

Long Dominating Cycles in Graphs

Yongga A

(Department of Mathematics of Inner Mongolia Normal University, Huhhot 010022, P.R.China)

Zhiren Sun

(Department of Mathematics of Nanjing Normal University, Nanjing 210097, P.R. China)

Abstract: Let G be a connected graph of order n , and $NC2(G)$ denote $\min\{|N(u) \cup N(v)| : \text{dist}(u, v) = 2\}$, where $\text{dist}(u, v)$ is the distance between u and v in G . A cycle C in G is called a *dominating cycle*, if $V(G) \setminus V(C)$ is an independent set in G . In this paper, we prove that if G contains a dominating cycle and $\delta \geq 2$, then G contains a dominating cycle of length at least $\min\{n, 2NC2(G) - 1\}$ and give a family of graphs showing our result is sharp, which proves a conjecture of R. Shen and F. Tian, also related with the cyclic structures of algebraically Smarandache multi-spaces.

Key words: Dominating cycle, neighborhood union, distance.

AMS(2000): 05C38, 05C45.

§1. Introduction

All graphs considered in this paper will be finite and simple. We use Bondy & Murty [1] for terminology and notations not defined here.

Let $G = (V, E)$ be a graph of order n and C be a cycle in G . C is called a *dominating cycle*, or briefly a *D-cycle*, if $V(G) \setminus V(C)$ is an independent set in G . For a vertex v in G , the neighborhood of v is denoted by $N(v)$, and the degree of v is denoted by $d(v)$. For two subsets S and T of $V(G)$, we set $N_T(S) = \{v \in T \setminus S : N(v) \cap S \neq \emptyset\}$. We write $N(u, v)$ instead of $N_{V(G)}(\{u, v\})$ for any $u, v \in V(G)$. If F and H are two subgraphs of G , we also write $N_F(H)$ instead of $N_{V(F)}(V(H))$. In the case $F = G$, if no ambiguity can arise, we usually omit the subscript G of $N_G(H)$. We denote by $G[S]$ the subgraph of G induced by any subset S of $V(G)$.

For a connected graph G and $u, v \in V(G)$, we define the *distance between u and v* in G , denoted by $\text{dist}(u, v)$, as the minimum value of the lengths of all paths joining u and v in G . If G is non-complete, let $NC(G)$ denote $\min\{|N(u, v)| : uv \notin E(G)\}$ and $NC2(G)$ denote $\min\{|N(u, v)| : \text{dist}(u, v) = 2\}$; if G is complete, we set $NC(G) = n - 1$ and $NC2(G) = n - 1$.

In [2], Broersma and Veldman gave the following result.

Theorem 1 ([2]) *If G is a 2-connected graph of order n and G contains a D-cycle, then G has a D-cycle of length at least $\min\{n, 2NC(G)\}$ unless G is the Petersen graph.*

For given positive integers n_1, n_2 and n_3 , let $K(n_1, n_2, n_3)$ denote the set of all graphs

¹Received August 6, 2007. Accepted September 8, 2007

²Supported by the National Science Foundation of China(10671095) and the Tian Yuan Foundation on Mathematics.

of order $n_1 + n_2 + n_3$ consisting of three disjoint complete graphs of order n_1 , n_2 and n_3 , respectively. For any integer $p \geq 3$, let $\mathcal{J}_1^*$ (resp. $\mathcal{J}_2^*$) denote the family of all graphs of order $2p + 3$ (resp. $2p + 4$) which can be obtained from a graph H in $K(3, p, p)$ (resp. $K(3, p, p + 1)$) by adding the edges of two triangles between two disjoint triples of vertices, each containing one vertex of each component of H . Let $\mathcal{J}_1 = \{G : G \text{ is a spanning subgraph of some graph in } \mathcal{J}_1^*\}$ and $\mathcal{J}_2 = \{G : G \text{ is a spanning subgraph of some graph in } \mathcal{J}_2^*\}$. In [5], Tian and Zhang got the following result.

Theorem 2([5]) *If G is a 2-connected graph of order n such that every longest cycle in G is a D -cycle, then G contains a D -cycle of length at least $\min\{n, 2NC2(G)\}$ unless G is the Petersen graph or $G \in \mathcal{J}_1 \cup \mathcal{J}_2$.*

In [4], Shen and Tian weakened the conditions of Theorem 2 and obtained the following theorem.

Theorem 3([4]) *If G contains a D -cycle and $\delta \geq 2$, then G contains a D -cycle of length at least $\min\{n, 2NC2(G) - 3\}$.*

Theorem 4([6]) *If G contains a D -cycle and $\delta \geq 2$, then G contains a D -cycle of length at least $\min\{n, 2NC2(G) - 2\}$.*

In [4], Shen and Tian believed the followings are true.

Conjecture 1 *If G satisfies the conditions of Theorem 3, then G contains a D -cycle of length at least $\min\{n, 2NC2(G) - \epsilon(n)\}$, where $\epsilon(n) = 1$ if n is even, and $\epsilon(n) = 2$ if n is odd.*

Conjecture 2 *If G contains a D -cycle and $\delta \geq 2$, then G contains a D -cycle of length at least $\min\{n, 2NC2(G)\}$ unless G is one of the exceptional graphs listed in Theorem 2. And the complete bipartite graphs $K_{m, m+q}$ ($q \geq 1$) show that the bound $2NC2(G)$ is sharp.*

In this paper, we prove the following result, which solves Conjecture 1 due to Shen and Tian, also related with the cyclic structures of algebraically Smarandache multi-spaces (see [3] for details).

Theorem 5 *If G contains a D -cycle and $\delta \geq 2$, then G contains a D -cycle of length at least $\min\{n, 2NC2(G) - 1\}$ unless $G \in \mathcal{J}_1$.*

Remark The Petersen graph shows that our bound $2NC2(G) - 1$ is sharp.

§2. Proof of Theorem 5

In order to prove Theorem 5, we introduce some additional notations.

Let C be a cycle in G . We denote by $\vec{C}$ the cycle C with a given orientation. If $u, v \in V(C)$, then $u\vec{C}v$ denotes the consecutive vertices on C from u to v in the direction specified by $\vec{C}$. The same vertices, in reverse order, are given by $v\overleftarrow{C}u$. We will consider $u\vec{C}v$ and $v\overleftarrow{C}u$ both as paths and as vertex sets. We use u^+ to denote the successor of u on $\vec{C}$ and u^- to

denote its predecessor. We write $u^{+2} := (u^+)^+$ and $u^{-2} := (u^-)^-$, etc. If $A \subseteq V(C)$, then $A^+ = \{v^+ : v \in A\}$ and $A^- = \{v^- : v \in A\}$. For any subset S of $V(G)$, we write $N^+(S)$ and $N^-(S)$ instead of $(N(S))^+$ and $(N(S))^-$, respectively.

Let G be a graph satisfying the conditions of Theorem 4, i.e. G contains a D-cycle and $\delta \geq 2$. Throughout, we suppose that

- G is non-hamiltonian and C is a longest D-cycle in G ,
- $|V(C)| \leq 2NC2(G) - 2$,
- $R = G \setminus V(C)$ and $x \in R$, such that $d(x)$ is as large as possible.

First of all, we prove some claims.

By the maximality of C and the definition of D-cycle, we have

Claim 1 $N(x) \subseteq V(C)$.

Claim 2 $N(x) \cap N^+(x) = N(x) \cap N^-(x) = \emptyset$.

Let $v_1, v_2, \dots, v_k$ be the vertices of $N(x)$, in cyclic order around $\vec{C}$. Then $k \geq 2$ since $\delta \geq 2$. For any $i \in \{1, 2, \dots, k\}$, we have $v_i^+ \neq v_{i+1}$ (indices taken modulo k) by Claim 2. Let $u_i = v_i^+$, $w_i = v_{i+1}^-$ (indices taken modulo k), $T_i = u_i \vec{C} w_i$, $t_i = |T_i|$.

Claim 3 $N_R(y_1) \cap N_R(y_2) = \emptyset$, if $y_1, y_2 \in N^+(x)$ or $y_1, y_2 \in N^-(x)$. In particular, $N^+(x) \cap N(u_i) = N^-(x) \cap N(w_i) = \emptyset$.

For any $i, j \in \{1, 2, \dots, k\} (i \neq j)$, we also have the following Claims.

Claim 4 Each of the followings does not hold :

(1) There are two paths $P_1[w_j, z]$ and $P_2[u_i, z^-]$, ($z \in v_{j+1} \vec{C} v_i$) of length at most two that are internally disjoint from C and each other ;

(2) There are two paths $P_1[w_j, z]$ and $P_2[u_i, z^+]$ ($z \in v_{j+1} \vec{C} v_i$) of length at two that are internally disjoint from C and each other ;

(3) There are two paths $P_1[u_i, z]$ and $P_2[u_j, z^+]$ ($z \in u_j^+ \vec{C} v_i$) of length at most two that are internally disjoint from C and each other, and similarly for $P_1[u_i, z]$ and $P_2[u_j, z^-]$ ($z \in u_i^+ \vec{C} v_j$).

Claim 5 For any $v \in V(G)$, we have $d_R(v) \leq 1$.

If not, then by Claim 1, there exists a vertex, say v , in C such that $d_R(v) > 1$. Let $x_1, x_2 \in N_R(v)$, then $|N(x_1, x_2)| \geq NC2(G)$.

First, we prove that $|N(x_1, x_2) \cap N^+(x_1, x_2)| \leq 2$. Otherwise, let y_1, y_2 and y_3 be three distinct vertices in $N(x_1, x_2) \cap N^+(x_1, x_2)$. By Claim 2, we know $y_i \in N(x_1) \cap N^+(x_2)$ or $y_i \in N(x_2) \cap N^+(x_1)$ for any $i \in \{1, 2, 3\}$. Thus, there must exist i and j ($i \neq j, i, j \in \{1, 2, 3\}$) such that $y_i, y_j \in N(x_1) \cap N^+(x_2)$ or $y_i, y_j \in N(x_2) \cap N^+(x_1)$. In either case, it contradicts Claim 3. So we have that $|N(x_1, x_2) \cap N^+(x_1, x_2)| \leq 2$.

Now we have

$$\begin{aligned}
 |V(C)| &\geq |N(x_1, x_2) \cup N^+(x_1, x_2)| \\
 &\geq 2|N(x_1, x_2)| - 2 \\
 &\geq 2NC2(G) - 2,
 \end{aligned}$$

so $V(C) = N(x_1, x_2) \cup N^+(x_1, x_2)$ by assumption on $|V(C)|$, and in particular, $N(x_1, x_2) \cap N^+(x_1, x_2) = \{y_1, y_2\}$. Therefore $y_1 \in N(x_1) \cap N^+(x_2)$ and $y_2 \in N^+(x_1) \cap N(x_2)$.

Now, we prove that $d_R(v^+) \leq 1, d_R(v^-) \leq 1$. If not, suppose $d_R(v^-) > 1$, let $z_1, z_2 \in N_R(v^-)$, by Claim 1 and $V(C) = N(x_1, x_2) \cup N^+(x_1, x_2)$, $N(z_1, z_2) \subseteq N^+(x_1, x_2)$, so we have x_1 (or x_2) $\in N(v^-)$. Using a similar argument as above, we have z_1 (or z_2) $\in N(v^{-3})$, which contradicts Claim 3. Thus, we have $d_R(v^-) \leq 1$; similarly, $d_R(v^+) \leq 1$.

Now, we consider $N(x_2, v^-) \cup N^-(x_1, v^+)$. Since $\text{dist}(x_2, v^-) = \text{dist}(x_1, v^+) = 2$ and $|N(x_2, v^-)| \geq NC2(G), |N^-(x_1, v^+)| = |N(x_1, v^+)| \geq NC2(G)$. We prove that $|N_C(x_2, v^-) \cap N_C^-(x_1, v^+)| \leq 1$. Let $z \in \{N_C(x_2, v^-) \cap N_C^-(x_1, v^+)\} \setminus \{y_1^-\}$.

We consider following cases.

(i) Let $z \in y_1^+ \vec{C} y_2^{-2}$, if $zx_2 \in E(G)$ and $x_1z^+ \in E(G)$, or $zx_2 \in E(G)$ and $v^+z^+ \in E(G)$, or $v^-z \in E(G)$ and $x_1z^+ \in E(G)$, each case contradicts Claim 3; if $v^-z \in E(G)$ and $v^+z^+ \in E(G)$, then $C' = x_1y_2^- \overleftarrow{C} z^+v^+ \overrightarrow{C} zv^- \overleftarrow{C} y_2x_2vx_1$ is a D -cycle longer than C , a contradiction.

(ii) Let $z \in y_2^+ \vec{C} y_1^{-2}$, if $x_2z \in E(G)$ and $x_1z^+ \in E(G)$, or $x_2z \in E(G)$ and $v^+z^+ \in E(G)$, both contradict Claim 3; if $v^-z \in E(G)$ and $x_1z^+ \in E(G)$, it contradicts Claim 3; if $v^-x_1 \in E(G)$ and $z^+v^+ \in E(G)$, then $C' = x_1y_1 \overrightarrow{C} v^-z \overleftarrow{C} v^+z^+ \overrightarrow{C} y_1^-x_2vx_1$ is a D -cycle longer than C , for $z \in v \overrightarrow{C} y_1^-$; and $C' = x_1y_2^- \overleftarrow{C} v^+z^+ \overrightarrow{C} v^-z \overleftarrow{C} y_2x_2vx_1$ is a D -cycle longer than C for $z \in y_2 \overrightarrow{C} v^-$.

So, we have $|N_C(x_2, v^-) \cap N_C^-(x_1, v^+)| \leq 1$. Moreover, $y_1, y_2^- \notin N(x_2, v^-) \cup N^-(x_1, v^+)$. Otherwise, if $y_1 \in N(v^-)$, then $C' = x_1y_2^- \overleftarrow{C} y_1v^- \overleftarrow{C} y_2x_2y_1^- \overleftarrow{C} vx_1$ is a D -cycle longer than C . By Claim 2, $y_1 \notin N(x_2) \cup N^-(x_1, v^+)$, so we have $y_1 \notin N(x_2, v^-) \cup N^-(x_1, v^+)$. By Claims 1 and 3 we have $y_2^- \notin N(x_2, v^-) \cup N^-(x_1, v^+)$. Thus, we have

$$\begin{aligned}
 |V(C)| &\geq |N_C(x_2, v^-) \cup N_C^-(x_1, v^+)| + 2 \\
 &\geq |N_C(x_2, v^-)| + |N_C^-(x_1, v^+)| - 1 + 2 \\
 &= |N(x_2, v^-) \setminus N_R(x_2, v^-)| + |N(x_1, v^+) \setminus N_R(x_1, v^+)| + 1 \\
 &\geq 2NC2(G) - 2 + 1 \\
 &= 2NC2(G) - 1,
 \end{aligned}$$

a contradiction with $|V(C)| \leq 2NC2(G) - 2$. So, we have $d_R(v) \leq 1$, for any $v \in V(G)$.

Claim 6 $t_i \geq 2$.

If $t_i = 1$ for all of i , then $N_R(u_i) = \emptyset$ for all of i (if not, let $z \in N_R(u_i)$ for some i , by Claim 1 and Claim 5 $N(z) \subseteq V(C)$ and $u_jz \in E(G)$ for some j . then, $z \in N_R(u_i) \cap N_R(u_j)$, a contradiction). Then $N(u_i) \cap N^+(u_i) = \emptyset$ (otherwise, $y \in N(u_i) \cap N^+(u_i)$, then $C' =$

$xv_{i+1}\overrightarrow{C}y^-u_iy\overrightarrow{C}v_ix$ is a D -cycle longer than C). Moreover, we have $N(x) \cap N^+(x) = \emptyset$ by Claim 2, $N^+(x) \cap N(u_i) = N^+(u_i) \cap N(x) = \emptyset$ by Claim 3. Hence, $N(x, u_i) \cap N^+(x, u_i) = \emptyset$. So we have

$$|V(C)| \geq |N(x, u_i) \cup N^+(x, u_i)| \geq 2|N(x, u_i)| \geq 2NC2(G),$$

a contradiction. So we may assume $t_i = 1$ for some i , without loss of generality, suppose $t_1 = 1$ and $N_R(w_k) \neq \emptyset$. Let $y \in N_R(w_k)$, choose $y_1 \in N(y)$ such that $N(y) \cap (y_1^+ \overrightarrow{C} w_k^-) = \emptyset$. Using a similar argument as above and $d_R(u_1) \leq 1$, by Claim 5, we have

$$|V(C)| = |N_C(x, u_1) \cup N_C^+(x, u_1)| \geq 2NC2(G) - 2.$$

So $V(C) = N_C(x, u_1) \cup N_C^+(x, u_1)$. Similarly, we know that $V(C) = N_C(x, u_1) \cup N_C^-(x, u_1)$. Moreover, $u_1 w_k^- \in E(G)$. If $|y_1^+ \overrightarrow{C} w_k^-| = 1$, then $C' = xv_2 \overrightarrow{C} y_1 y w_k w_k^- u_1 v_1 x$ is a D -cycle longer than C , a contradiction. So we may assume that $|y_1^+ \overrightarrow{C} w_k^-| \geq 2$.

Now, we consider $N_C(y, y_1^+) \cup N_C^-(x, u_1)$. Since $\text{dist}(y, y_1^+) = \text{dist}(x, u_1) = 2$, $|N(y, y_1^+)| \geq NC2(G)$, $|N^-(x, u_1)| = |N(x, u_1)| \geq NC2(G)$. Moreover, we have $v_1, v_2 \notin N_C(y, y_1^+) \cup N_C^-(x, u_1)$ and $N_C(y, y_1^+) \cap N_C^-(x, u_1) \subseteq \{w_k\}$. In fact, $v_1 \notin N(y, y_1^+)$ by Claims 3 and 5, if $v_1 \in N^-(x, u_1)$, then $v_1^+ x \in E(G)$ or $v_1^+ u_1 \in E(G)$, which contradicts to Claims 2 and 3. So $v_1 \notin N_C(y, y_1^+) \cup N_C^-(x, u_1)$; if $v_2 \in N_C(y, y_1^+)$, then $v_2 y^+ \in E(G)$ by Claim 5, which contradicts to Claim 4. If $v_2 \in N_C^-(x, u_1)$ then $v_2^+ \in N(x, u_1)$, which contradicts to Claims 2 and 3. So $v_2 \notin N_C(y, y_1^+) \cup N_C^-(x, u_1)$. Suppose $z \in N_C(y, y_1^+) \cap N_C^-(x, u_1) \setminus \{w_k\}$. Now, we consider the following cases.

(i) $z \in v_2 \overrightarrow{C} y_1^-$. If $yz \in E(G)$ and $xz^+ \in E(G)$, then, it contradicts to Claim 3. Put

$$C' = \begin{cases} yz \overrightarrow{C} v_2 x v_1 u_1 z^+ \overleftarrow{C} w_k y & \text{if } yz \in E(G) \text{ and } u_1 z^+ \in E(G); \\ xz^+ \overrightarrow{C} y_1 y w_k \overleftarrow{C} y_1^+ z \overleftarrow{C} v_1 x & \text{if } y_1^+ z \in E(G) \text{ and } xz^+ \in E(G); \\ x v_2 \overrightarrow{C} z y_1^+ \overrightarrow{C} w_k y y_1 \overleftarrow{C} z^+ u_1 v_1 x & \text{if } y_1^+ z \in E(G) \text{ and } u_1 z^+ \in E(G). \end{cases}$$

(ii) $z \in y_1 \overrightarrow{C} w_k^-$, then $z \in N(y_1^+)$ since $N(y) \cap (y_1^+ \overrightarrow{C} w_k^-) = \emptyset$. Let $zy_1^+ \in E(G)$ and $z^+ \in N_C(x, u_1)$. Since $V(C) = N_C(x, u_1) \cup N_C^-(x, u_1)$, So $y_1^+ \in N_C(x, u_1) \cup N_C^-(x, u_1)$. If $u_1 y_1^+ \in E(G)$ then $C' = x v_2 \overrightarrow{C} y_1 y w_k \overleftarrow{C} y_1^+ u_1 v_1 x$ is a D -cycle longer than C , a contradiction; if $xy_1^+ \in E(G)$, then it contradicts with Claim 3. Then, $y_1^+ \in N^-(x, u_1)$. If $xz^+ \in E(G)$ and $y_1^{+2} x \in E(G)$, then it contradicts to Claim 3; Put

$$C' = \begin{cases} x y_1^{+2} \overrightarrow{C} z y_1^+ \overleftarrow{C} u_1 z^+ \overleftarrow{C} v_1 x & \text{if } y_1^{+2} x \in E(G) \text{ and } u_1 z^+ \in E(G); \\ x v_2 \overrightarrow{C} y_1^+ z \overleftarrow{C} y_1^{+2} u_1 \overleftarrow{C} z^+ x & \text{if } y_1^{+2} u_1 \in E(G) \text{ and } xz^+ \in E(G); \\ x v_2 \overrightarrow{C} y_1^+ z \overleftarrow{C} y_1^{+2} u_1 z^+ \overleftarrow{C} v_1 x & \text{if } y_1^{+2} u_1 \in E(G) \text{ and } u_1 z^+ \in E(G). \end{cases}$$

In any cases, C' is a D -cycle longer than C , a contradiction. Therefore, $v_1, v_2 \notin N_C(y, y_1^+) \cup N_C^-(x, u_1)$, $N_C(y, y_1^+) \cap N_C^-(x, u_1) \subseteq \{w_k\}$. Hence, we have

$$\begin{aligned} |V(C)| &\geq |N_C(y, y_1^+) \cup N_C^-(x, u_1)| + 2 \\ &\geq |N_C(y, y_1^+)| + |N_C^-(x, u_1)| - 1 + 2 \\ &= |N(y, y_1^+) \setminus N_R(y, y_1^+)| + |N(x, u_1) \setminus N_R(x, u_1)| + 1 \\ &\geq 2NC2(G) - 2 + 1 \\ &= 2NC2(G) - 1, \end{aligned}$$

a contradiction with $|V(C)| \leq 2NC2(G) - 2$.

Claim 7 If $\bigcup_{i=1}^k N_R(y_i) \neq \emptyset$, then $N_R(y_i) \neq \emptyset$ for all $i \in \{1, 2, \dots, k\}$, where $y_i = u_i$ (w_i , respectively).

If not, without loss of generality, we assume that $N_R(u_1) \neq \emptyset$ and $N_R(u_k) = \emptyset$. Suppose $x_1 \in N_R(u_1)$ and $y \in N(x_1)$ ($y \neq u_1$). Then $\text{dist}(x_1, y^+) = \text{dist}(x_1, y^-) = 2$ and $|N(x_1, y^+)| \geq NC2(G)$, $|N(x_1, y^-)| \geq NC2(G)$.

Case 1 $N(x_1) \cap (u_1^+ \vec{C} v_k) = \emptyset$.

If not, we may choose $y, y \in N(x_1) \cap (u_1^+ \vec{C} v_k)$, such that $N(x_1) \cap (u_1^+ \vec{C} y^-) = \emptyset$. We define a mapping f on $V(C)$ as follows:

$$f(v) = \begin{cases} v^- & \text{if } v \in u_k \vec{C} y^-; \\ v^+ & \text{if } v \in y \vec{C} w_{k-1}; \\ y^- & \text{if } v = v_k. \end{cases}$$

Then $|f(N_C(x, u_k))| = |N_C(x, u_k)| = |N(x, u_k)| \geq NC2(G)$ by Claim 1 and the assumption $N_R(u_k) = \emptyset$. Moreover, we have $f(N_C(x, u_k)) \cap N(x_1, y^-) \subseteq \{w_k, u_1\}$. In fact, suppose that $z \in f(N_C(x, u_k)) \cap N(x_1, y^-) \setminus \{w_k, u_1\}$. Obviously, $z \neq v_1, y^-$ by Claims 2 and 4. Now we consider the following cases.

(i) If $z \in u_k \vec{C} w_k^-$, then $z \in N_C^-(u_k)$ since $N(x) \cap (u_k \vec{C} w_k) = \emptyset$. Put

$$C' = \begin{cases} u_k z^+ \vec{C} v_1 x v_k \overleftarrow{C} u_1 x_1 z \overleftarrow{C} u_k & \text{if } x_1 z \in E(G); \\ u_k z^+ \vec{C} v_1 x v_k \overleftarrow{C} y x_1 u_1 \vec{C} y^- z \overleftarrow{C} u_k & \text{if } y^- z \in E(G). \end{cases}$$

(ii) If $z \in u_1^+ \vec{C} y^-$, then $zy^- \in E(G)$ since $N(x_1) \cap (u_1^+ \vec{C} y^-) = \emptyset$. Put

$$C' = \begin{cases} u_1 \vec{C} z y^- \overleftarrow{C} z^+ x v_1 \overleftarrow{C} y x_1 u_1 & \text{if } x z^+ \in E(G); \\ u_1 \vec{C} z y^- \overleftarrow{C} z^+ u_k \vec{C} v_1 x v_k \overleftarrow{C} y x_1 u_1 & \text{if } u_k z^+ \in E(G). \end{cases}$$

(iii) If $z \in y^+ \vec{C} v_k$, we put

$$C' = \begin{cases} u_1 \vec{C} z^- x v_1 \overleftarrow{C} z x_1 u_1 & \text{if } x z^- \in E(G) \text{ and } x_1 z \in E(G); \\ u_1 \vec{C} y^- z \overleftarrow{C} v_1 x z^- \overleftarrow{C} y x_1 u_1 & \text{if } x z^- \in E(G) \text{ and } y^- z \in E(G); \\ u_1 \vec{C} z^- u_k \vec{C} v_1 x v_k \overleftarrow{C} z x_1 u_1 & \text{if } u_k z^- \in E(G) \text{ and } x_1 z \in E(G); \\ u_1 \vec{C} y^- z \overleftarrow{C} v_k x v_1 \overleftarrow{C} u_k z^- \overleftarrow{C} y x_1 u_1 & \text{if } u_k z^- \in E(G) \text{ and } y^- z \in E(G). \end{cases}$$

In any cases, C' is a D-cycle longer than C , a contradiction. Therefore, we have $f(N_C(x, u_k)) \cap N(x_1, y^-) \subseteq \{w_k, u_1\}$. By Claims 2 and 4, we have $u_1 \notin N(x, u_k)$ and $v_1 \notin N(x_1, y^-)$. Then $v_1 \notin f(N_C(x, u_k)) \cup N(x_1, y^-)$. Hence, by Claim 6 we have

$$\begin{aligned} |V(C)| &\geq |f(N_C(x, u_k)) \cup N_C(x_1, y^-)| + 1 \\ &\geq |f(N_C(x, u_k))| + |N_C(x_1, y^-)| - 2 + 1 \\ &\geq 2NC2(G) - 2. \end{aligned}$$

So, we have $V(C) = N_C(x_1, y^-) \cup f(N_C(x, u_k)) \cup \{v_1\}$, $N_C(x_1, y^-) \cap f(N_C(x, u_k)) = \{w_k, u_1\}$. Hence, $y^-w_k \in E(G)$ and $u_k u_1^+ \in E(G)$ since $t_i \geq 2$.

Now, we prove that $N_R(y^-) = \emptyset$. If not, there exist $y_1 \in N_R(y^-)$, $z \in N_C(y_1)$ ($z \neq y^-$) by Claim 1 and $\delta \geq 2$.

Subcase 1 $N(y_1) \cap (u_1 \overrightarrow{C} y^{-2}) = \emptyset$.

If not, we choose $z \in N(y_1)$, such that $N(y_1) \cap (z^+ \overrightarrow{C} y^{-2}) = \emptyset$. Therefore we can define a mapping f_1 on $V(C)$ as follows:

$$f_1(v) = \begin{cases} v^- & \text{if } v \in u_k^+ \overrightarrow{C} z^+; \\ v^+ & \text{if } v \in z^{+2} \overrightarrow{C} w_{k-1}; \\ z^{+2} & \text{if } v = v_k; \\ z^+ & \text{if } v = u_k. \end{cases}$$

Using an argument as above, we have $|f_1(N_C(x, u_k))| \geq NC2(G)$. Moreover, we have $z^+, v_1, y \notin N_C(y_1, z^+) \cup f_1(N_C(x, u_k))$ and $N_C(y_1, z^+) \cap f_1(N_C(x, u_k)) \subseteq \{z^{+2}, y^-, w_k\}$. Clearly, $z^+ \notin N_C(y_1, z^+)$. If $z^+ \in f_1(N_C(x, u_k))$, then, $u_k \in N_C(x, u_k)$, a contradiction. $y_1 v_1 \notin E(G)$ by Claim 5. If $v_1 z^+ \in E(G)$, since $y, z^+ \in N^+(y_1)$, the two paths $yx_1 u_1$ and $z^+ v_1$ contradict with Claim 4; By Claims 2 and 4, we have $y \notin N(y_1, z^+)$, if $y \in f_1(N_C(x, u_k))$ then $y^- \in N_C(x, u_k)$, by Claim 3 $y^- \notin N(x)$, so $y^- \in N(u_k)$, then $C' = xv_k \overleftarrow{C} yx_1 u_1 \overrightarrow{C} y^- u_k \overrightarrow{C} v_1 x$ is a D -cycle longer than C , a contradiction. So we have $z^+, v_1, y \notin N_C(y_1, z^+) \cup f_1(N_C(x, u_k))$. Suppose $s \in N_C(y_1, z^+) \cap f_1(N_C(x, u_k)) \setminus \{z^{+2}, y^-, w_k\}$.

Now, we consider the following cases.

(i) $s \in y^+ \overrightarrow{C} v_k$. If $y_1 s \in E(G)$ and $xs^- \in E(G)$ then it contradicts with Claim 4. We put

$$C' = \begin{cases} xv_k \overleftarrow{C} sy_1 y^- \overleftarrow{C} u_1 x_1 y \overleftarrow{C} s^- u_k \overrightarrow{C} v_1 x & \text{if } y_1 s, u_k s^- \in E(G); \\ xs^- \overleftarrow{C} yx_1 u_1 \overrightarrow{C} zy_1 y^- \overleftarrow{C} z^+ s \overrightarrow{C} v_1 x & \text{if } z^+ s, xs^- \in E(G); \\ xv_k \overleftarrow{C} sz^+ \overrightarrow{C} y^- y_1 z \overleftarrow{C} u_1 x_1 y \overleftarrow{C} s^- u_k \overrightarrow{C} v_1 x & \text{if } z^+ s, u_k s^- \in E(G). \end{cases}$$

(ii) $s \in u_k \overrightarrow{C} w_{k-1}$. We have $s \in N^-(u_k)$ since $N(x) \cap (u_k \overrightarrow{C} w_k) = \emptyset$. Put

$$C' = \begin{cases} xv_k \overleftarrow{C} yx_1 u_1 \overrightarrow{C} y^- y_1 s \overleftarrow{C} u_k s^+ \overrightarrow{C} v_1 x & \text{if } y_1 s, u_k s^+ \in E(G); \\ xv_k \overleftarrow{C} yx_1 u_1 \overrightarrow{C} zy_1 y^- \overleftarrow{C} z^+ s \overleftarrow{C} u_k s^+ \overrightarrow{C} v_1 x & \text{if } z^+ s, u_k s^+ \in E(G); \end{cases}$$

(iii) $s \in u_1 \overrightarrow{C} y^{-2}$. If $y_1 s, xs^+ \in E(G)$ then contradicts to Claim 4. If $y_1 s, u_k s^+ \in E(G)$, then

$$C' = xv_k \overleftarrow{C} yx_1 u_1 \overrightarrow{C} sy_1 y^- \overleftarrow{C} s^+ u_k \overrightarrow{C} v_1 x$$

is a D -cycle longer than C , a contradiction. If $s \in z^+ \overrightarrow{C} y^-$, we put

$$C' = \begin{cases} xs^- \overleftarrow{C} z^+ s \overrightarrow{C} y^- y_1 z \overleftarrow{C} u_1 x_1 y \overrightarrow{C} v_1 x & \text{if } z^+ s, s^- x \in E(G); \\ xv_k \overleftarrow{C} yx_1 u_1 \overrightarrow{C} zy_1 y^- \overleftarrow{C} sz^+ \overrightarrow{C} s^- u_k \overrightarrow{C} v_1 x & \text{if } z^+ s, s^- u_k \in E(G). \end{cases}$$

If $s \in u_1 \vec{C} z$, we put

$$C' = \begin{cases} xs^+ \vec{C} zy_1 y^- \overleftarrow{C} z^+ s \overleftarrow{C} u_1 x_1 y \vec{C} v_1 x & \text{if } z^+ s, xs^+ \in E(G); \\ xv_k \overleftarrow{C} yx_1 u_1 \vec{C} sz^+ \vec{C} y^- y_1 z \overleftarrow{C} s^+ u_k \vec{C} v_1 x & \text{if } z^+ s, u_k s^+ \in E(G). \end{cases}$$

In any cases, C' is a D -cycle longer than C , a contradiction. Hence, by Claim 5 we have

$$\begin{aligned} |V(C)| &\geq |f_1(N_C(x, u_k)) \cup N_C(y_1, z^+)| + 3 \\ &\geq |f_1(N_C(x, u_k))| + |N_C(y_1, z^+)| - 3 + 3 \\ &\geq 2NC2(G) - 1, \end{aligned}$$

a contradiction. So $N(y_1) \cap (u_1 \vec{C} y^{-2}) = \emptyset$,

Subcase 2 $N(y_1) \cap (y \vec{C} v_k) = \emptyset$.

If not, we may choose $z \in N(y_1) \cap (y \vec{C} v_k)$, such that $N(y_1) \cap (y \vec{C} z^-) = \emptyset$. Therefore, we can define a mapping f_2 on $V(C)$ as follows:

$$f_2(v) = \begin{cases} v^+ & \text{if } v \in u_1 \vec{C} y^{-2} \cup z^- \vec{C} w_{k-1}; \\ v^- & \text{if } v \in y^+ \vec{C} z^{-2} \cup u_k^+ \vec{C} v_1; \\ z^- & \text{if } v = v_k; \\ v_1 & \text{if } v = u_k; \\ z^{-2} & \text{if } v = y; \\ u_1 & \text{if } v = y^- \end{cases}$$

Using a similar argument as above, we have $|f_2(N_C(x, u_k))| \geq NC2(G)$. We consider $N_C(y_1, z^-) \cup f_2(N_C(x, u_k))$, then $v_1, u_1^+ \notin N_C(y_1, z^-) \cup f_2(N_C(x, u_k))$, and $N_C(y_1, z^-) \cap f_2(N_C(x, u_k)) \subseteq \{y^-, w_k\}$. In fact, $v_1 \notin N(y_1, z^-)$ by Claims 4, 5; if $v_1 \in f_2(N_C(x, u_k))$ then $u_k \in N(x, u_k)$, a contradiction; if $u_1^+ \in N(z^-)$, then the paths $yx_1 u_1$ and $z^- u_1^+$ contradict with Claim 5; if $u_1^+ \in f_2(N_C(x, u_k))$, then $u_1 \in N(x, u_k)$, a contradiction. So we have $v_1, u_1^+ \notin N_C(y_1, z^-) \cup f_2(N_C(x, u_k))$. For $s \in N_C(y_1, z^-) \cap f_2(N_C(x, u_k)) \setminus \{y^-, w_k\}$, we consider the following cases.

(i) If $s \in u_1 \vec{C} y$. We have $s \in N(z^-)$ since $N(y_1) \cap (u_1 \vec{C} y^{-2}) = \emptyset$. Put

$$C' = \begin{cases} xs^- \overleftarrow{C} u_1 x_1 y \vec{C} z^- s \vec{C} y^- y_1 z \vec{C} v_1 x & \text{if } s^- x \in E(G); \\ xv_k \overleftarrow{C} zy_1 y^- \overleftarrow{C} sz \overleftarrow{C} yx_1 u_1 \vec{C} s^- u_k \vec{C} v_1 x & \text{if } s^- u_k \in E(G). \end{cases}$$

(ii) If $s \in u_k \vec{C} v_1$, then $s^+ \in N(u_k)$ since $N(x) \cap (u_k \vec{C} w_k) = \emptyset$. Put

$$C' = \begin{cases} xv_k \overleftarrow{C} zy_1 y^- \overleftarrow{C} u_1 x_1 y \vec{C} z^- s \overleftarrow{C} u_k s^+ \vec{C} v_1 x & \text{if } z^- s \in E(G); \\ xv_k \overleftarrow{C} yx_1 u_1 \vec{C} y^- y_1 s \overleftarrow{C} u_k s^+ \vec{C} v_1 x & \text{if } y_1 s \in E(G). \end{cases}$$

(iii) If $s \in y \vec{C} z^{-2}$, then we have $s \in N(z^-)$ since $N(y_1) \cap (y \vec{C} z^{-2}) = \emptyset$. Put

$$C' = \begin{cases} x_1 y \vec{C} sz^- \overleftarrow{C} s^+ x v_1 \overleftarrow{C} zy_1 y^- \overleftarrow{C} u_1 x_1 & \text{if } xs^+ \in E(G); \\ xv_k \overleftarrow{C} zy_1 y^- \overleftarrow{C} u_1 x_1 y \vec{C} sz^- s^+ u_k \vec{C} v_1 x & \text{if } u_k s^+ \in E(G). \end{cases}$$

(iv) If $s \in z^- \vec{C} v_k$. If $y_1 s, x s^- \in E(G)$ then it contradicts to Claim 4. We put

$$C' = \begin{cases} x v_k \overleftarrow{C} s y_1 y^- \overleftarrow{C} u_1 x_1 y \overrightarrow{C} s^- u_k \overrightarrow{C} v_1 x & \text{if } y_1 s, u_k s^- \in E(G); \\ x s^- \overleftarrow{C} z y_1 y^- \overleftarrow{C} u_1 x_1 y \overrightarrow{C} z^- s \overrightarrow{C} v_1 x & \text{if } z^- s, s^- x \in E(G); \\ x v_k \overleftarrow{C} s z^- \overleftarrow{C} y x_1 u_1 \overrightarrow{C} y^- y_1 z \overrightarrow{C} s^- u_k \overrightarrow{C} v_1 x & \text{if } z^- s, s^- u_k \in E(G). \end{cases}$$

In any cases, C' is a D -cycle longer than C , a contradiction. Therefore, we have $v_1, u_1^+ \notin N_C(y_1, z^-) \cup f_2(N_C(x, u_k))$, and $N_C(y_1, z^-) \cap f_2(N_C(x, u_k)) \subseteq \{y^-, w_k\}$. So

$$\begin{aligned} |V(C)| &\geq |N_C(y_1, z^-) \cup f_2(N_C(x, u_k))| + 2 \\ &\geq |N_C(y_1, z^-)| + |N_C(x, u_k)| - 2 + 2 \\ &\geq 2NC2(G) - 1, \end{aligned}$$

a contradiction with $|V(C)| \leq 2NC2(G) - 2$. Hence, $N(y_1) \setminus \{y^-\} \subseteq (u_k \vec{C} u_1)$.

Subcase 3 $N(y_1) \cap (u_k \vec{C} u_1) = \emptyset$.

If not, we may choose $z \in N(y_1) \cap (u_k \vec{C} u_1)$, such that $N(y_1) \cap (z^+ \vec{C} u_1) = \emptyset$. We define a mapping f_3 on $V(C)$ as follows:

$$f_3(v) = \begin{cases} v^- & \text{if } v \in y^+ \vec{C} v_k \cup u_k^+ \vec{C} z^+; \\ v^+ & \text{if } v \in z^{+2} \vec{C} y^{-2}; \\ z^+ & \text{if } v = u_k; \\ v_k & \text{if } v = y; \\ z^{+2} & \text{if } v = y^-. \end{cases}$$

Using a similar argument as above, we have $|f_3(N_C(x, u_k))| \geq NC2(G)$. Moreover, $z^+, u_1^+ \notin N_C(y_1, z^+) \cup f_3(N_C(x, u_k))$, $N_C(y_1, z^+) \cap f_3(N_C(x, u_k)) \subseteq \{y^-, w_k\}$. In fact, clearly, $z^+ \notin N_C(y_1, z^+)$, if $z^+ \in f_3(N_C(x, u_k))$, then $u_k \in N_C(x, u_k)$, a contradiction; if $u_1^+ \in N_C(y_1, z^+)$, then $u_1^+ \in N(z^+)$ since $N_C(y_1) \cap (y^{-2} \vec{C} u_k) = \emptyset$, so $C' = x_1 y \overrightarrow{C} z y_1 y^- \overleftarrow{C} u_1^+ z^+ \overrightarrow{C} u_1 x_1$ is a D -cycle longer than C , a contradiction; if $u_1^+ \in f_3(N_C(x, u_k))$ then $u_1 \in N_C(x, u_k)$, a contradiction; so we have $z^+, u_1^+ \notin N_C(y_1, z^+) \cup f_3(N_C(x, u_k))$. Suppose $s \in N_C(y_1, z^+) \cap f_3(N_C(x, u_k)) \setminus \{y^-, w_k\}$. Now, we consider the following cases.

(i) If $s \in v_k \vec{C} z^+$, then We have $s^+ u_k \in E(G)$ since $N(x) \cap (u_k \vec{C} w_k) = \emptyset$. Put

$$C' = \begin{cases} x v_k \overleftarrow{C} y x_1 u_1 \overrightarrow{C} y^- y_1 s \overleftarrow{C} u_k s^+ \overrightarrow{C} v_1 x & \text{if } y_1 s \in E(G); \\ x v_k \overleftarrow{C} y x_1 u_1 \overrightarrow{C} y^- y_1 z \overleftarrow{C} s^+ u_k \overrightarrow{C} s z^+ \overrightarrow{C} v_1 x & \text{if } z^+ s \in E(G). \end{cases}$$

(ii) If $s \in z^{+2} \vec{C} w_k^-$, then we have $s^- u_k, s^+ \in E(G)$ since $N(x) \cap (u_k \vec{C} w_k) = N(y_1) \cap (z^+ \vec{C} v_1) = \emptyset$. Put

$$C' = x v_k \overleftarrow{C} y x_1 u_1 \overrightarrow{C} y^- y_1 z \overleftarrow{C} u_k s^- \overleftarrow{C} z^+ s \overrightarrow{C} v_1 x$$

(iii) If $s \in u_1 \vec{C} y^{-2}$, then we have $s z^+ \in E(G)$ since $N(y_1) \cap (u_1 \vec{C} y^{-2}) = \emptyset$. Put

$$C' = \begin{cases} xs^- \overleftarrow{C} u_1 x_1 y \overrightarrow{C} z y_1 y^- \overleftarrow{C} s z^+ \overrightarrow{C} v_1 x & \text{if } xs^- \in E(G); \\ xv_k \overleftarrow{C} y x_1 u_1 \overleftarrow{C} s^- u_k \overrightarrow{C} z y_1 y^- \overleftarrow{C} s z^+ \overrightarrow{C} v_1 x & \text{if } u_k s^- \in E(G). \end{cases}$$

(iv) If $s \in y \overrightarrow{C} v_k$, then we have $sz^+ \in E(G)$ since $N(y_1) \cap (y \overrightarrow{C} v_k) = \emptyset$. Put

$$C' = \begin{cases} xs^+ \overrightarrow{C} z y_1 y^- \overleftarrow{C} u_1 x_1 y \overrightarrow{C} s z^+ \overrightarrow{C} v_1 x & \text{if } xs^+ \in E(G); \\ xv_k \overleftarrow{C} s^+ u_k \overrightarrow{C} z y_1 y^- \overleftarrow{C} u_1 x_1 y \overrightarrow{C} s z^+ \overrightarrow{C} v_1 x & \text{if } u_k s^+ \in E(G). \end{cases}$$

In any cases, C' is a D -cycle longer than C , a contradiction. Therefore we have $N_C(y_1, z^+) \cap f_3(N_C(x, u_k)) \subseteq \{y^-, w_k\}$. So we have

$$\begin{aligned} |V(C)| &\geq |N_C(y_1, z^+) \cup f_3(N_C(x, u_k))| + 2 \\ &\geq |N_C(y_1, z^+)| + |N_C(x, u_k)| - 2 + 2 \\ &\geq 2NC2(G) - 1, \end{aligned}$$

a contradiction with $|V(C)| \leq 2NC2(G) - 2$. Hence, $N(y_1) \cap (u_k \overrightarrow{C} v_1) = \emptyset$.

Thus, $N(y_1) = \{y^-\}$, which contradicts to $\delta \geq 2$. Therefore, we know that $N_R(y^-) = \emptyset$.

So we have

$$\begin{aligned} |V(C)| &\geq |f(N_C(x, u_k)) \cup N_C(x_1, y^-)| + 1 \\ &\geq |f(N_C(x, u_k))| + |N_C(x_1, y^-)| - 2 + 1 \\ &= |N(x, u_k) \setminus N_R(x, u_k)| + |N(x_1, y^-) \setminus N_R(x_1, y^-)| - 1 \\ &= |N(x, u_k)| + |N(x_1, y^-)| - 1 \\ &\geq 2NC2(G) - 1, \end{aligned}$$

a contradiction. So we have $N(x_1) \cap (u_1^+ \overrightarrow{C} v_k) = \emptyset$, hence, $N(x_1) \subseteq u_k \overrightarrow{C} u_1$.

Case 2 $N(x_1) \cap (u_k \overrightarrow{C} v_1) = \emptyset$.

Otherwise, since $v_1 x_1 \notin E(G)$, we can choose $y, y \in u_k \overrightarrow{C} w_k$, such that $N(x_1) \cap (y^+ \overrightarrow{C} v_1) = \emptyset$. Therefore, we can define a mapping g on $V(C)$ as follows:

$$g(v) = \begin{cases} v^- & \text{if } v \in u_1^+ \overrightarrow{C} y; \\ v^+ & \text{if } v \in y^+ \overrightarrow{C} w_k; \\ y^+ & \text{if } v = u_1, \\ y & \text{if } v = v_1. \end{cases}$$

Using a similar argument as before, we have $|g(N_C(x, u_k))| \geq NC2(G)$, $y^+ \notin g(N_C(x, u_k)) \cup N(x_1, y^+)$ and $g(N_C(x, u_k)) \cap N(x_1, y^+) \subseteq \{u_1\}$. Hence, by Claim 6 we have

$$\begin{aligned} |V(C)| &\geq |g(N_C(x, u_k)) \cup N(x_1, y^+)| + 1 \\ &\geq |g(N_C(x, u_k))| + |N(x_1, y^+)| - 1 + 1 \\ &\geq 2NC2(G) - 1, \end{aligned}$$

a contradiction. So $N(x_1) \cap (u_k \vec{C} v_1) = \emptyset$. Then $N(x_1) = \{u_1\}$, which contradicts to $\delta \geq 2$.

Claim 8 If $x_1 \in N_R(u_1)$ and $N(x_1) \cap (u_1^+ \vec{C} v_k) \neq \emptyset$, then $|\{u_k u_1^+, y^- w_k\} \cap E(G)| = 1$ for $y \in N(x_1) \cap (u_1^+ \vec{C} v_k)$ with $N(x_1) \cap (u_1^+ \vec{C} y^-) = \emptyset$.

First we have $d(x_1, y^-) = 2$ and $|N(x_1, y^-)| \geq NC2(G)$. Let $u_k u_1^+ \notin E(G)$. Now we define a mapping f on $V(C)$ as follows:

$$f(v) = \begin{cases} v^- & \text{if } v \in u_k^{+2} \vec{C} v_1 \cup u_1^{+2} \vec{C} y^-; \\ v^+ & \text{if } v \in y \vec{C} w_{k-1}; \\ y^- & \text{if } v = u_k; \\ y & \text{if } v = v_k; \\ u_1 & \text{if } v = u_k^+; \\ v_1 & \text{if } v = u_1^+; \\ u_k & \text{if } v = u_1. \end{cases}$$

Then $|f(N_C(x, u_k))| = |N_C(x, u_k)| \geq NC2(G) - 1$ by Claim 5. Moreover using a similar argument as in Claim 7, we have $f(N_C(x, u_k)) \cap N(x_1, y^-) \subseteq \{w_k, u_1, y\}$. But we have $y^-, v_1, u_k \notin f(N_C(x, u_k)) \cup N(x_1, y^-)$ by the choice of y Claims 2 and 4, respectively. Therefore, by Claim 5 we have

$$\begin{aligned} |V(C)| &\geq |f(N_C(x, u_k)) \cup N_C(x_1, y^-)| + 3 \\ &\geq |f(N_C(x, u_k))| + |N_C(x_1, y^-)| - 3 + 3 \\ &\geq 2NC2(G) - 2. \end{aligned}$$

So $V(C) = f(N_C(x, u_k)) \cup N_C(x_1, y^-) \cup \{v_1, y^-, u_k\}$ by the assumption on $|V(C)|$, and in particular, $f(N_C(x, u_k)) \cap N_C(x_1, y^-) = \{w_k, u_1, y\}$. Therefore, $y^- w_k \in E(G)$. Using a similar argument as above, we have if $y^- w_k \notin E(G)$, then $u_k u_1^+ \in E(G)$.

Claim 9 There exists a vertex x with $x \notin V(C)$ such that $N_R(u_i) = N_R(w_i) = \emptyset$.

We only prove $N_R(u_i) = \emptyset$. If not, we may choose $x \notin V(C)$ such that $\min\{t_i\}$ is as small as possible. By Claim 7, without loss of generality, suppose that $t_k = \min\{t_i\}$ for the vertex x . Let $x_1 \in N_R(u_1), x_2 \in N_R(u_k)$. By Claims 2 and 3, $x \neq x_1, x_2; x_1 \neq x_2$. And by Claim 5 and the choice of x , we have $N(x_i) \cap (u_k \vec{C} v_1) = \emptyset$, for $i = 1, 2$. Since $\delta \geq 2$, $N(x_1) \cap (u_1^+ \vec{C} v_k) \neq \emptyset$. Choose $y \in N(x_1) \cap (u_k \vec{C} v_k)$ such that $N(x_1) \cap (u_1^+ \vec{C} y^-) = \emptyset$, then $d(x_1, y^-) = 2$ and $|N(x_1, y^-)| \geq NC2(G)$. By Claim 8, we have $u_k u_1^+$ or $y^- w_k \in E(G)$.

First we prove that $N(x_2) \cap (y \vec{C} v_k) = \emptyset$. If not, we may choose $z \in y^+ \vec{C} v_k^-$ such that $N(x_2) \cap (z^+ \vec{C} v_k) = \emptyset$ by Claim 5. Then $d(x_2, z^+) = 2$ and $|N(x_2, z^+)| \geq NC2(G)$. Now we define a mapping f on $V(C)$ as follows:

$$f(v) = \begin{cases} v^- & \text{if } v \in u_1^+ \vec{C} y^- \cup z^{+2} \vec{C} v_k; \\ v^+ & \text{if } v \in y \vec{C} z^- \cup u_k \vec{C} w_k; \\ y & \text{if } v = z; \\ v_k & \text{if } v = z^+; \\ u_k & \text{if } v = v_1; \\ y^- & \text{if } v = u_1. \end{cases}$$

Then $|f(N_C(x_2, z^+))| = |N_C(x_2, z^+)| \geq NC2(G) - 1$ by Claim 5. Moreover using a similar argument as in Claim 7, we have $f(N_C(x_2, z^+)) \cap N(x_1, y^-) \subseteq \{u_1, y\}$. But $y^-, v_k, v_1 \notin f(N_C(x_2, z^+)) \cup N(x_1, y^-)$, otherwise, $u_1 z^+ \in E(G)$ or $y^- v_k \in E(G)$ or $z^+ w_k \in E(G)$ by Claim 5, and hence the D-cycle

$$C' = \begin{cases} u_1 \vec{C} z x_2 u_k \vec{C} v_1 x v_k \overleftarrow{C} z^+ u_1 & \text{if } u_1 z^+ \in E(G); \\ u_1 x_1 y \vec{C} v_k y^- \overleftarrow{C} u_1^+ u_k \vec{C} u_1 & \text{if } y^- v_k \in E(G); \\ x v_k \overleftarrow{C} z^+ w_k \overleftarrow{C} u_k x_2 z \overleftarrow{C} v_1 x & \text{if } z^+ w_k \in E(G). \end{cases}$$

is longer than C , a contradiction. Therefore, by Claim 5 we have

$$\begin{aligned} |V(C)| &\geq |f(N_C(x_2, z^+)) \cup N_C(x_1, y^-)| + 3 \\ &\geq |f(N_C(x_2, z^+))| + |N_C(x_1, y^-)| - 2 + 3 \\ &\geq 2NC2(G) - 1, \end{aligned}$$

which contradicts to that $|V(C)| \leq 2NC2(G) - 2$. So we have $N(x_2) \cap (y \vec{C} v_k) = \emptyset$. Hence $N(x_2) \cap (u_1^+ \vec{C} y^-) \cup \{u_k\}$.

Now, we prove that $N(x_2) \cap (u_1^+ \vec{C} y^-) = \emptyset$. In fact, we may choose $z \in u_1^+ \vec{C} y^{-2}$ with $z \in N(x_2)$ such that $N(x_2) \cap (u_1^+ \vec{C} z^-) = \emptyset$. (Since $x_2 y^- \notin E(G)$, otherwise, $C' = u_1 \vec{C} y^- x_2 u_k \vec{C} v_1 x v_k \overleftarrow{C} y x_1 u_1$ is a D-cycle longer than C , a contradiction.) Then $d(x_2, z^-) = 2$ and $|N(x_2, z^-)| \geq NC2(G)$. We define a mapping g on $V(C)$ as follows:

$$g(v) = \begin{cases} v^- & \text{if } v \in z^+ \vec{C} v_k; \\ v^+ & \text{if } v \in u_k \vec{C} z^{-2}; \\ v_k & \text{if } v = z; \\ u_k & \text{if } v = z^-. \end{cases}$$

Then we have $|g(N_C(x_2, z^-))| \geq NC2(G) - 1$ by Claim 5. Moreover using a similar argument as in Claim 7, we have $g(N_C(x_2, z^-)) \cap N(x_1, y^-) \subseteq \{u_1\}$. But $v_1, u_k \notin g(N_C(x_2, z^-)) \cup N(x_1, y^-)$, otherwise since $u_k \notin g(N_C(x_2, z^-)) \cup N(x_1, y^-)$, $w_k z^- \in E(G)$ by Claims 2 and 4, and hence the D-cycle $u_1 \vec{C} z^- w_k \overleftarrow{C} u_k x_2 z \vec{C} v_k x v_1 u_1$ is longer than C , a contradiction. Therefore, by Claim 5 we have

$$\begin{aligned}
|V(C)| &\geq |g(N_C(x_2, z^-)) \cap N(x_1, y^-)| + 2 \\
&\geq |g(N_C(x_2, z^-))| + |N(x_1, y^-)| - 1 + 2 \\
&\geq 2NC2(G) - 1,
\end{aligned}$$

which contradicts to that $|V(C)| \leq 2NC2(G) - 2$. So we have $N(x_2) \cap (u_1^+ \vec{C} y^-) = \emptyset$.

Therefore, $N(x_2) = \{u_k\}$, which contradicts to $\delta \geq 2$.

Claim 10 For any $x \notin V(C)$, $t_i \geq 3$.

Otherwise, there exists a vertex $x, x \notin V(C)$, such that $\min\{t_i\} = 2$ by Claim 6. Note that the choice of the vertex x in Claim 9, we have $N_R(u_i) = N_R(w_i) = \emptyset$ for the vertex x . Without loss of generality, suppose $t_1 = 2$, then $N_{\vec{C}}(u_1) \cap N_C(w_1) = \{u_1\}$ by Claim 4, $N(x) \cap N^+(x) = \emptyset$ by Claim 2, and $N_{\vec{C}}(u_1) \cap N(x) = N^-(x) \cap N_C(w_1) = \emptyset$ by Claim 3. Hence, $N_{\vec{C}}(x, u_1) \cap N_C(x, w_1) = \{u_1\}$. We also have $|N_C(x, u_1)| \geq NC2(G)$ and $|N_C(x, w_1)| \geq NC2(G)$ since $d(x, u_1) = d(x, w_1) = 2$. Then

$$\begin{aligned}
|V(C)| &\geq |N_{\vec{C}}(x, u_1) \cup N_C(x, w_1)| \\
&\geq |N_{\vec{C}}(x, u_1)| + |N_C(x, w_1)| - 1 \\
&\geq 2NC2(G) - 1,
\end{aligned}$$

which contradicts to that $|V(C)| \leq 2NC2(G) - 2$.

By Claim 10, we have $|V(C)| = k + \sum_{i=1}^k t_i \geq 4k$. Thus we get the following.

Claim 11 For any $x, x \notin V(C)$,

$$d(x) \leq \frac{|V(C)|}{4} \leq \frac{2NC2(G) - 2}{4} = (NC2(G) - 1)/2.$$

Claim 12 $u_i^+ u_j \notin E(G)$, for the vertex x as in Claim 9.

In fact, if $u_i^+ u_j \in E(G)$, then the cycle $u_i^+ \vec{C} v_j x v_i \overleftarrow{C} u_j u_i^+$ is a longest D-cycle not containing u_i , by Claim 9. Thus $d(u_i) \leq (NC2(G) - 1)/2$ by Claim 11. So we have

$$NC2(G) \leq |N(x, u_i)| \leq d(x) + d(u_i) \leq NC2(G) - 1,$$

a contradiction. We choose x as in Claim 9, and define a mapping f on $V(C)$ as follows:

$$f(v) = \begin{cases} v^+ & \text{if } v \in u_1 \vec{C} v_k^-; \\ v^- & \text{if } v \in u_k^+ \vec{C} v_1; \\ u_1 & \text{if } v = v_k; \\ v_1 & \text{if } v = u_k. \end{cases}$$

Then $|f(N_C(x, u_k))| \geq NC2(G)$ and $|N_C(x, u_1)| \geq NC2(G)$ by Claim 10. Moreover, we have $f(N_C(x, u_k)) \cap N_C(x, u_1) \subseteq \{v_2, v_3, \dots, v_k, w_k\}$. By Claims 2, 4, and 12, we also have $u_2^+, u_3^+, \dots, u_{k-1}^+ \notin f(N_C(x, u_k)) \cup N_C(x, u_1)$. Therefore, we have

$$\begin{aligned}
|V(C)| &\geq |f(N_C(x, u_k)) \cup N_C(x, u_1)| + k - 2 \\
&\geq |f(N_C(x, u_k))| + |N_C(x, u_1)| - k + k - 2 \\
&\geq 2NC2(G) - 2.
\end{aligned}$$

So

$$V(C) = f(N_C(x, u_k)) \cup N_C(x, u_1) \cup \{u_2^+, u_3^+, \dots, u_{k-1}^+\}$$

by the assumption on $|V(C)|$, and in particular,

$$f(N_C(x, u_k)) \cap N_C(x, u_1) = \{v_2, v_3, \dots, v_k, w_k\}.$$

Then $u_1 w_k, u_k w_{k-1} \in E(G)$.

Claim 13 $k = 2$.

If there exists $v \in V(C) \setminus \{v_1, v_k\}$, by partition of $V(C)$, we have $v^{+2} \in f(N_C(x, u_k)) \cup N_C(x, u_1) \cup \{u_2^+, u_3^+, \dots, u_{k-1}^+\}$. If $v^{+2} \in N_C(x, u_1)$, then $v^{+2} u_1 \in E(G)$, and the cycle $u_1 v^{+2} \vec{C} v_1 x v \vec{C} u_1$ is a D-cycle not containing v^+ by Claim 9. Thus $d(v^+) \leq (NC2(G) - 1)/2$ by Claim 11. So we have

$$NC2(G) \leq |N(x, v^+)| \leq d(x) + d(v^+) \leq NC2(G) - 1,$$

a contradiction. So $v^+ \in N(x, u_k)$, which contradicts to Claims 2,3. Hence we have $k = 2$.

Claim 14 Each of the followings does not hold :

- (1) There is $u \in u_1 \vec{C} v_2$, such that $u^+ u_1 \in E(G)$ and $u^- u_2 \in E(G)$.
- (2) There is $u \in u_2 \vec{C} v_1$, such that $u^- u_1 \in E(G)$ and $u^+ u_2 \in E(G)$.
- (3) There is $u \in u_2 \vec{C} v_1$, such that $u^+ w_1 \in E(G)$ and $u^- w_2 \in E(G)$.
- (4) There is $u \in u_1 \vec{C} v_2$, such that $u^+ w_2 \in E(G)$ and $u^- w_1 \in E(G)$.

If not, suppose there is $u \in u_1 \vec{C} v_2$, such that $u^+ u_1 \in E(G)$ and $u^- u_2 \in E(G)$. We define a mapping h on $V(C)$ as follows :

$$h(v) = \begin{cases} v^+ & \text{if } v \in u_1 \vec{C} u^- u_2 \cup u^+ \vec{C} w_1; \\ v^- & \text{if } v \in u_2^+ \vec{C} v_1; \\ u^+ & \text{if } v = v_2; \\ v_1 & \text{if } v = u_2; \\ u_1 & \text{if } v = u; \\ u & \text{if } v = u_2^+. \end{cases}$$

Then $|h(N_C(x, u_2))| \geq NC2(G)$ and $|N_C(x, u_1)| \geq NC2(G)$. Moreover we have $u_1 \notin N(x, u_1) \cup h(N(x, u_2))$, and $N(x, u_1) \cap h(N(x, u_2)) \subseteq \{v_2, u^+\}$. In fact, clearly $u_1 \notin N(x, u_1)$, if $u_1 \in h(N(x, u_2))$, then $u \in N(x, u_2)$, a contradiction. Let $s \in N(x, u_1) \cap h(N(x, u_2)) \setminus \{v_2, u^+\}$, if $s \in u_1^+ \vec{C} v_2 \cap N(x, u_1) \cap h(N(x, u_2)) \setminus \{v_2, u^+\}$ then $su_1 \in E(G)$ and $s^- u_2 \in E(G)$; or if

$s \in u_2 \vec{C} w_2 \cap N(x, u_1) \cap h(N(x, u_2))$, then $su_1 \in E(G)$ and $s^+u_2 \in E(G)$, both cases contradict to Claim 3. So $u_1 \notin N(x, u_1) \cup h(N(x, u_2))$, $N(x, u_1) \cap h(N(x, u_2)) \subseteq \{v_2, u^+\}$. Hence

$$\begin{aligned} |V(C)| &\geq |h(N_C(x, u_2)) \cup N_C(x, u_1)| + 1 \\ &\geq |h(N_C(x, u_2))| + |N_C(x, u_1)| - 2 + 1 \\ &\geq 2NC2(G) - 1, \end{aligned}$$

a contradiction. Similarly, (2), (3) and (4) are true.

Claim 15 $N(u_2) \cap (u_1 \vec{C} w_1^-) = N(u_1) \cap (u_2 \vec{C} w_2^-) = \emptyset$.

If not, we may choose $z \in N(u_2) \cap (u_1 \vec{C} w_1^-)$, such that $N(u_2) \cap (u_1 \vec{C} z^-) = \emptyset$. then $u_1 z \in E(G)$ (if not, $u_1 z \notin E(G)$ then $u_2 z^- \in E(G)$ by partition of $V(G)$, which contradicts the choice of z) and $N(u_1) \cap (z^+ \vec{C} w_1) = \emptyset$ (if not, we may choose $s \in N(u_1) \cap (z^+ \vec{C} w_1)$, such that $N(u_1) \cap (z^+ \vec{C} s^-) = \emptyset$ since $z^+u_1 \notin E(G)$. So $s^-u_1 \notin E(G)$, by partition of the $V(C)$, $s^-u_2 \in E(G)$. Which contradicts Claim 14) Moreover $u_1^+ \vec{C} z \subseteq N(u_1)$, and $z \vec{C} v_2 \subseteq N(u_2)$. Similarly , we have $y \in u_2 \vec{C} w_2$, such that $u_2 y, u_1 y \in E(G)$ and $N(u_1) \cap (u_2 \vec{C} y^-) = N(u_2) \cap (y^+ \vec{C} w_2) = \emptyset$, $y \vec{C} v_1 \subseteq N(u_1)$ and $u_2^+ \vec{C} y \subseteq N(u_2)$.

Now we define a mapping g on $V(C)$ as follows:

$$g(v) = \begin{cases} v^+ & \text{if } v \in v_2 \vec{C} w_2^-; \\ v^- & \text{if } v \in u_1 \vec{C} w_1; \\ v_2 & \text{if } v = w_2; \\ w_1 & \text{if } v = v_1. \end{cases}$$

Using similar argument as above , consider $N(x, w_1) \cup g(N(x, w_2))$, there exists $u \in V(C)$, such that $w_1 u, w_2 u \in E(G)$. Without loss generality, we may assume $u \in u_1 \vec{C} w_1$, Moreover then $N(w_2) \cap (u^+ \vec{C} w_1) = N(w_1) \cap (u_1 \vec{C} u^-) = \emptyset$, and $v_1 \vec{C} u \subseteq N(w_2)$, $u \vec{C} v_2 \subseteq N(w_1)$. Let $u \neq z$. If $u \in z \vec{C} w_1^-$, $u^-u_2 \in E(G)$ by partition of $V(C)$ since $uu_1 \notin E(G)$, which contradicts to Claim 4 ; if $u \in u_1 \vec{C} z$, then $C' = xv_2w_1u \vec{C} w_1^-u_2 \vec{C} w_2u^- \vec{C} v_1x$ is a D -cycle longer than C , a contradiction. If $u = z$, since $z^{+2}u_1 \notin E(G)$, $z^+u_2 \in E(G)$ by partition of $V(C)$, which contradicts to Claim 4. Hence $N(u_2) \cap (u_1 \vec{C} w_2^-) = \emptyset$. Similarly $N(u_1) \cap (u_2 \vec{C} w_1^-) = \emptyset$.

By Claim 15 we have

Claim 16 If there exists $z \in v_1 \vec{C} v_2$, such that $u_2 z \in E(G)$, then $u_1 z \in E(G)$ and $u_1^+ \vec{C} z \subseteq N(u_1)$, $z \vec{C} w_1 \subseteq N(u_2)$. similarly if there exists $z \in v_2 \vec{C} v_1$, such that $u_2 z \in E(G)$, then $u_1 z \in E(G)$ and $u_2^+ \vec{C} z \subseteq N(u_2)$, $z \vec{C} w_2 \subseteq N(u_1)$.

Proof of Theorem 5

Now we are going to complete the proof of Theorem 5. We choose x as in Claim 9. By Claim 13, we know that $k = 2$.

First we prove that there exists $u \in V(C)$ such that $u_1, u_2 \in N(u)$. If there is not any $u \in V(C) \setminus \{v_2, w_1, u_2^+\}$ such that $u_2 u \notin E(G)$, then $w_1^-u_1 \in E(G)$ (if not, $w_1^{-2}u_2 \in E(G)$ by

partition of $V(C)$). If $u_1w_1 \notin E(G)$ then $u_2w_1^- \in E(G)$, so we have $u_1, u_2 \in N(w_1^-)$; if there is $u \in V(C)$, such that $u_2u \in E(G)$ then, by Claim 16, $u_1u \in E(G)$, hence $u_1, u_2 \in N(u)$.

By Claim 16, clearly, there are not $z \in u_1\vec{C}w_1, y \in u_2\vec{C}w_2$, such that $yz \in E(G)$.

So we have $G \in \mathcal{J}_1$. The proof of Theorem 5 is finished.

Acknowledgment

We were helped in completing this paper by conversations with Prof.F.Tian and L.Zhang.

References

- [1] J.A. Bondy and U.S.R. Murty, *Graph Theory with Applications*, Macmillan, London and Elsevier, New York(1976).
- [2] H.J. Broersma and H.J. Veldman, *Long dominating cycles and paths in graphs with large neighborhood unions*, Journal of Graph Theory 15(1991), 29-38.
- [3] L.F.Mao, *Smarandache multi-space theory*, Hexis, Phoenix, AZ2006.
- [4] R.Q.Shen and F.Tian, *Long dominating cycles in graphs*, *Discrete Mathematics*, Vol.177(1997), 287-294.
- [5] F. Tian and L.Z.Zhang *Long dominating cycle in a kind of 2-connected graphs*, Systems Science and Mathematical Science 8(1995), 66-74.
- [6] Z.R.Sun, *Long dominating cycle in graphs* (Submitted).