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A Combinatorially Generalized Stokes Theorem on Integrations

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Abstract: As an immediately application of Smarandache multi-spaces, a combinatorial manifold $\widetilde{M}$ with a given integer $m \geq 1$ is defined to be a geometrical object $\widetilde{M}$ such that for $\forall p \in \widetilde{M}$, there is a local chart (U_p, φ_p) enable $\varphi_p : U_p \to B^{n_{i_1}} \bigcup B^{n_{i_2}} \bigcup \cdots \bigcup B^{n_{i_{s(p)}}}$ with $B^{n_{i_1}} \bigcap B^{n_{i_2}} \bigcap \cdots \bigcap B^{n_{i_{s(p)}}} \neq \emptyset$, where $B^{n_{i_j}}$ is an n_{i_j}-ball for integers $1 \leq j \leq s(p) \leq m$. Integral theory on these smoothly combinatorial manifolds are introduced. Some classical results, such as those of *Stokes'* theorem and *Gauss'* theorem are generalized to smoothly combinatorial manifolds. By a relation of smoothly combinatorial manifolds with vertex-edge labeled graphs, counterparts of these conception and results are also established on graphs in this paper.

Key Words: combinatorial manifold, integration, Stokes' theorem, Gauss' theorem, vertex-edge labeled graph.

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§1. Introduction

As a localized Euclidean space, an *n-manifold* M^n is a Hausdorff space M^n, i.e., a space that satisfies the T_2 separation axiom such that for $\forall p \in M^n$, there is an open neighborhood $U_p, p \in U_p \subset M^n$ and a homeomorphism $\varphi_p : U_p \to \mathbf{R}^n$. These manifolds, particularly, differential manifolds are very important to modern geometries and mechanics. As an immediately application of Smarandache multi-spaces ([8]), also the application of the combinatorial speculation for classical mathematics, i.e. *mathematics can be reconstructed from or turned into combinatorialization*([3]), combinatorial manifolds were introduced in [4], which are the generalization of classical manifolds and can be also endowed with a topological or differential structure as geometrical objects.

Now for an integer $s \geq 1$, let $n_1, n_2, \cdots, n_s$ be an integer sequence with $0 < n_1 < n_2 < \cdots < n_s$. Choose s open unit balls $B_1^{n_1}, B_2^{n_2}, \cdots, B_s^{n_s}$, where $\bigcap\limits_{i=1}^{s} B_i^{n_i} \neq \emptyset$ in $\mathbf{R}^{n_1+n_2+\cdots n_s}$. A *unit open combinatorial ball of degree s* is a union

$$\widetilde{B}(n_1, n_2, \cdots, n_s) = \bigcup_{i=1}^{s} B_i^{n_i}.$$

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Then a combinatorial manifold $\widetilde{M}$ is defined in the next.

Definition 1.1 *For a given integer sequence* $n_1, n_2, \cdots, n_m, m \geq 1$ *with* $0 < n_1 < n_2 < \cdots < n_m$*, a combinatorial manifold* $\widetilde{M}$ *is a Hausdorff space such that for any point* $p \in \widetilde{M}$*, there is a local chart* (U_p, φ_p) *of* p*, i.e., an open neighborhood* U_p *of* p *in* $\widetilde{M}$ *and a homoeomorphism* $\varphi_p : U_p \to \widetilde{B}(n_1(p), n_2(p), \cdots, n_{s(p)}(p))$ *with* $\{n_1(p), n_2(p), \cdots, n_{s(p)}(p)\} \subseteq \{n_1, n_2, \cdots, n_m\}$ *and* $\bigcup\limits_{p \in \widetilde{M}} \{n_1(p), n_2(p), \cdots, n_{s(p)}(p)\} = \{n_1, n_2, \cdots, n_m\}$*, denoted by* $\widetilde{M}(n_1, n_2, \cdots, n_m)$ *or* $\widetilde{M}$ *on the context and*

$$\widetilde{\mathcal{A}} = \{(U_p, \varphi_p) | p \in \widetilde{M}(n_1, n_2, \cdots, n_m))\}$$

an atlas on $\widetilde{M}(n_1, n_2, \cdots, n_m)$*. The maximum value of* $s(p)$ *and the dimension* $\widehat{s}(p)$ *of* $\bigcap\limits_{i=1}^{s(p)} B_i^{n_i}$ *are called the dimension and the intersectional dimensional of* $\widetilde{M}(n_1, n_2, \cdots, n_m)$ *at the point* p*, denoted by* $d(p)$ *and* $\widehat{d}(p)$*, respectively.*

A combinatorial manifold $\widetilde{M}$ is called *finite* if it is just combined by finite manifolds without one manifold is contained in the union of others, is called *smooth* if it is finite endowed with a C^∞ differential structure. For a smoothly combinatorial manifold $\widetilde{M}$ and a point $p \in \widetilde{M}$, it has been shown in [4] that $\dim T_p\widetilde{M}(n_1, n_2, \cdots, n_m) = \widehat{s}(p) + \sum\limits_{i=1}^{s(p)} (n_i - \widehat{s}(p))$ and $\dim T_p^*\widetilde{M}(n_1, n_2, \cdots, n_m) = \widehat{s}(p) + \sum\limits_{i=1}^{s(p)} (n_i - \widehat{s}(p))$ with a basis

$$\{\frac{\partial}{\partial x^{hj}}|_p | 1 \leq j \leq \widehat{s}(p)\} \bigcup (\bigcup_{i=1}^{s(p)} \bigcup_{j=\widehat{s}(p)+1}^{n_i} \{\frac{\partial}{\partial x^{ij}}|_p \mid 1 \leq j \leq s\})$$

or

$$\{dx^{hj}|_p|\} 1 \leq j \leq \widehat{s}(p)\} \bigcup (\bigcup_{i=1}^{s(p)} \bigcup_{j=\widehat{s}(p)+1}^{n_i} \{dx^{ij}|_p \mid 1 \leq j \leq s\}$$

for a given integer $h, 1 \leq h \leq s(p)$. Denoted all k-forms of $\widetilde{M}(n_1, n_2, \cdots, n_m)$ by $\Lambda^k(\widetilde{M})$ and $\Lambda(\widetilde{M}) = \bigoplus\limits_{k=0}^{\widehat{s}(p)+\sum_{i=1}^{s(p)}(n_i - \widehat{s}(p))} \Lambda^k(\widetilde{M})$, then there is a unique exterior differentiation $\widetilde{d} : \Lambda(\widetilde{M}) \to \Lambda(\widetilde{M})$ such that for any integer $k \geq 1$, $\widetilde{d}(\Lambda^k) \subset \Lambda^{k+1}(\widetilde{M})$ with conditions following hold similar to the classical tensor analysis([1]).

(i) $\widetilde{d}$ is linear, i.e., for $\forall \varphi, \psi \in \Lambda(\widetilde{M})$, $\lambda \in \mathbf{R}$,

$$\widetilde{d}(\varphi + \lambda\psi) = \widetilde{d}\varphi \wedge \psi + \lambda\widetilde{d}\psi$$

and for $\varphi \in \Lambda^k(\widetilde{M}), \psi \in \Lambda(\widetilde{M})$,

$$\widetilde{d}(\varphi \wedge \psi) = \widetilde{d}\varphi + (-1)^k \varphi \wedge \widetilde{d}\psi.$$

(ii) For $f \in \Lambda^0(\widetilde{M})$, $\widetilde{d}f$ is the differentiation of f.

(*iii*) $\widetilde{d}^2 = \widetilde{d} \cdot \widetilde{d} = 0$.

(*iv*) $\widetilde{d}$ is a local operator, i.e., if $U \subset V \subset \widetilde{M}$ are open sets and $\alpha \in \Lambda^k(V)$, then $\widetilde{d}(\alpha|_U) = (\widetilde{d}\alpha)|_U$.

Therefore, smoothly combinatorial manifolds poss a local structure analogous smoothly manifolds. But notes that this local structure maybe different for neighborhoods of different points. Whence, geometries on combinatorial manifolds are *Smarandache* geometries([6]-[8]).

There are two well-known theorems in classical tensor analysis, i.e., *Stokes'* and *Gauss'* theorems for the integration of differential n-forms on an n-manifold M, which enables us knowing that

$$\int_M d\omega = \int_{\partial M} \omega$$

for a $\omega \in \Lambda^{n-1}(M)$ with compact supports and

$$\int_M (\mathrm{div} X)\mu = \int_{\partial M} \mathbf{i}_X \mu$$

for a vector field X, where $\mathbf{i}_X : \Lambda^{k+1}(M) \rightarrow \Lambda^k(M)$ defined by $\mathbf{i}_X\varpi(X_1, X_2, \cdots, X_k) = \varpi(X, X_1, \cdots, X_k)$ for $\varpi \in \Lambda^{k+1}(M)$. The similar local properties for combinatorial manifolds with manifolds naturally forward the following questions: *wether the Stokes' or Gauss' theorem is still valid on smoothly combinatorial manifolds?* or if invalid, *What are their modified forms for smoothly combinatorial manifolds?*.

The main purpose of this paper is to find the revised Stokes' or Gauss' theorem for combinatorial manifolds, namely, the Stokes' or Gauss' theorem is still valid for $\widetilde{n}$-forms on smoothly combinatorial manifolds $\widetilde{M}$ if $\widetilde{n} \in \mathscr{H}_{\widetilde{M}}(n, m)$, where $\mathscr{H}_{\widetilde{M}}(n, m)$ is an integer set determined by its structure of a given smoothly combinatorial manifold $\widetilde{M}$. For this objective, we first consider a particular case of combinatorial manifolds, i.e., the combinatorial Euclidean spaces in the next section, establish a relation for finitely combinatorial manifolds with vertex-edge labeled graphs and calculate the integer set $\mathscr{H}_{\widetilde{M}}(n, m)$ for a given vertex-edge labeled graph in Section 3, then generalize the definition of integration on manifolds to combinatorial manifolds in Section 4. The generalized form for Stokes' or Gauss' theorem, also their counterparts on graphs can be found in Section 5. Terminologies and notations used in this paper are standard and can be found in [1] − [2] or [4] for those of manifolds and combinatorial manifolds and [6] for graphs, respectively.

§2. Combinatorially Euclidean Spaces

As a simplest case of combinatorial manifolds, we characterize combinatorially Euclidean spaces of finite and generalize some results in Euclidean spaces in this section.

Definition 2.1 *For a given integer sequence* $n_1, n_2, \cdots, n_m, m \geq 1$ *with* $0 < n_1 < n_2 < \cdots < n_m$*, a combinatorially Euclidean space* $\widetilde{\mathbf{R}}(n_1, \cdots, n_m)$ *is a union of finitely Euclidean spaces* $\bigcup_{i=1}^{m} \mathbf{R}^{n_i}$ *such that for* $\forall p \in \widetilde{\mathbf{R}}(n_1, \cdots, n_m)$, $p \in \bigcap_{i=1}^{m} \mathbf{R}^{n_i}$ *with* $\widehat{m} = \dim(\bigcap_{i=1}^{m} \mathbf{R}^{n_i})$ *a constant.*

By definition, we can present a point p of $\widetilde{\mathbf{R}}$ by an $m \times n_m$ coordinate matrix $[\overline{x}]$ following with $x^{il} = \frac{x^l}{m}$ for $1 \leq i \leq m, 1 \leq l \leq \widehat{m}$.

$$[\overline{x}] = \begin{bmatrix} x^{11} & \cdots & x^{1\widehat{m}} & x^{1(\widehat{m})+1)} & \cdots & x^{1n_1} & \cdots & 0 \\ x^{21} & \cdots & x^{2\widehat{m}} & x^{2(\widehat{m}+1)} & \cdots & x^{2n_2} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & \\ x^{m1} & \cdots & x^{m\widehat{m}} & x^{m(\widehat{m}+1)} & \cdots & \cdots & x^{mn_m-1} & x^{mn_m} \end{bmatrix}$$

For making a combinatorially Euclidean space to be a metric space, we introduce *inner product of matrixes* similar to that of vectors in the next.

Definition 2.2 *Let $(A) = (a_{ij})_{m\times n}$ and $(B) = (b_{ij})_{m\times n}$ be two matrixes. The inner product $\langle (A), (B)\rangle$ of (A) and (B) is defined by*

$$\langle (A), (B)\rangle = \sum_{i,j} a_{ij} b_{ij}.$$

Theorem 2.1 *Let $(A), (B), (C)$ be $m \times n$ matrixes and α a constant. Then*

(1) $\langle A, B\rangle = \langle B, A\rangle$;

(2) $\langle A + B, C\rangle = \langle A, C\rangle + \langle B, C\rangle$;

(3) $\langle \alpha A, B\rangle = \alpha \langle B, A\rangle$;

(4) $\langle A, A\rangle \geq 0$ with equality hold if and only if $(A) = O_{m\times n}$.

Proof (1)-(3) can be gotten immediately by definition. Now calculation shows that

$$\langle A, A\rangle = \sum_{i,j} a_{ij}^2 \geq 0$$

and with equality hold if and only if $a_{ij} = 0$ for any integers $i, j, 1 \leq i \leq m, 1 \leq j \leq n$, namely, $(A) = O_{m\times n}$. □

Theorem 2.2 *$(A), (B)$ be $m \times n$ matrixes. Then*

$$\langle (A), (B)\rangle^2 \leq \langle (A), (A)\rangle \langle (B), (B)\rangle$$

and with equality hold only if $(A) = \lambda(B)$, where λ is a real constant.

Proof If $(A) = \lambda(B)$, then $\langle A, B\rangle^2 = \lambda^2 \langle B, B\rangle^2 = \langle A, A\rangle \langle B, B\rangle$. Now if there are no constant λ enabling $(A) = \lambda(B)$, then $(A) - \lambda(B) \neq O_{m\times n}$ for any real number λ. According to Theorem 2.1, we know that

$$\langle (A) - \lambda(B), (A) - \lambda(B)\rangle > 0,$$

i.e.,

$$\langle (A), (A)\rangle - 2\lambda \langle (A), (B)\rangle + \lambda^2 \langle (B), (B)\rangle > 0.$$

Therefore, we find that

$$\Delta = (-2\langle (A),(B)\rangle)^2 - 4\langle (A),(A)\rangle \langle (B),(B)\rangle < 0,$$

namely,

$$\langle (A),(B)\rangle^2 < \langle (A),(A)\rangle \langle (B),(B)\rangle .\square$$

Corollary 2.1 *For given real numbers a_{ij}, b_{ij}, $1 \leq i \leq m, 1 \leq j \leq n$,*

$$(\sum_{i,j} a_{ij}b_{ij})^2 \leq (\sum_{i,j} a_{ij}^2)(\sum_{i,j} b_{ij}^2).$$

Let O be the original point of $\widetilde{\mathbf{R}}(n_1, \cdots, n_m)$. Then $[O] = O_{m\times n_m}$. Now for $\forall p, q \in \widetilde{\mathbf{R}}(n_1, \cdots, n_m)$, we also call $\overrightarrow{Op}$ the vector correspondent to the point p similar to that of classical Euclidean spaces, Then $\overrightarrow{pq} = \overrightarrow{Oq} - \overrightarrow{Op}$. Theorem 2.2 enables us to introduce an angle between two vectors $\overrightarrow{pq}$ and $\overrightarrow{uv}$ for points $p, q, u, v \in \widetilde{\mathbf{R}}(n_1, \cdots, n_m)$.

Definition 2.3 *Let $p, q, u, v \in \widetilde{\mathbf{R}}(n_1, \cdots, n_m)$. Then the angle θ between vectors $\overrightarrow{pq}$ and $\overrightarrow{uv}$ is determined by*

$$\cos\theta = \frac{\langle [p]-[q],[u]-[v]\rangle}{\sqrt{\langle [p]-[q],[p]-[q]\rangle \langle [u]-[v],[u]-[v]\rangle}}$$

under the condition that $0 \leq \theta \leq \pi$.

Corollary 2.2 *The conception of angle between two vectors is well defined.*

Proof Notice that

$$\langle [p]-[q],[u]-[v]\rangle^2 \leq \langle [p]-[q],[p]-[q]\rangle \langle [u]-[v],[u]-[v]\rangle$$

by Theorem 2.2. Thereby, we know that

$$-1 \leq \frac{\langle [p]-[q],[u]-[v]\rangle}{\sqrt{\langle [p]-[q],[p]-[q]\rangle \langle [u]-[v],[u]-[v]\rangle}} \leq 1.$$

Therefore there is a unique angle θ with $0 \leq \theta \leq \pi$ enabling Definition 2.3 hold. $\square$

For two points p, q in $\widetilde{\mathbf{R}}(n_1, \cdots, n_m)$, the *distance* $d(p, q)$ between points p and q is defined to be $\sqrt{\langle [p]-[q],[p]-[q]\rangle}$. We get the following result.

Theorem 2.3 *For a given integer sequence $n_1, n_2, \cdots, n_m, m \geq 1$ with $0 < n_1 < n_2 < \cdots < n_m$, $(\widetilde{\mathbf{R}}(n_1, \cdots, n_m); d)$ is a metric space.*

Proof We need to verify that each condition for a metric space holds in $(\widetilde{\mathbf{R}}(n_1, \cdots, n_m); d)$. For two point $p, q \in \widetilde{\mathbf{R}}(n_1, \cdots, n_m)$, by definition we know that

$$d(p,q) = \sqrt{\langle [p]-[q],[p]-[q]\rangle} \geq 0$$

with equality hold if and only if $[p]=[q]$, namely, $p=q$ and

$$d(p,q)=\sqrt{\langle [p]-[q],[p]-[q]\rangle}=\sqrt{\langle [q]-[p],[q]-[p]\rangle}=d(q,p).$$

Now let $u\in \widetilde{\mathbf{R}}(n_1,\cdots,n_m)$. By Theorem 2.2, we then find that

$$\begin{aligned}&(d(p,u)+d(u,p))^2\\&=\langle [p]-[u],[p]-[u]\rangle+2\sqrt{\langle [p]-[u],[p]-[u]\rangle\,\langle [u]-[q],[u]-[q]\rangle}\\&+\langle [u]-[q],[u]-[q]\rangle\\&\geq \langle [p]-[u],[p]-[u]\rangle+2\,\langle [p]-[u],[u]-[q]\rangle+\langle [u]-[q],[u]-[q]\rangle\\&=\langle [p]-[q],[p]-[q]\rangle=d^2(p,q).\end{aligned}$$

Whence, $d(p,u)+d(u,p)\geq d(p,q)$ and $(\widetilde{\mathbf{R}}(n_1,\cdots,n_m);d)$ is a metric space. □

By previous discussions, a combinatorially Euclidean space $\widetilde{R}(n_1,n_2,\cdots,n_m)$ can be turned to an Euclidean space $\mathbf{R}^n$ with $n=\widehat{m}+\sum\limits_{i=1}^{m}(n_i-\widehat{m})$. It is the same the other way round, namely we can also decompose an Euclidean space into a combinatorially Euclidean space.

Theorem 2.4 *Let $\mathbf{R}^n$ be an Euclidean space and $n_1,n_2,\cdots,n_m$ integers with $\widehat{m}<n_i<n$ for $1\leq i\leq m$ and the equation*

$$\widehat{m}+\sum_{i=1}^{m}(n_i-\widehat{m})=n$$

hold for an integer $\widehat{m}, 1\leq \widehat{m}\leq n$. Then there is a combinatorially Euclidean space $\widetilde{\mathbf{R}}(n_1,n_2,\cdots,n_m)$ such that

$$\mathbf{R}^n\cong \widetilde{\mathbf{R}}(n_1,n_2,\cdots,n_m).$$

Proof Not loss of generality, assume the coordinate system of $\mathbf{R}^n$ is $(x_1,x_2,\cdots,x_n)$ with a basis $\{\mathbf{e}_1,\mathbf{e}_2,\cdots,\mathbf{e}_n\}$. Since

$$n-\widehat{m}=\sum_{i=1}^{m}(n_i-\widehat{m}),$$

Choose

$$\mathbf{R}_1=\langle \mathbf{e}_1,\mathbf{e}_2,\cdots,\mathbf{e}_{\widehat{m}},\mathbf{e}_{\widehat{m}+1},\cdots,\mathbf{e}_{n_1}\rangle\,;$$

$$\mathbf{R}_2=\langle \mathbf{e}_1,\mathbf{e}_2,\cdots,\mathbf{e}_{\widehat{m}},\mathbf{e}_{n_1+1},\mathbf{e}_{n_1+2},\cdots,\mathbf{e}_{n_2}\rangle\,;$$

$$\mathbf{R}_3=\langle \mathbf{e}_1,\mathbf{e}_2,\cdots,\mathbf{e}_{\widehat{m}},\mathbf{e}_{n_2+1},\mathbf{e}_{n_2+2},\cdots,\mathbf{e}_{n_3}\rangle\,;$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\,;$$

$$\mathbf{R}_m = \left\langle \mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_{\widehat{m}}, \mathbf{e}_{n_{m-1}+1}, \mathbf{e}_{n_{m-1}+2}, \cdots, \mathbf{e}_{n_m} \right\rangle.$$

Calculation shows $\dim\mathbf{R}_i = n_i$ and $\dim(\bigcap_{i=1}^{m} \mathbf{R}_i) = \widehat{m}$. Whence $\widetilde{\mathbf{R}}(n_1, n_2, \cdots, n_m)$ is a combinatorially Euclidean space. By Definitions $2.1-2.2$ and Theorems $2.1-2.3$, we then get that

$$\mathbf{R}^n \cong \widetilde{\mathbf{R}}(n_1, n_2, \cdots, n_m). \qquad \square$$

§3. Determining $\mathscr{H}_{\widetilde{M}}(n,m)$

Let $\widetilde{M}(n_1, \cdots, n_m)$ be a smoothly combinatorial manifold. Then there exists an atlas $\mathscr{C} = \{(\widetilde{U}_\alpha, [\varphi_\alpha])|\alpha \in \widetilde{I}\}$ on $\widetilde{M}(n_1, \cdots, n_m)$ consisting of positively oriented charts such that for $\forall \alpha \in \widetilde{I}$, $\widehat{s}(p) + \sum_{i=1}^{s(p)} (n_i - \widehat{s}(p))$ is an constant $n_{\widetilde{U}_\alpha}$ for $\forall p \in \widetilde{U}_\alpha$ ([4]). The integer set $\mathscr{H}_{\widetilde{M}}(n,m)$ is then defined by

$$\mathscr{H}_{\widetilde{M}}(n,m) = \{n_{\widetilde{U}_\alpha} | \alpha \in \widetilde{I}\}.$$

Notice that $\widetilde{M}(n_1, \cdots, n_m)$ is smoothly. We know that $\mathscr{H}_{\widetilde{M}}(n,m)$ is finite. This set is important to the definition of integral and the establishing of Stokes' or Gauss' theorems on smoothly combinatorial manifolds. We characterize it by a combinatorial manner in this section.

A *vertex-edge labeled graph* $G([1,k],[1,l])$ is a connected graph $G = (V, E)$ with two mappings

$$\tau_1 : V \to \{1, 2, \cdots, k\},$$

$$\tau_2 : E \to \{1, 2, \cdots, l\}$$

for integers k and l. For example, two vertex-edge labeled graphs with an underlying graph K_4 are shown in Fig.3.1.

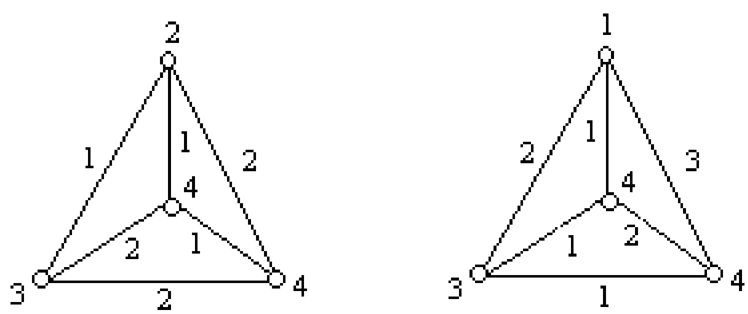

Fig.3.1

For a combinatorial finite manifold $\widetilde{M}(n_1, n_2, \cdots, n_m)$ with $1 \le n_1 < n_2 < \cdots < n_m, m \ge 1$, there is a natural $1-1$ mapping $\theta : \widetilde{M}(n_1, n_2, \cdots, n_m) \to G([0, n_m], [0, n_m])$ determined in the following. Define

$$V(G([0,n_m],[0,n_m])) = V_1 \bigcup V_2,$$

where $V_1 = \{n_i -$ manifolds M^{n_i} in $\widetilde{M}(n_1,n_2,\cdots,n_m)|1 \leq i \leq m\}$ and $V_2 = \{$isolated inters−ection points $O_{M^{n_i},M^{n_j}}$ ofM^{n_i}, M^{n_j} in $\widetilde{M}(n_1,n_2,\cdots,n_m)$ for $1 \leq i,j \leq m\}$, and label each n_i-manifold M^{n_i} in V_1 or O in V_2 by $\tau_1(M^{n_i}) = n_i$, $\tau_1(O) = 0$. Choose

$$E(G([0,n_m],[0,n_m])) = E_1 \bigcup E_2,$$

where $E_1 = \{(M^{n_i}, M^{n_j})|\dim(M^{n_i} \bigcap M^{n_j}) \geq 1, 1 \leq i,j \leq m\}$ and $E_2 = \{(O_{M^{n_i},M^{n_j}}, M^{n_i}),$ $(O_{M^{n_i},M^{n_j}}, M^{n_j})|M^{n_i}$ tangent M^{n_j} at the point $O_{M^{n_i},M^{n_j}}$ for $1 \leq i,j \leq m\}$, and for an edge $(M^{n_i}, M^{n_j}) \in E_1$ or $(O_{M^{n_i},M^{n_j}}, M^{n_i}) \in E_2$, label it by $\tau_2(M^{n_i}, M^{n_j}) = \dim(M^{n_i} \bigcap M^{n_j})$ or 0, respectively. This construction then enables us getting a $1-1$ mapping $\theta : \widetilde{M}(n_1,n_2,\cdots,n_m) \rightarrow G([0,n_m],[0,n_m])$.

Now let $\mathcal{H}(n_1,n_2,\cdots,n_m)$ denote all finitely combinatorial manifolds $\widetilde{M}(n_1,n_2,\cdots,n_m)$ and let $\mathcal{G}[0,n_m]$ denote all vertex-edge labeled graphs $G([0,n_m],[0,n_m])$ with conditions following hold.

(1) Each induced subgraph by vertices labeled with 1 in G is a union of complete graphs and vertices labeled with 0 can only be adjacent to vertices labeled with 1.

(2) For each edge $e = (u,v) \in E(G)$, $\tau_2(e) \leq \min\{\tau_1(u), \tau_1(v)\}$.

Then we know a relation between sets $\mathcal{H}(n_1,n_2,\cdots,n_m)$ and $\mathcal{G}([0,n_m],[0,n_m])$.

Theorem 3.1 *Let $1 \leq n_1 < n_2 < \cdots < n_m, m \geq 1$ be a given integer sequence. Then every finitely combinatorial manifold $\widetilde{M} \in \mathcal{H}(n_1,n_2,\cdots,n_m)$ defines a vertex-edge labeled graph $G([0,n_m],[0,n_m]) \in \mathcal{G}[0,n_m]$. Conversely, every vertex-edge labeled graph $G([0,n_m],[0,n_m]) \in \mathcal{G}[0,n_m]$ defines a finitely combinatorial manifold $\widetilde{M} \in \mathcal{H}(n_1,n_2,\cdots,n_m)$ with a $1-1$ mapping $\theta : G([0,n_m],[0,n_m]) \rightarrow \widetilde{M}$ such that $\theta(u)$ is a $\theta(u)$-manifold in $\widetilde{M}$, $\tau_1(u) = \dim\theta(u)$ and $\tau_2(v,w) = \dim(\theta(v) \bigcap \theta(w))$ for $\forall u \in V(G([0,n_m],[0,n_m]))$ and $\forall (v,w) \in E(G([0,n_m],[0,n_m]))$.*

Proof By definition, for $\forall \widetilde{M} \in \mathcal{H}(n_1,n_2,\cdots,n_m)$ there is a vertex-edge labeled graph $G([0,n_m],[0,n_m]) \in \mathcal{G}([0,n_m],[0,n_m])$ and a $1-1$ mapping $\theta : \widetilde{M} \rightarrow G([0,n_m],[0,n_m])$ such that $\theta(u)$ is a $\theta(u)$-manifold in $\widetilde{M}$. For completing the proof, we need to construct a finitely combinatorial manifold $\widetilde{M} \in \mathcal{H}(n_1,n_2,\cdots,n_m)$ for $\forall G([0,n_m],[0,n_m]) \in \mathcal{G}[0,n_m]$ with $\tau_1(u) = \dim\theta(u)$ and $\tau_2(v,w) = \dim(\theta(v) \bigcap \theta(w))$ for $\forall u \in V(G([0,n_m],[0,n_m]))$ and $\forall (v,w) \in E(G([0,n_m],[0,n_m]))$. The construction is carried out by programming following.

STEP 1. Choose $|G([0,n_m],[0,n_m])| - |V_0|$ manifolds correspondent to each vertex u with a dimensional n_i if $\tau_1(u) = n_i$, where $V_0 = \{u|u \in V(G([0,n_m],[0,n_m]))$ and $\tau_1(u) = 0\}$. Denoted by $V_{\geq 1}$ all these vertices in $G([0,n_m],[0,n_m])$ with label≥ 1.

STEP 2. For $\forall u_1 \in V_{\geq 1}$ with $\tau_1(u_1) = n_{i_1}$, if its neighborhood set $N_{G([0,n_m],[0,n_m])}(u_1) \bigcap V_{\geq 1} = \{v_1^1, v_1^2, \cdots, v_1^{s(u_1)}\}$ with $\tau_1(v_1^1) = n_{11}$, $\tau_1(v_1^2) = n_{12}$, $\cdots$, $\tau_1(v_1^{s(u_1)}) = n_{1s(u_1)}$, then let the manifold correspondent to the vertex u_1 with an intersection dimension $\tau_2(u_1v_1^i)$ with manifold correspondent to the vertex v_1^i for $1 \leq i \leq s(u_1)$ and define a vertex set $\Delta_1 = \{u_1\}$.

STEP 3. If the vertex set $\Delta_l = \{u_1, u_2, \cdots, u_l\} \subseteq V_{\geq 1}$ has been defined and $V_{\geq 1} \setminus \Delta_l \neq \emptyset$, let $u_{l+1} \in V_{\geq 1} \setminus \Delta_l$ with a label $n_{i_{l+1}}$. Assume

$$(N_{G([0,n_m],[0,n_m])}(u_{l+1}) \bigcap V_{\geq 1}) \setminus \Delta_l = \{v_{l+1}^1, v_{l+1}^2, \cdots, v_{l+1}^{s(u_{l+1})}\}$$

with $\tau_1(v_{l+1}^1) = n_{l+1,1}$, $\tau_1(v_{l+1}^2) = n_{l+1,2}$, $\cdots$,$\tau_1(v_{l+1}^{s(u_{l+1})}) = n_{l+1,s(u_{l+1})}$. Then let the manifold correspondent to the vertex u_{l+1} with an intersection dimension $\tau_2(u_{l+1}v_{l+1}^i)$ with the manifold correspondent to the vertex $v_{l+1}^i, 1 \leq i \leq s(u_{l+1})$ and define a vertex set $\Delta_{l+1} = \Delta_l \bigcup \{u_{l+1}\}$.

STEP 4. Repeat steps 2 and 3 until a vertex set $\Delta_t = V_{\geq 1}$ has been constructed. This construction is ended if there are no vertices $w \in V(G)$ with $\tau_1(w) = 0$, i.e., $V_{\geq 1} = V(G)$. Otherwise, go to the next step.

STEP 5. For $\forall w \in V(G([0,n_m],[0,n_m])) \setminus V_{\geq 1}$, assume $N_{G([0,n_m],[0,n_m])}(w) = \{w_1, w_2, \cdots, w_e\}$. Let all these manifolds correspondent to vertices $w_1, w_2, \cdots, w_e$ intersects at one point simultaneously and define a vertex set $\Delta_{t+1}^* = \Delta_t \bigcup \{w\}$.

STEP 6. Repeat STEP 5 for vertices in $V(G([0,n_m],[0,n_m])) \setminus V_{\geq 1}$. This construction is finally ended until a vertex set $\Delta_{t+h}^* = V(G[n_1, n_2, \cdots, n_m])$ has been constructed.

A finitely combinatorial manifold $\widetilde{M}$ correspondent to $G([0,n_m],[0,n_m])$ is gotten when Δ_{t+h}^* has been constructed. By this construction, it is easily verified that $\widetilde{M} \in \mathcal{H}(n_1, n_2, \cdots, n_m)$ with $\tau_1(u) = \dim\theta(u)$ and $\tau_2(v,w) = \dim(\theta(v) \bigcap \theta(w))$ for $\forall u \in V(G([0,n_m],[0,n_m]))$ and $\forall (v,w) \in E(G([0,n_m],[0,n_m]))$. This completes the proof. □

Now we determine the integer set $\mathscr{H}_{\widetilde{M}}(n,m)$ for a given smoothly combinatorial manifold $\widetilde{M}(n_1, n_2, \cdots, n_m)$. Notice the relation between sets $\mathcal{H}(n_1, n_2, \cdots, n_m)$ and $\mathcal{G}([0,n_m],[0,n_m])$ established in Theorem 2.4. We can determine it under its vertex-edge labeled graph $G([0,n_m],$ $[0,n_m])$.

Theorem 3.2 *Let $\widetilde{M}$ be a smoothly combinatorial manifold with a correspondent vertex-edge labeled graph $G([0,n_m],[0,n_m])$. Then*

$$\mathscr{H}_{\widetilde{M}}(n,m) \subseteq \{n_1, n_2, \cdots, n_m\} \bigcup_{\widehat{d}(p) \geq 3, p \in \widetilde{M}} \{\widehat{d}(p) + \sum_{i=1}^{d(p)} (n_i - \widehat{d}(p))\}$$
$$\bigcup \{\tau_1(u) + \tau_1(v) - \tau_2(u,v) | \forall (u,v) \in E(G([0,n_m],[0,n_m]))\}.$$

Particularly, if $G([0,n_m],[0,n_m])$ is K_3-free, then

$$\mathscr{H}_{\widetilde{M}}(n,m) = \{\tau_1(u) | u \in V(G([0,n_m],[0,n_m]))\}$$
$$\bigcup \{\tau_1(u) + \tau_1(v) - \tau_2(u,v) | \forall (u,v) \in E(G([0,n_m],[0,n_m]))\}.$$

Proof Notice that the dimension of a point $p \in \widetilde{M}$ is

$$n_p = \widehat{d}(p) + \sum_{i=1}^{d(p)} (n_i - \widehat{d}(p))$$

by definition. If $d(p) = 1$, then $n_p = n_j, 1 \leq j \leq m$. If $d(p) = 2$, namely, $p \in M^{n_i} \cap M^{n_j}$ for $1 \leq i, j \leq m$, we know that its dimension is

$$n_i + n_j - \widehat{d}(p) = \tau_1(M^{n_i}) + \tau_1(M^{n_j}) - \widehat{d}(p).$$

Whence, we get that

$$\begin{aligned}\mathscr{H}_{\widetilde{M}}(n,m) \subseteq \quad & \{n_1, n_2, \cdots, n_m\} \bigcup_{\widehat{d}(p) \geq 3, p \in \widetilde{M}} \{\widehat{d}(p) + \sum_{i=1}^{d(p)} (n_i - \widehat{d}(p))\} \\ & \bigcup \{\tau_1(u) + \tau_1(v) - \tau_2(u,v) | \forall (u,v) \in E(G([0,n_m],[0,n_m]))\}.\end{aligned}$$

Now if $G([0,n_m],[0,n_m])$ is K_3-free, then there are no points with intersectional dimension$\geq$ 3. In this case, there are really existing points $p \in M^{n_i}$ for any integer $i, 1 \leq i \leq m$ and $q \in M^{n_i} \cap M^{n_j}$ for $1 \leq i, j \leq m$ by definition. Therefore, we get that

$$\begin{aligned}\mathscr{H}_{\widetilde{M}}(n,m) = \quad & \{\tau_1(u) | u \in V(G([0,n_m],[0,n_m]))\} \\ & \bigcup \{\tau_1(u) + \tau_1(v) - \tau_2(u,v) | \forall (u,v) \in E(G([0,n_m],[0,n_m]))\}. \square\end{aligned}$$

For some special graphs, we get the following interesting results for the integer set $\mathscr{H}_{\widetilde{M}}(n,m)$.

Corollary 3.1 *Let $\widetilde{M}$ be a smoothly combinatorial manifold with a correspondent vertex-edge labeled graph $G([0,n_m],[0,n_m])$. If $G([0,n_m],[0,n_m]) \cong P^s$, then*

$$\mathscr{H}_{\widetilde{M}}(n,m) = \{\tau_1(u_i), 1 \leq i \leq p\} \bigcup \{\tau_1(u_i) + \tau_1(u_{i+1}) - \tau_2(u_i, u_{i+1}) | 1 \leq i \leq p-1\}$$

and if $G([0,n_m],[0,n_m]) \cong C^p$ with $p \geq 4$, then

$$\mathscr{H}_{\widetilde{M}}(n,m) = \{\tau_1(u_i), 1 \leq i \leq p\} \bigcup \{\tau_1(u_i) + \tau_1(u_{i+1}) - \tau_2(u_i, u_{i+1}) | 1 \leq i \leq p, i \equiv (modp)\}.$$

§4. Integration on combinatorial manifolds

We generalize the integration on manifolds to combinatorial manifolds and show it is independent on the choice of local charts and partition of unity in this section.

4.1 Partition of unity

Definition 4.1 *Let $\widetilde{M}$ be a smoothly combinatorial manifold and $\omega \in \Lambda(\widetilde{M})$. A support set* Supp$\omega$ *of ω is defined by*

$$\mathrm{Supp}\omega = \overline{\{p \in \widetilde{M}; \omega(p) \neq 0\}}$$

and say ω has compact support if Suppω *is compact in $\widetilde{M}$. A collection of subsets $\{C_i | i \in \widetilde{I}\}$ of $\widetilde{M}$ is called locally finite if for each $p \in \widetilde{M}$, there is a neighborhood U_p of p such that $U_p \cap C_i = \emptyset$ except for finitely many indices i.*

A *partition of unity* on a combinatorial manifold $\widetilde{M}$ is defined in the next.

Definition 4.2 *A partition of unity on a combinatorial manifold $\widetilde{M}$ is a collection $\{(U_i, g_i) | i \in \widetilde{I}\}$, where*

(1) $\{U_i | i \in \widetilde{I}\}$ is a locally finite open covering of $\widetilde{M}$;
(2) $g_i \in \mathscr{X}(\widetilde{M})$, $g_i(p) \geq 0$ for $\forall p \in \widetilde{M}$ and supp$g_i \in U_i$ for $i \in \widetilde{I}$;
(3) For $p \in \widetilde{M}$, $\sum\limits_i g_i(p) = 1$.

For a smoothly combinatorial manifold $\widetilde{M}$, denoted by $G[\widetilde{M}]$ the underlying graph of its correspondent vertex-edge labeled graph. We get the next result for a partition of unity on smoothly combinatorial manifolds.

Theorem 4.1 *Let $\widetilde{M}$ be a smoothly combinatorial manifold. Then $\widetilde{M}$ admits partitions of unity.*

Proof For $\forall M \in V(G[\widetilde{M}])$, since $\widetilde{M}$ is smooth we know that M is a smoothly submanifold of $\widetilde{M}$. As a byproduct, there is a partition of unity $\{(U_M^\alpha, g_M^\alpha) | \alpha \in I_M\}$ on M with conditions following hold.

(1) $\{U_M^\alpha | \alpha \in I_M\}$ is a locally finite open covering of M;

(2) $g_M^\alpha(p) \geq 0$ for $\forall p \in M$ and supp$g_M^\alpha \in U_M^\alpha$ for $\alpha \in I_M$;

(3) For $p \in M$, $\sum\limits_i g_M^i(p) = 1$.

By definition, for $\forall p \in \widetilde{M}$, there is a local chart $(U_p, [\varphi_p])$ enable $\varphi_p : U_p \to B^{n_{i_1}} \bigcup B^{n_{i_2}} \bigcup \cdots \bigcup B^{n_{i_{s(p)}}}$ with $B^{n_{i_1}} \bigcap B^{n_{i_2}} \bigcap \cdots \bigcap B^{n_{i_{s(p)}}} \neq \emptyset$. Now let $U_{M_{i_1}}^\alpha$, $U_{M_{i_2}}^\alpha$, $\cdots$, $U_{M_{i_{s(p)}}}^\alpha$ be $s(p)$ open sets on manifolds $M, M \in V(G[\widetilde{M}])$ such that

$$p \in U_p^\alpha = \bigcup_{h=1}^{s(p)} U_{M_{i_h}}^\alpha. \quad (4.1)$$

We define

$$\widetilde{S}(p) = \{U_p^\alpha | \text{ all integers } \alpha \text{ enabling (4.1) hold}\}.$$

Then

$$\widetilde{\mathcal{A}} = \bigcup_{p \in \widetilde{M}} \widetilde{S}(p) = \{U_p^\alpha | \alpha \in \widetilde{I}(p)\}$$

is locally finite covering of the combinatorial manifold $\widetilde{M}$ by properties (1) − (3). For $\forall U_p^\alpha \in \widetilde{S}(p)$, define

$$\sigma_{U_p^\alpha} = \sum_{s\geq 1} \sum_{\{i_1,i_2,\cdots,i_s\}\subset\{1,2,\cdots,s(p)\}} (\prod_{h=1}^{s} g_{M_{i_h}^s})$$

and

$$g_{U_p^\alpha} = \frac{\sigma_{U_p^\alpha}}{\sum\limits_{\widetilde{V}\in\widetilde{S}(p)} \sigma_{\widetilde{V}}}.$$

Then it can be checked immediately that $\{(U_p^\alpha, g_{U_p^\alpha})|p \in \widetilde{M}, \alpha \in \widetilde{I}(p)\}$ is a partition of unity on $\widetilde{M}$ by properties (1)-(3) on g_M^α and the definition of $g_{U_p^\alpha}$. □

Corollary 4.1 *Let $\widetilde{M}$ be a smoothly combinatorial manifold with an atlas $\widetilde{\mathcal{A}} = \{(V_\alpha, [\varphi_\alpha])|\alpha \in \widetilde{I}\}$ and t_α be a C^k tensor field, $k \geq 1$, of field type (r,s) defined on V_α for each α, and assume that there exists a partition of unity $\{(U_i, g_i)|i \in J\}$ subordinate to $\widetilde{\mathcal{A}}$, i.e., for $\forall i \in J$, there exists $\alpha(i)$ such that $U_i \subset V_{\alpha(i)}$. Then for $\forall p \in \widetilde{M}$,*

$$t(p) = \sum_i g_i t_{\alpha(i)}$$

is a C^k tensor field of type (r,s) on $\widetilde{M}$

Proof Since $\{U_i|i \in J\}$ is locally finite, the sum at each point p is a finite sum and $t(p)$ is a type (r,s) for every $p \in \widetilde{M}$. Notice that t is C^k since the local form of t in a local chart $(V_{\alpha(i)}, [\varphi_{\alpha(i)}])$ is

$$\sum_j g_i t_{\alpha(j)},$$

where the summation taken over all indices j such that $V_{\alpha(i)} \bigcap V_{\alpha(j)} \neq \emptyset$. Those number j is finite by the local finiteness. □

4.2 **Integration on combinatorial manifolds**

First, we introduce integration on combinatorial Euclidean spaces. Let $\widetilde{\mathbf{R}}(n_1,\cdots,n_m)$ be a combinatorially Euclidean space and

$$\tau : \widetilde{\mathbf{R}}(n_1,\cdots,n_m) \to \widetilde{\mathbf{R}}(n_1,\cdots,n_m)$$

a C^1 differential mapping with

$$[\overline{y}] = [y^{\kappa\lambda}]_{m\times n_m} = [\tau^{\kappa\lambda}([x^{\mu\nu}])]_{m\times n_m}.$$

The *Jacobi matrix* of f is defined by

$$\frac{\partial[\overline{y}]}{\partial[\overline{x}]} = [A_{(\kappa\lambda)(\mu\nu)}],$$

where $A_{(\kappa\lambda)(\mu\nu)} = \frac{\partial \tau^{\kappa\lambda}}{\partial x^{\mu\nu}}$.

Now let $\omega \in T_k^0(\widetilde{\mathbf{R}}(n_1, \cdots, n_m))$, a *pull-back* $\tau^*\omega \in T_k^0(\widetilde{\mathbf{R}}(n_1, \cdots, n_m))$ is defined by

$$\tau^*\omega(a_1, a_2, \cdots, a_k) = \omega(f(a_1), f(a_2), \cdots, f(a_k))$$

for $\forall a_1, a_2, \cdots, a_k \in \widetilde{R}$.

Denoted by $n = \widehat{m} + \sum_{i=1}^{m}(n_i - \widehat{m})$. If $0 \leq l \leq n$, recall([4]) that the basis of $\Lambda^l(\widetilde{\mathbf{R}}(n_1, \cdots, n_m))$ is

$$\{\mathbf{e}^{i_1} \wedge \mathbf{e}^{i_2} \wedge \cdots \wedge \mathbf{e}^{i_l} | 1 \leq i_1 < i_2 \cdots < i_l \leq n\}$$

for a basis $\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n$ of $\widetilde{\mathbf{R}}(n_1, \cdots, n_m)$ and its dual basis $\mathbf{e}^1, \mathbf{e}^2, \cdots, \mathbf{e}^n$. Thereby the dimension of $\Lambda^l(\widetilde{\mathbf{R}}(n_1, \cdots, n_m))$ is

$$\begin{pmatrix} n \\ l \end{pmatrix} = \frac{(\widehat{m} + \sum\limits_{i=1}^{m}(n_i - \widehat{m}))!}{l!(\widehat{m} + \sum\limits_{i=1}^{m}(n_i - \widehat{m}) - l)!}.$$

Whence $\Lambda^n(\widetilde{\mathbf{R}}(n_1, \cdots, n_m))$ is one-dimensional. Now if ω_0 is a basis of $\Lambda^n(\widetilde{R})$, we then know that its each element ω can be represented by $\omega = c\omega_0$ for a number $c \in \mathbf{R}$. Let $\tau : \widetilde{\mathbf{R}}(n_1, \cdots, n_m) \to \widetilde{\mathbf{R}}(n_1, \cdots, n_m)$ be a linear mapping. Then

$$\tau^* : \Lambda^n(\widetilde{\mathbf{R}}(n_1, \cdots, n_m)) \to \Lambda^n(\widetilde{\mathbf{R}}(n_1, \cdots, n_m))$$

is also a linear mapping with $\tau^*\omega = c\tau^*\omega_0 = b\omega$ for a unique constant $b = \det\tau$, called the determinant of τ. It has been known that ([1])

$$\det\tau = \det(\frac{\partial[\overline{y}]}{\partial[\overline{x}]})$$

for a given basis $\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n$ of $\widetilde{\mathbf{R}}(n_1, \cdots, n_m)$ and its dual basis $\mathbf{e}^1, \mathbf{e}^2, \cdots, \mathbf{e}^n$.

Definition 4.3 *Let $\widetilde{\mathbf{R}}(n_1, n_2, \cdots, n_m)$ be a combinatorial Euclidean space,$n = \widehat{m} + \sum_{i=1}^{m}(n_i - \widehat{m})$, $\widetilde{U} \subset \widetilde{\mathbf{R}}(n_1, n_2, \cdots, n_m)$ and $\omega \in \Lambda^n(U)$ have compact support with*

$$\omega(x) = \omega_{(\mu_{i_1}\nu_{i_1})\cdots(\mu_{i_n}\nu_{i_n})} dx^{\mu_{i_1}\nu_{i_1}} \wedge \cdots \wedge dx^{\mu_{i_n}\nu_{i_n}}$$

relative to the standard basis $\mathbf{e}^{\mu\nu}, 1 \leq \mu \leq m, 1 \leq \nu \leq n_m$ of $\widetilde{\mathbf{R}}(n_1, n_2, \cdots, n_m)$ with $\mathbf{e}^{\mu\nu} = e^{\nu}$ for $1 \leq \mu \leq \widehat{m}$. An integral of ω on $\widetilde{U}$ is defined to be a mapping $\int_{\widetilde{U}} : f \to \int_{\widetilde{U}} f \in \mathbf{R}$ with

$$\int_{\widetilde{U}} \omega = \int \omega(x) \prod_{\nu=1}^{\widehat{m}} dx^{\nu} \prod_{\mu \geq \widehat{m}+1, 1 \leq \nu \leq n_i} dx^{\mu\nu}, \qquad (4.2)$$

where the right hand side of (4.2) *is the Riemannian integral of ω on $\widetilde{U}$.*

For example, consider the combinatorial Euclidean space $\widetilde{\mathbf{R}}(3,5)$ with $\mathbf{R}^3 \cap \mathbf{R}^5 = \mathbf{R}$. Then the integration of an $\omega \in \Lambda^7(\widetilde{U})$ for an open subset $\widetilde{U} \in \widetilde{\mathbf{R}}(3,5)$ is

$$\int_{\widetilde{U}} \omega = \int_{\widetilde{U}\cap(\mathbf{R}^3\cup\mathbf{R}^5)} \omega(x)dx^1dx^{12}dx^{13}dx^{22}dx^{23}dx^{24}dx^{25}.$$

Theorem 4.2 *Let U and V be open subsets of $\widetilde{\mathbf{R}}(n_1,\cdots,n_m)$ and $\tau: U\to V$ is an orientation-preserving diffeomorphism. If $\omega\in\Lambda^n(V)$ has a compact support for $n=\widehat{m}+\sum\limits_{i=1}^{m}(n_i-\widehat{m})$, then $\tau^*\omega\in\Lambda^n(U)$ has compact support and*

$$\int\tau^*\omega=\int\omega.$$

Proof Let $\omega(x)=\omega_{(\mu_{i_1}\nu_{i_1})\cdots(\mu_{i_n}\nu_{i_n})}dx^{\mu_{i_1}\nu_{i_1}}\wedge\cdots\wedge dx^{\mu_{i_n}\nu_{i_n}}\in\Lambda^n(V)$. Since τ is a diffeomorphism, the support of $\tau^*\omega$ is $\tau^{-1}(\mathrm{supp}\omega)$, which is compact by that of suppω compact.

By the usual change of variables formula, since $\tau^*\omega=(\omega\circ\tau)(\det\tau)\omega_0$ by definition, where $\omega_0=dx^1\wedge\cdots\wedge dx^{\widehat{m}}\wedge dx^{1(\widehat{m}+1)}\wedge dx^{1(\widehat{m}+2)}\wedge\cdots\wedge dx^{1n_1}\wedge\cdots\wedge dx^{mn_m}$, we then get that

$$\begin{aligned}\int\tau^*\omega &= \int(\omega\circ\tau)(\det\tau)\prod_{\nu=1}^{\widehat{m}}dx^{\nu}\prod_{\mu\geq\widehat{m}+1,1\leq\nu\leq n_\mu}dx^{\mu\nu}\\ &= \int\omega.\end{aligned}$$

□

Definition 4.4 *Let $\widetilde{M}$ be a smoothly combinatorial manifold. If there exists a family $\{(U_\alpha,[\varphi_\alpha]|\alpha\in\widetilde{I})\}$ of local charts such that*

(1) $\bigcup\limits_{\alpha\in\widetilde{I}}U_\alpha=\widetilde{M}$;

(2) *for $\forall\alpha,\beta\in\widetilde{I}$, either $U_\alpha\bigcap U_\beta=\emptyset$ or $U_\alpha\bigcap U_\beta\neq\emptyset$ but for $\forall p\in U_\alpha\bigcap U_\beta$, the Jacobi matrix*

$$det(\frac{\partial[\varphi_\beta]}{\partial[\varphi_\alpha]})>0,$$

then $\widetilde{M}$ is called an oriently combinatorial manifold and $(U_\alpha,[\varphi_\alpha])$ an oriented chart for $\forall\alpha\in\widetilde{I}$.

Now for any integer $\widetilde{n}\in\mathscr{H}_{\widetilde{M}}(n,m)$, we can define an integral of $\widetilde{n}$-forms on a smoothly combinatorial manifold $\widetilde{M}(n_1,\cdots,n_m)$.

Definition 4.5 *Let $\widetilde{M}$ be a smoothly combinatorial manifold with orientation $\mathscr{O}$ and $(\widetilde{U};[\varphi])$ a positively oriented chart with a constant $n_{\widetilde{U}}\in\mathscr{H}_{\widetilde{M}}(n,m)$. Suppose $\omega\in\Lambda^{n_{\widetilde{U}}}(\widetilde{M}),\widetilde{U}\subset\widetilde{M}$ has compact support $\widetilde{C}\subset\widetilde{U}$. Then define*

$$\int_{\widetilde{C}}\omega=\int\varphi_*(\omega|_{\widetilde{U}}).\qquad(4.3)$$

Now if $\mathscr{C}_{\widetilde{M}}$ is an atlas of positively oriented charts with an integer set $\mathscr{H}_{\widetilde{M}}(n,m)$, let $\widetilde{P}=\{(\widetilde{U}_\alpha,\varphi_\alpha,g_\alpha)|\alpha\in\widetilde{I}\}$ be a partition of unity subordinate to $\mathscr{C}_{\widetilde{M}}$. For $\forall\omega\in\Lambda^{\widetilde{n}}(\widetilde{M})$, $\widetilde{n}\in\mathscr{H}_{\widetilde{M}}(n,m)$, an integral of ω on $\widetilde{P}$ is defined by

$$\int_{\widetilde{P}} \omega = \sum_{\alpha \in \widetilde{I}} \int g_\alpha \omega. \qquad (4.4)$$

The next result shows that the integral of $\widetilde{n}$-forms for $\forall \widetilde{n} \in \mathscr{H}_{\widetilde{M}}(n,m)$ is well-defined.

Theorem 4.3 *Let $\widetilde{M}(n_1, \cdots, n_m)$ be a smoothly combinatorial manifold. For $\widetilde{n} \in \mathscr{H}_{\widetilde{M}}(n,m)$, the integral of $\widetilde{n}$-forms on $\widetilde{M}(n_1, \cdots, n_m)$ is well-defined, namely, the sum on the right hand side of* (4.4) *contains only a finite number of nonzero terms, not dependent on the choice of $\mathscr{C}_{\widetilde{M}}$ and if P and Q are two partitions of unity subordinate to $\mathscr{C}_{\widetilde{M}}$, then*

$$\int_{\widetilde{P}} \omega = \int_{\widetilde{Q}} \omega.$$

Proof By definition for any point $p \in \widetilde{M}(n_1, \cdots, n_m)$, there is a neighborhood $\widetilde{U}_p$ such that only a finite number of g_α are nonzero on $\widetilde{U}_p$. Now by the compactness of suppω, only a finite number of such neighborhood cover suppω. Therefore, only a finite number of g_α are nonzero on the union of these $\widetilde{U}_p$, namely, the sum on the right hand side of (4.4) contains only a finite number of nonzero terms.

Notice that the integral of $\widetilde{n}$-forms on a smoothly combinatorial manifold $\widetilde{M}(n_1, \cdots, n_m)$ is well-defined for a local chart $\widetilde{U}$ with a constant $n_{\widetilde{U}} = \widehat{s}(p) + \sum\limits_{i=1}^{s(p)} (n_i - \widehat{s}(p))$ for $\forall p \in \widetilde{U} \subset \widetilde{M}(n_1, \cdots, n_m)$ by (4.3) and Definition 4.3. Whence each term on the right hand side of (4.4) is well-defined. Thereby $\int_{\widetilde{P}} \omega$ is well-defined.

Now let $\widetilde{P} = \{(\widetilde{U}_\alpha, \varphi_\alpha, g_\alpha) | \alpha \in \widetilde{I}\}$ and $\widetilde{Q} = \{(\widetilde{V}_\beta, \varphi_\beta, h_\beta) | \beta \in \widetilde{J}\}$ be partitions of unity subordinate to atlas $\mathscr{C}_{\widetilde{M}}$ and $\mathscr{C}^*_{\widetilde{M}}$ with respective integer sets $\mathscr{H}_{\widetilde{M}}(n,m)$ and $\mathscr{H}^*_{\widetilde{M}}(n,m)$. Then these functions $\{g_\alpha h_\beta\}$ satisfy $g_\alpha h_\beta(p) = 0$ except only for a finite number of index pairs (α, β) and

$$\sum_\alpha \sum_\beta g_\alpha h_\beta(p) = 1, \quad \text{for } \forall p \in \widetilde{M}(n_1, \cdots, n_m).$$

Since $\sum\limits_\beta = 1$, we then get that

$$\begin{aligned} \int_{\widetilde{P}} &= \sum_\alpha \int g_\alpha \omega = \sum_\beta \sum_\alpha \int h_\beta g_\alpha \omega \\ &= \sum_\alpha \sum_\beta \int g_\alpha h_\beta \omega = \int_{\widetilde{Q}} \omega. \end{aligned}$$

□

By the relation of smoothly combinatorial manifolds with these vertex-edge labeled graphs established in Theorem 3.1, we can also get the integration on a vertex-edge labeled graph $G([0,n_m],[0,n_m])$ by viewing it that of the correspondent smoothly combinatorial manifold $\widetilde{M}$ with $\Lambda^l(G) = \Lambda^l(\widetilde{M})$, $\mathscr{H}_G(n,m) = \mathscr{H}_{\widetilde{M}}(n,m)$, namely define the *integral of an $\widetilde{n}$-form ω on $G([0,n_m],[0,n_m])$ for $\widetilde{n} \in \mathscr{H}_G(n,m)$* by

$$\int_{G([0,n_m],[0,n_m])}\omega=\int_{\widetilde{M}}\omega.$$

Then each result in this paper can be restated by combinatorial words, such as Theorem 5.1 and its corollaries in next section.

Now let $n_1, n_2, \cdots, n_m$ be a positive integer sequence. For any point $p \in \widetilde{M}$, if there is a local chart $(\widetilde{U}_p, [\varphi_p])$ such that $[\varphi_p] : U_p \to B^{n_1} \bigcup B^{n_2} \bigcup \cdots \bigcup B^{n_m}$ with $\dim(B^{n_1} \bigcap B^{n_2} \bigcap \cdots \bigcap B^{n_m}) = \widehat{m}$, then $\widetilde{M}$ is called a *homogenously combinatorial manifold*. Particularly, if $m = 1$, a homogenously combinatorial manifold is nothing but a manifold. We then get consequences for the integral of $(\widehat{m} + \sum\limits_{i=1}^{m}(n_i - \widehat{m}))$-forms on homogenously combinatorial manifolds.

Corollary 4.2 *The integral of $(\widehat{m} + \sum\limits_{i=1}^{m}(n_i - \widehat{m}))$-forms on a homogenously combinatorial manifold $\widetilde{M}(n_1, n_2, \cdots, n_m)$ is well-defined, particularly, the integral of n-forms on an n-manifold is well-defined.*

Similar to Theorem 4.2 for the *change of variables formula of integral* in a combinatorial Euclidean space, we get that of formula in smoothly combinatorial manifolds.

Theorem 4.4 *Let $\widetilde{M}(n_1, n_2, \cdots, n_m)$ and $\widetilde{N}(k_1, k_2, \cdots, k_l)$ be oriently combinatorial manifolds and $\tau : \widetilde{M} \to \widetilde{N}$ an orientation-preserving diffeomorphism. If $\omega \in \Lambda^{\widetilde{k}}(\widetilde{N})$, $\widetilde{k} \in \mathscr{H}_{\widetilde{N}}(k,l)$ has compact support, then $\tau^*\omega$ has compact support and*

$$\int \omega = \int \tau^*\omega.$$

Proof Notice that $\mathrm{supp}\tau^*\omega = \tau^{-1}(\mathrm{supp}\omega)$. Thereby $\tau^*\omega$ has compact support since ω has so. Now let $\{(U_i, \varphi_i)|i \in \widetilde{I}\}$ be an atlas of positively oriented charts of $\widetilde{M}$ and $\widetilde{P} = \{g_i|i \in \widetilde{I}\}$ a subordinate partition of unity with an integer set $\mathscr{H}_{\widetilde{M}}(n,m)$. Then $\{(\tau(U_i), \varphi_i \circ \tau^{-1})|i \in \widetilde{I}\}$ is an atlas of positively oriented charts of $\widetilde{N}$ and $\widetilde{Q} = \{g_i \circ \tau^{-1}\}$ is a partition of unity subordinate to the covering $\{\tau(U_i)|i \in \widetilde{I}\}$ with an integer set $\mathscr{H}_{\tau(\widetilde{M})}(k,l)$. Whence, we get that

$$\begin{aligned}\int \tau^*\omega &= \sum_i \int g_i\tau^*\omega = \sum_i \int \varphi_{i*}(g_i\tau^*\omega)\\ &= \sum_i \int \varphi_{i*}(\tau^{-1})_*(g_i \circ \tau^{-1})\omega = \sum_i \int (\varphi_i \circ \tau^{-1})_*(g_i \circ \tau^{-1})\omega = \int \omega.\end{aligned}$$

□

§5. A generalized of Stokes' or Gauss' theorem

Definition 5.1 *Let $\widetilde{M}$ be a smoothly combinatorial manifold. A subset $\widetilde{D}$ of $\widetilde{M}$ is with boundary if its points can be classified into two classes following.*

Class 1(**interior point Int**$\widetilde{D}$) *For* $\forall p \in \mathrm{Int}D$*, there is a neighborhood* $\widetilde{V}_p$ *of* p *enable* $\widetilde{V}_p \subset \widetilde{D}$.

Case 2(**boundary** $\partial\widetilde{D}$) *For* $\forall p \in \partial\widetilde{D}$*, there is integers* μ, ν *for a local chart* $(U_p; [\varphi_p])$ *of* p *such that* $x^{\mu\nu}(p) = 0$ *but*

$$\widetilde{U}_p \cap \widetilde{D} = \{q | q \in U_p, x^{\kappa\lambda} \geq 0 \ for \ \forall \{\kappa, \lambda\} \neq \{\mu, \nu\}\}.$$

Then we generalize the famous Stokes' theorem on manifolds in the next.

Theorem 5.1 *Let* $\widetilde{M}$ *be a smoothly combinatorial manifold with an integer set* $\mathscr{H}_{\widetilde{M}}(n, m)$ *and* $\widetilde{D}$ *a boundary subset of* $\widetilde{M}$*. For* $\forall \widetilde{n} \in \mathscr{H}_{\widetilde{M}}(n, m)$ *if* $\omega \in \Lambda^{\widetilde{n}}(\widetilde{M})$ *has a compact support, then*

$$\int_{\widetilde{D}} \widetilde{d}\omega = \int_{\partial\widetilde{D}} \omega$$

with the convention $\int_{\partial\widetilde{D}} \omega = 0$ *while* $\partial\widetilde{D} = \emptyset$.

Proof By Definition 4.5, the integration on a smoothly combinatorial manifold was constructed with partitions of unity subordinate to an atlas. Let $\mathscr{C}_{\widetilde{M}}$ be an atlas of positively oriented charts with an integer set $\mathscr{H}_{\widetilde{M}}(n, m)$ and $\widetilde{P} = \{(\widetilde{U}_\alpha, \varphi_\alpha, g_\alpha) | \alpha \in \widetilde{I}\}$ a partition of unity subordinate to $\mathscr{C}_{\widetilde{M}}$. Since supp$\omega$ is compact, we know that

$$\int_{\widetilde{D}} \widetilde{d}\omega = \sum_{\alpha \in \widetilde{I}} \int_{\widetilde{D}} \widetilde{d}(g_\alpha \omega),$$

$$\int_{\partial\widetilde{D}} \omega = \sum_{\alpha \in \widetilde{I}} \int_{\partial\widetilde{D}} g_\alpha \omega.$$

and there are only finite nonzero terms on the right hand side of the above two formulae. Thereby, we only need to prove

$$\int_{\widetilde{D}} \widetilde{d}(g_\alpha \omega) = \int_{\partial\widetilde{D}} g_\alpha \omega$$

for $\forall \alpha \in \widetilde{I}$.

Not loss of generality we can assume that ω is an $\widetilde{n}$-forms on a local chart $(\widetilde{U}, [\varphi])$ with a compact support for $\widetilde{n} \in \mathscr{H}_{\widetilde{M}}(n, m)$. Now write

$$\omega = \sum_{h=1}^{\widetilde{n}} (-1)^{h-1} \omega_{\mu_{i_h} \nu_{i_h}} dx^{\mu_{i_1} \nu_{i_1}} \wedge \cdots \wedge \widehat{dx^{\mu_{i_h} \nu_{i_h}}} \wedge \cdots \wedge dx^{\mu_{i_{\widetilde{n}}} \nu_{i_{\widetilde{n}}}},$$

where $\widehat{dx^{\mu_{i_h} \nu_{i_h}}}$ means that $dx^{\mu_{i_h} \nu_{i_h}}$ is deleted, where

$$i_h \in \{1, \cdots, \widehat{n}_U, (1(\widehat{n}_U + 1)), \cdots, (1n_1), (2(\widehat{n}_U + 1)), \cdots, (2n_2), \cdots, (mn_m)\}.$$

Then

$$\widetilde{d}\omega=\sum_{h=1}^{\widetilde{n}}\frac{\partial\omega_{\mu_{i_h}\nu_{i_h}}}{\partial x^{\mu_{i_h}\nu_{i_h}}}dx^{\mu_{i_1}\nu_{i_1}}\wedge\cdots\wedge dx^{\mu_{i_{\widetilde{n}}}\nu_{i_{\widetilde{n}}}}. \quad (5.1)$$

Consider the appearance of neighborhood $\widetilde{U}$. There are two cases must be considered.

Case 1 $\widetilde{U}\bigcap\partial\widetilde{D}=\emptyset$

In this case, $\int_{\partial\widetilde{D}}\omega=0$ and $\widetilde{U}$ is in $\widetilde{M}\setminus\widetilde{D}$ or in $\mathbf{Int}\widetilde{D}$. The former is naturally implies that $\int_{\widetilde{D}}\widetilde{d}(g_\alpha\omega)=0$. For the later, we find that

$$\int_{\widetilde{D}}\widetilde{d}\omega=\sum_{h=1}^{\widetilde{n}}\int_{\widetilde{U}}\frac{\partial\omega_{\mu_{i_h}\nu_{i_h}}}{\partial x^{\mu_{i_h}\nu_{i_h}}}dx^{\mu_{i_1}\nu_{i_1}}\cdots dx^{\mu_{i_{\widetilde{n}}}\nu_{i_{\widetilde{n}}}}. \quad (5.2)$$

Notice that $\int_{-\infty}^{+\infty}\frac{\partial\omega_{\mu_{i_h}\nu_{i_h}}}{\partial x^{\mu_{i_h}\nu_{i_h}}}dx^{\mu_{i_h}\nu_{i_h}}=0$ since $\omega_{\mu_{i_h}\nu_{i_h}}$ has compact support. Thus $\int_{\widetilde{D}}\widetilde{d}\omega=0$ as desired.

Case 2 $\widetilde{U}\bigcap\partial\widetilde{D}\neq\emptyset$

In this case we can do the same trick for each term except the last. Without loss of generality, assume that

$$\widetilde{U}\bigcap\widetilde{D}=\{q|q\in U, x^{\mu_{i_{\widetilde{n}}}\nu_{i_{\widetilde{n}}}}(q)\geq 0\}$$

and

$$\widetilde{U}\bigcap\partial\widetilde{D}=\{q|q\in U, x^{\mu_{i_{\widetilde{n}}}\nu_{i_{\widetilde{n}}}}(q)=0\}.$$

Then we get that

$$\begin{aligned}\int_{\partial\widetilde{D}}\omega &= \int_{U\cap\partial\widetilde{D}}\omega\\ &= \sum_{h=1}^{\widetilde{n}}(-1)^{h-1}\int_{U\cap\partial\widetilde{D}}\omega_{\mu_{i_h}\nu_{i_h}}dx^{\mu_{i_1}\nu_{i_1}}\wedge\cdots\wedge\widehat{dx^{\mu_{i_h}\nu_{i_h}}}\wedge\cdots\wedge dx^{\mu_{i_{\widetilde{n}}}\nu_{i_{\widetilde{n}}}}\\ &= (-1)^{\widetilde{n}-1}\int_{U\cap\partial\widetilde{D}}\omega_{\mu_{i_{\widetilde{n}}}\nu_{i_{\widetilde{n}}}}dx^{\mu_{i_1}\nu_{i_1}}\wedge\cdots\wedge dx^{\mu_{i_{\widetilde{n}-1}}\nu_{i_{\widetilde{n}-1}}}\end{aligned}$$

since $dx^{\mu_{i_n}\nu_{i_{\widetilde{n}}}}(q)=0$ for $q\in\widetilde{U}\cap\partial\widetilde{D}$. Notice that $\mathbf{R}^{\widetilde{n}-1}=\partial\mathbf{R}_+^{\widetilde{n}}$ but the usual orientation on $\mathbf{R}^{\widetilde{n}-1}$ is not the boundary orientation, whose outward unit normal is $-\mathbf{e}_{\widetilde{n}}=(0,\cdots,0,-1)$. Hence

$$\int_{\partial\widetilde{D}}\omega=-\int_{\partial\mathbf{R}_+^{\widetilde{n}}}\omega_{\mu_{i_{\widetilde{n}}}\nu_{i_{\widetilde{n}}}}(x^{\mu_{i_1}\nu_{i_1}},\cdots,x^{\mu_{i_{\widetilde{n}-1}}\nu_{i_{\widetilde{n}-1}}},0)dx^{\mu_{i_1}\nu_{i_1}}\cdots dx^{\mu_{i_{\widetilde{n}-1}}\nu_{i_{\widetilde{n}-1}}}.$$

On the other hand, by the fundamental theorem of calculus,

$$\int_{\mathbf{R}^{\tilde{n}-1}}(\int_0^\infty \frac{\partial\omega_{\mu_{i_{\tilde{n}}}\nu_{i_{\tilde{n}}}}}{\partial x^{\mu_{i_{\tilde{n}}}\nu_{i_{\tilde{n}}}}})dx^{\mu_{i_1}\nu_{i_1}}\cdots dx^{\mu_{i_{\tilde{n}-1}}\nu_{i_{\tilde{n}-1}}}$$
$$= -\int_{\mathbf{R}^{\tilde{n}-1}}\omega_{\mu_{i_{\tilde{n}}}\nu_{i_{\tilde{n}}}}(x^{\mu_{i_1}\nu_{i_1}},\cdots,x^{\mu_{i_{\tilde{n}-1}}\nu_{i_{\tilde{n}-1}}},0)dx^{\mu_{i_1}\nu_{i_1}}\cdots dx^{\mu_{i_{n-1}}\nu_{i_{n-1}}}.$$

Since $\omega_{\mu_{i_{\tilde{n}}}\nu_{i_{\tilde{n}}}}$ has a compact support, thus

$$\int_U \omega = -\int_{\mathbf{R}^{\tilde{n}-1}}\omega_{\mu_{i_{\tilde{n}}}\nu_{i_{\tilde{n}}}}(x^{\mu_{i_1}\nu_{i_1}},\cdots,x^{\mu_{i_{\tilde{n}-1}}\nu_{i_{\tilde{n}-1}}},0)dx^{\mu_{i_1}\nu_{i_1}}\cdots dx^{\mu_{i_{\tilde{n}-1}}\nu_{i_{\tilde{n}-1}}}.$$

Therefore, we get that

$$\int_{\widetilde{D}}\tilde{d}\omega = \int_{\partial\widetilde{D}}\omega$$

This completes the proof. □

Corollaries following are immediately obtained by Theorem 5.1

Corollary 5.1 *Let $\widetilde{M}$ be a homogenously combinatorial manifold with an integer set $\mathscr{H}_{\widetilde{M}}(n,m)$ and $\widetilde{D}$ a boundary subset of $\widetilde{M}$. For $\widetilde{n}\in\mathscr{H}_{\widetilde{M}}(n,m)$ if $\omega\in\Lambda^{\widetilde{n}}(\widetilde{M})$ has a compact support, then*

$$\int_{\widetilde{D}}\tilde{d}\omega = \int_{\partial\widetilde{D}}\omega,$$

particularly, if $\widetilde{M}$ is nothing but a manifold, the Stokes' theorem holds.

Corollary 5.2 *Let $\widetilde{M}$ be a smoothly combinatorial manifold with an integer set $\mathscr{H}_{\widetilde{M}}(n,m)$. For $\widetilde{n}\in\mathscr{H}_{\widetilde{M}}(n,m)$, if $\omega\in\Lambda^{\widetilde{n}}(\widetilde{M})$ has a compact support, then*

$$\int_{\widetilde{M}}\omega = 0.$$

By the definition of integration on vertex-edge labeled graphs $G([0,n_m],[0,n_m])$, let a boundary subset of $G([0,n_m],[0,n_m])$ mean that of its correspondent combinatorial manifold $\widetilde{M}$. Theorem 5.1 and Corollary 5.2 then can be restated by a combinatorial manner as follows.

Theorem 5.2 *Let $G([0,n_m],[0,n_m])$ be a vertex-edge labeled graph correspondent with an integer set $\mathscr{H}_G(n,m)$ and $\widetilde{D}$ a boundary subset of $G([0,n_m],[0,n_m])$. For $\forall\widetilde{n}\in\mathscr{H}_G(n,m)$ if $\omega\in\Lambda^{\widetilde{n}}(G([0,n_m],[0,n_m]))$ has a compact support, then*

$$\int_{\widetilde{D}}\tilde{d}\omega = \int_{\partial\widetilde{D}}\omega$$

with the convention $\int_{\partial\widetilde{D}}\omega = 0$ while $\partial\widetilde{D}=\emptyset$.

Corollary 5.3 *Let $G([0,n_m],[0,n_m])$ be a vertex-edge labeled graph correspondent with an integer set $\mathscr{H}_G(n,m)$. For $\forall\widetilde{n}\in\mathscr{H}_G(n,m)$ if $\omega\in\Lambda^{\widetilde{n}}(G([0,n_m],[0,n_m]))$ has a compact support, then*

$$\int_{G([0,n_m][0,n_m])} \omega = 0.$$

Similar to the case of manifolds, we find a generalization for Gauss' theorem on smoothly combinatorial manifolds in the next.

Theorem 5.3 *Let $\widetilde{M}$ be a smoothly combinatorial manifold with an integer set $\mathscr{H}_{\widetilde{M}}(n,m)$, $\widetilde{D}$ a boundary subset of $\widetilde{M}$ and $\mathbf{X}$ a vector field on $\widetilde{M}$ with a compact support. Then*

$$\int_{\widetilde{D}} (\mathrm{div}\mathbf{X})\mathbf{v} = \int_{\partial\widetilde{D}} \mathbf{i}_{\mathbf{X}}\mathbf{v},$$

where $\mathbf{v}$ is a volume form on $\widetilde{M}$, i.e., nonzero elements in $\Lambda^{\widetilde{n}}(\widetilde{M})$ for $\widetilde{n} \in \mathscr{H}_{\widetilde{M}}(n,m)$.

Proof This result is also a consequence of Theorem 5.1. Notice that

$$(\mathrm{div}\mathbf{X})\mathbf{v} = \widetilde{d}\mathbf{i}_{\mathbf{X}}\mathbf{v} + \mathbf{i}_{\mathbf{X}}\widetilde{d}\mathbf{v} = \widetilde{d}\mathbf{i}_{\mathbf{X}}\mathbf{v}.$$

According to Theorem 5.1, we then get that

$$\int_{\widetilde{D}} (\mathrm{div}\mathbf{X})\mathbf{v} = \int_{\partial\widetilde{D}} \mathbf{i}_{\mathbf{X}}\mathbf{v}.$$

□

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