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A Note on 4-Ordered Hamiltonicity of Cayley Graphs

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Abstract: A hamiltonian graph G of order n is k-ordered for an integer $k, 2 \leq k \leq n$ if for every sequence $(v_1, v_2, ..., v_k)$ of k distinct vertices of G, there exists a hamiltonian cycle that encounters $(v_1, v_2, ..., v_k)$ in order. For any integer $k \geq 1$, let $G = \mathbb{Z}_{3k-1}$ denote the additive group of integers modulo $3k-1$ and C the subset of $\mathbb{Z}_{3k-1}$ consisting of these elements congruent to 1 modulo 3. Denote by $And(k)$ the Cayley graph $Cay(G : C)$. In this note, we show that $And(k)$ is a 4-ordered hamiltonian graph.

Keywords: Cayley graph, k-ordered, hamiltonicity.

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§1. Introduction

All groups and graphs considered in this paper are finite. For any integers $n \geq 3$ and $k, 2 \leq k \leq n$, a hamiltonian graph G of order n is k-ordered if for every sequence $(v_1, v_2, ..., v_k)$ of k distinct vertices of G, there exists a hamiltonian cycle that encounters $(v_1, v_2, ..., v_k)$ in order. Let $G = \mathbb{Z}_{3k-1}$ denote the additive group of integers modulo $3k-1$ with $k \geq 1$ and C the subset of $\mathbb{Z}_{3k-1}$ consisting of these elements congruent to 1 modulo 3. We denote the Cayley graph $Cay(G : C)$ by $And(k)$ in this note.

For $\forall v_i, v_j \in V(And(k))$, d$(v_i)$=d$(v_j)$=$k$, $v_i \sim v_j$ if and only if $j - i \equiv \pm 1 (\mathrm{mod}\ 3)$. We have known that the diameter of $And(k)$ is 2 and the subgraph of $And(k)$ induced by $\{0, 1, 2, ..., 3(k-1)-2\}$ is $And(k-1)$ by results in references $[2]-[3]$. Therefore, we can get $And(k-1)$ from $And(k)$ by deleting the path $3k-4 \sim 3k-3 \sim 3k-2$. As it has been shown also in [2], there exist 4-regular, 4-ordered graphs of order n for any integer $n \geq 5$. In this note, we research 4-ordered property of $And(k)$.

§2. Main result and its proof

Theorem *$And(k)$ is a 4-ordered hamiltonian graph.*

Proof We have known that $And(k)$ is a hamiltonian graph. For any $S = (x, u, v, w) \subseteq V[And(k)] = \{0, 1, 2, ..., 3k-2\}$, it is obvious that there is a hamiltonian cycle C that encounters the vertices of S, not loss of generality, we can assume it passing through these vertices in the order (x, u, v, w). By a reverse traversing, we also get a hamiltonian cycle that encounters the

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vertices of S in the order (x, w, v, u). Notice that there are six cyclic orders for (x, u, v, w) as follows:

$$(x, u, v, w), \qquad (x, w, v, u);$$
$$(x, w, u, v), \qquad (x, v, u, w);$$
$$(x, u, w, v), \qquad (x, v, w, u).$$

Here, in each row, one is a reversion of another.

Our proof is divided into following discussions.

Firstly, we show that there is a hamiltonian cycle C that encounters the vertices of S in the order (x, v, u, w).

Case 1 $\quad v - u \equiv 0(\mathrm{mod}3)$

Notice that $v-(u-1) = v-u+1 \equiv 1(\mathrm{mod}3)$, $v \sim (u-1)$, $(v+1)-u = v-u+1 \equiv 1(\mathrm{mod}3)$, $(v+1) \sim u$ in this case. There exists a hamiltonian cycle

$$x = 0, 1, 2, 3, ..., u-1, v, v-1, v-2, ..., u, v+1, v+2, ...w, ..., 3k-2$$

in $And(k)$ encountering vertices of S in the order (x, v, u, w).

Case 2 $\quad v - u \equiv 1(\mathrm{mod}3)$

In this case, $v-(u-3) = v-u+3 \equiv 1(\mathrm{mod}3)$, $v \sim (u-3)$, $(v+1)-(u-2) = v-u+3 \equiv 1(\mathrm{mod}3)$, $(v+1) \sim (u-2)$. We find a hamiltonian cycle

$$x = 0, 1, 2, ..., u-3, v, v-1, ..., u, u-1, u-2, v+1, v+2, ...w, ..., 3k-2$$

in $And(k)$ encountering vertices of S in the order (x, v, u, w).

Case 3 $\quad v - u \equiv 2(\mathrm{mod}3)$

Since $v-(u-2) = v-u+2 \equiv 1(\mathrm{mod}3)$, $v \sim (u-2)$, $(v+1)-(u-1) = v-u+2 \equiv 1(\mathrm{mod}3)$, $(v+1) \sim (u-1)$ in this case. We have a hamiltonian cycle

$$x = 0, 1, 2, ..., u-2, v, v-1, ..., u, u-1, v+1, v+2, ...w, ..., 3k-2$$

in $And(k)$ encountering vertices of S in the order (x, v, u, w).

By traversing this cycle in a reverse direction, there is also a hamiltonian cycle that encounters the vertices of S in the order (x, w, u, v).

Next, we show that there is also a hamiltonian cycle C that encounters the vertices of S in the order (x, u, w, v).

Case 1 $\quad w - v \equiv 0(\mathrm{mod}3)$

Notice that $w-(v-1) = w-v+1 \equiv 1(\mathrm{mod}3)$, $w \sim (v-1)$, $(w+1)-v = w-v+1 \equiv 1(\mathrm{mod}3)$, $(w+1) \sim v$ in this case. We find a hamiltonian cycle

$$x = 0, 1, 2, ..., u, ..., v-1, w, w-1, ..., v, w+1, w+2, ..., 3k-2$$

in $And(k)$ encountering vertices of S in the order (x, u, w, v).

Case 2 $w - v \equiv 1 \mathrm{mod} 3$

In this case, $(w+1)-(v-2) = w-v+3 \equiv 1(\mathrm{mod}3)$, $(w+1) \sim (v-2)$, $(w+2)-(v-1) = w-v+3 \equiv 1(\mathrm{mod}3)$, $(w+2) \sim (v-1)$. There exists a hamiltonian cycle

$$x = 0, 1, 2, ..., u, ..., v-2, w+1, w, w-1, ..., v, v-1, w+2, ..., 3k-2$$

in the graph $And(k)$ encountering vertices of S in the order (x, u, w, v) if $w \neq 3k-2, u \neq v-1$. While $w = 3k-2, u = v-1$, notice that $(3k-2)-v \equiv 1(\mathrm{mod}3)$, $3k-v \equiv 0(\mathrm{mod}3)$ and $v \equiv 0(\mathrm{mod}3)$. So $u+5 = (v-1)+5 = v+4 \equiv 1(\mathrm{mod}3)$, $u+5 \sim 0$. The cycle

$$x = 0, 1, 2, 3, ..., u, u+4, u+3, u+2, u+6, u+7, ..., 3k-2, v(u+1), u+5, 0$$

in $And(k)$ is a hamiltonian cycle encountering vertices of S in the order (x, u, w, v).

Case 3 $w - v \equiv 2(\mathrm{mod}3)$

By assumption, $(w+1)-(v-1) = w-v+2 \equiv 1(\mathrm{mod}3)$, $(w+1) \sim (v-1)$, $(w+2)-v = w-v+2 \equiv 1(\mathrm{mod}3)$, $(w+2) \sim v$. We get a hamiltonian cycle

$$x = 0, 1, 2, ..., u, ..., v-1, w+1, w, w-1, ..., v, w+2, ..., 3k-2$$

in $And(k)$ encountering all vertices of S in the order (x, u, w, v) if $w \neq 3k-2, u \neq v-1$. Now if $w = 3k-2, u = v-1$, $w-v \equiv 2(\mathrm{mod}3)$, notice that $(w-2)-u = (w-2)-(v-1) = w-v-1 \equiv 1(\mathrm{mod}3)$, $(w-2) \sim u$, $w-(u-1) = w-(v-1-1) = w-v+2 \equiv 1(\mathrm{mod}3)$, $w \sim (u-1)$, $(w-3)-0 = (3k-2)-3 = 3k-5 \equiv 1(\mathrm{mod}3)$, $(w-3) \sim 0$, $(3k-2)-(u+1) = 3k-3-u \equiv 2(\mathrm{mod}3)$, $u-0 \equiv 1\mathrm{mod}3)$, $u \sim 0$. There is also a hamiltonian cycle

$$x = 0, u, w-2, w-1, w, u-1, ..., 1, v, v+1, v+2, ..., w-3, 0$$

in $And(k)$ encountering vertices of S in the order (x, u, w, v).

By traversing the cycle in a reverse direction, we also find a hamiltonian cycle that encounters the vertices of S in the order (x, v, w, u).

This completes the proof. □

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