cosmological term $\lambda g_{(\mu\nu)(\kappa\lambda)}$ in (4.8) and obtain these gravitational equations

$$\widetilde{\mathcal{R}}_{(\mu\nu)(\kappa\lambda)} - \frac{1}{2}\mathbf{R}g_{(\mu\nu)(\kappa\lambda)} + \lambda g_{(\mu\nu)(\kappa\lambda)} = -8\pi G\mathscr{E}_{(\mu\nu)(\kappa\lambda)}. \quad (4.9)$$

All of these equations (4.8) and (4.9) mean that there are multi-space solutions in classical *Einstein's* gravitational equations by a multi-spacetime view, which will be shown in the next section.

§5. Multi-Space Solutions of Einstein's Equations

For given integers $0 < n_1 < n_2 < \cdots < n_m, m \ge 1$, let $(\widetilde{M}, g, \widetilde{D})$ be a combinatorial Riemannian manifold with $\widetilde{M} = \widetilde{M}(n_1, n_2, \cdots, n_m)$ and $(U_p, [\varphi_p])$ a local chart for $p \in \widetilde{M}$. By definition, if $\varphi_p : U_p \to \bigcup\limits_{i=1}^{s(p)} B^{n_i(p)}$ and $\widehat{s}(p) = \dim(\bigcap\limits_{i=1}^{s(p)} B^{n_i(p)})$, then $[\varphi_p]$ is an $s(p) \times n_{s(p)}$ matrix shown following.

$$[\varphi_p] = \begin{bmatrix} \frac{x^{11}}{s(p)} & \cdots & \frac{x^{1\widehat{s}(p)}}{s(p)} & x^{1(\widehat{s}(p)+1)} & \cdots & x^{1n_1} & \cdots & 0 \\ \frac{x^{21}}{s(p)} & \cdots & \frac{x^{2\widehat{s}(p)}}{s(p)} & x^{2(\widehat{s}(p)+1)} & \cdots & x^{2n_2} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & \\ \frac{x^{s(p)1}}{s(p)} & \cdots & \frac{x^{s(p)\widehat{s}(p)}}{s(p)} & x^{s(p)(\widehat{s}(p)+1)} & \cdots & \cdots & x^{s(p)n_{s(p)}-1} & x^{s(p)n_{s(p)}} \end{bmatrix}$$

with $x^{is} = x^{js}$ for $1 \le i, j \le s(p), 1 \le s \le \widehat{s}(p)$.

For given non-negative integers $r, s, r+s \ge 1$, choose a type (r,s) tensor $\mathscr{F} \in T^r_s(\widetilde{M})$. Then how to get multi-space solutions of a tensor equation

$$\mathscr{F} = 0 \ ?$$

We need to apply the *projective principle* following.

[Projective Principle] *Let $(\widetilde{M}, g, \widetilde{D})$ be a combinatorial Riemannian manifold and $\mathscr{F} \in \left\langle T | T \in T^r_s(\widetilde{M}) \right\rangle$ with a local form $\mathscr{F}_{(\mu_1\nu_1)(\mu_2\nu_2)\cdots(\mu_s\nu_s)}\omega^{\mu_1\nu_1} \otimes \omega^{\mu_2\nu_2} \otimes \cdots \otimes \omega^{\mu_n\nu_n}$ in $(U_p, [\varphi_p])$. If*

$$\mathscr{F}_{(\mu_1\nu_1)(\mu_2\nu_2)\cdots(\mu_n\nu_n)} = 0$$

for integers $1 \le \mu_i \le s(p), 1 \le \nu_i \le n_{\mu_i}$ with $1 \le i \le s$, then for any integer $\mu, 1 \le \mu \le s(p)$, there must be

$$\mathscr{F}_{(\mu\nu_1)(\mu\nu_2)\cdots(\mu\nu_s)} = 0$$

for integers ν_i, $1 \leq \nu_i \leq n_\mu$ *with* $1 \leq i \leq s$.

Now we solve these vacuum *Einstein's* gravitational equations

$$\widetilde{R}_{(\mu\nu)(\kappa\lambda)} - \frac{1}{2}g_{(\mu\nu)(\kappa\lambda)}\mathbf{R} = 0 \quad (5.1)$$

by the projective principle on a combinatorially Riemannian manifold $(\widetilde{M}, g, \widetilde{D})$. For a given point $p \in \widetilde{M}$, we get $s(p)$ tensor equations

$$\widetilde{R}_{(\mu\nu)(\mu\lambda)} - \frac{1}{2}g_{(\mu\nu)(\mu\lambda)}\mathbf{R} = 0,\ 1 \leq \mu \leq s(p) \quad (5.2)$$

as these usual vacuum *Einstein's* equations in classical gravitational field, where $1 \leq \nu, \lambda \leq n_\mu$. For line elements in $\widetilde{M}$, the next result is easily obtained.

Theorem 5.1 *If each line element* ds_μ *is uniquely determined by equations* (5.2)*, Then* $\widetilde{ds}$ *is uniquely determined in* $\widetilde{M}$.

Proof For a given index μ, let

$$ds_\mu^2 = \sum_{i=1}^{n_\mu} a_{\mu i}^2 dx_{\mu i}^2.$$

Then we know that

$$\widetilde{ds}^2 = \sum_{i=1}^{\widehat{s}(p)}(\sum_{\mu=1}^{s(p)} a_{\mu i})^2 dx_{\mu i}^2 + \sum_{\mu=1}^{s(p)}\sum_{i=\widehat{s}(p)+1}^{n_\mu} a_{\mu i}^2 dx_{\mu i}^2.$$

Therefore, the line element $\widetilde{ds}$ is uniquely determined in $\widetilde{M}$ if $ds_{\mu i}$ is uniquely determined by (5.2). □.

We consider a special case for these *Einstein's* gravitational equations (5.1), solutions of combinatorially Euclidean spaces $\widetilde{M} = \bigcup_{i=1}^m \mathbf{R}^{n_i}$ with a matrix ([11])

$$[\overline{x}] = \begin{bmatrix} x^{11} & \cdots & x^{1\widehat{m}} & x^{1(\widehat{m})+1)} & \cdots & x^{1n_1} & \cdots & 0 \\ x^{21} & \cdots & x^{2\widehat{m}} & x^{2(\widehat{m}+1)} & \cdots & x^{2n_2} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & \\ x^{m1} & \cdots & x^{m\widehat{m}} & x^{m(\widehat{m}+1)} & \cdots & \cdots & x^{mn_m-1} & x^{mn_m} \end{bmatrix}$$

for any point $\overline{x} \in \widetilde{M}$, where $\widehat{m} = \dim(\bigcap\limits_{i=1}^{m} \mathbf{R}^{n_i})$ is a constant for $\forall p \in \bigcap\limits_{i=1}^{m} \mathbf{R}^{n_i}$ and $x^{il} = \frac{x^l}{m}$ for $1 \leq i \leq m, 1 \leq l \leq \widehat{m}$. In this case, we have a unifying solution for these equations (5.1), i.e.,

$$\widetilde{ds}^2 = \sum_{i=1}^{\widehat{m}}(\sum_{\mu=1}^{m} a_{\mu i})^2 dx_{\mu i}^2 + \sum_{\mu=1}^{m}\sum_{i=\widehat{m}+1}^{n_\mu} a_{\mu i}^2 dx_{\mu i}^2$$

for each point $p \in \widetilde{M}$ by Theorem 5.1.

For usually undergoing, we consider the case of $n_\mu = 4$ for $1 \leq \mu \leq m$ since line elements have been found concretely in classical gravitational field in these cases. Now establish m spherical coordinate subframe $(t_\mu; r_\mu, \theta_\mu, \phi_\mu)$ with its originality at the center of the mass space. Then we have known its a spherically symmetric solution for the line element ds_μ with a given index μ by *Schwarzschild* (see also [3]) for (5.2) to be

$$ds_\mu^2 = (1 - \frac{r_{\mu s}}{r_\mu})c^2 dt_\mu^2 - (1 - \frac{r_{\mu s}}{r_\mu})^{-1} dr_\mu^2 - r_\mu^2 (d\theta_\mu^2 + \sin^2\theta_\mu d\phi_\mu^2).$$

for $1 \leq \mu \leq m$, where $r_{\mu s} = 2Gm_\mu / c^2$. Applying Theorem 5.1, the line element $\widetilde{ds}$ in $\widetilde{M}$ is

$$\widetilde{ds} = (\sum_{\mu=1}^m \sqrt{1 - \frac{r_{\mu s}}{r_\mu}})^2 c^2 dt^2 - \sum_{\mu=1}^m (1 - \frac{r_{\mu s}}{r_\mu})^{-1} dr_\mu^2 - \sum_{\mu=1}^m r_\mu^2 (d\theta_\mu^2 + \sin^2\theta_\mu d\phi_\mu^2)$$

if $\widehat{m} = 1$, $t_\mu = t$ for $1 \leq \mu \leq m$ and

$$\widetilde{ds} = (\sum_{\mu=1}^m \sqrt{1 - \frac{r_{\mu s}}{r_\mu}})^2 c^2 dt^2 - (\sum_{\mu=1}^m \sqrt{(1 - \frac{r_{\mu s}}{r_\mu})^{-1}})^2 dr^2 - \sum_{\mu=1}^m r_\mu^2 (d\theta_\mu^2 + \sin^2\theta_\mu d\phi_\mu^2)$$

if $\widehat{m} = 2$, $t_\mu = t, r_\mu = r$ for $1 \leq \mu \leq m$ and

$$\widetilde{ds} = (\sum_{\mu=1}^m \sqrt{1 - \frac{r_{\mu s}}{r_\mu}})^2 c^2 dt^2 - (\sum_{\mu=1}^m \sqrt{(1 - \frac{r_{\mu s}}{r_\mu})^{-1}})^2 dr^2 - m^2 r^2 d\theta^2 - \sum_{\mu=1}^m r_\mu^2 \sin^2\theta_\mu d\phi_\mu^2$$

if $\widehat{m} = 3$, $t_\mu = t, r_\mu = r, \theta_\mu = \theta$ for $1 \leq \mu \leq m$ and

$$\widetilde{ds} = (\sum_{\mu=1}^m \sqrt{1 - \frac{r_{\mu s}}{r_\mu}})^2 c^2 dt^2 - (\sum_{\mu=1}^m \sqrt{(1 - \frac{r_{\mu s}}{r_\mu})^{-1}})^2 dr^2 - m^2 r^2 d\theta^2 - m^2 r^2 \sin^2\theta d\phi^2$$

if $\widehat{m} = 4$, $t_\mu = t, r_\mu = r, \theta_\mu = \theta$ and $\phi_\mu = \phi$ for $1 \leq \mu \leq m$.

For another interesting case, let $\widehat{m} = 3, r_\mu = r, \theta_\mu = \theta, \phi_\mu = \phi$ and

$$d\Omega^2(r, \theta, \phi) = (1 - \frac{r_s}{r})^{-1} dr^2 - r^2 (d\theta^2 + \sin^2\theta d\phi^2).$$

Then we can choose a multi-time system $\{t_1, t_2, \cdots, t_m\}$ to get a cosmic model of $m, m \geq 2$ combinatorially $\mathbf{R}^4$ spaces with line elements

$$ds_1^2 = -c^2 dt_1^2 + a^2(t_1) d\Omega^2(r, \theta, \phi),$$

$$ds_2^2 = -c^2 dt_2^2 + a^2(t_2) d\Omega^2(r, \theta, \phi),$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots ,$$

$$ds_m^2 = -c^2 dt_m^2 + a^2(t_m) d\Omega^2(r, \theta, \phi).$$

In this case, the line element $\widetilde{ds}$ is

$$\widetilde{ds} = \sum_{\mu=1}^{m} (1 - \frac{r_{\mu s}}{r_\mu}) c^2 dt_\mu^2 - (\sum_{\mu=1}^{m} \sqrt{(1 - \frac{r_{\mu s}}{r_\mu})^{-1}})^2 dr^2 - m^2 r^2 d\theta^2 - m^2 r^2 \sin^2\theta d\phi^2.$$

As a by-product for our universe $\mathbf{R}^3$, these formulas mean that these beings with time notion different from human being will recognize differently the structure of our universe if these beings are intellectual enough for the structure of the universe.

References

[1] R.Abraham and J.E.Marsden, *Foundation of Mechanics*(2nd edition), Addison-Wesley, Reading, Mass, 1978.

[2] G.Bassini and S.Capozziello, Multi-spaces and many worlds from conservation laws, *Progress in Physics*, Vol.4(2006), 65-72.

[3] M.Carmeli, *Classical Fields–General Relativity and Gauge Theory*, World Scientific, 2001.

[4] W.H.Chern and X.X.Li, *Introduction to Riemannian Geometry*, Peking University Press, 2002.

[5] L.Kuciuk and M.Antholy, An Introduction to Smarandache Geometries, *JP Journal of Geometry and Topology*, 5(1), 2005,77-81.

[6] L.F.Mao, *Automorphism Groups of Maps, Surfaces and Smarandache Geometries*, American Research Press, 2005.

[7] L.F.Mao, *Smarandache multi-space theory*, Hexis, Phoenix, AZ2006.

[8] L.F.Mao, Combinatorial speculation and the combinatorial conjecture for mathematics, *Selected Papers on Mathematical Combinatorics*(I), 1-22, World Academic Union, 2006.

[9] L.F.Mao, Geometrical theory on combinatorial manifolds, *JP J.Geometry and Topology*, Vol.7, No.1(2007),65-114.

[10] L.F.Mao, An introduction to Smarandache multi-spaces and mathematical combinatorics, *Scientia Magna*, Vol.3, No.1(2007), 54-80.

[11] L.F.Mao, A combinatorially generalized Stokes' theorem on integrations, *International J.Mathematical Combinatorics*, Vol.1, No.1(2007), 1-26.

[12] F.Smarandache, Mixed noneuclidean geometries, *eprint arXiv: math.GM/0010119*, 10/2000.

[13] F.Smarandache, A unifying field in logic: neutrosophic field, *Multi-value Logic*, Vol.8, No.3(2002), 385-438.

[14] M.Tegmark, Parallel Universes, in *Science and Ultimate Reality: From Quantum to Cosmos*, ed. by J.D.Barrow, P.C.W.Davies and C.L.Harper, Cambridge University Press, 2003.