Analytical Solution of the Simplest Entropiece Inversion Problem

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Introduction

In advanced information fusion systems for decision-making under uncertainty, information to deal with are generally imprecise, incomplete and conflicting.

An interesting mathematical framework to model imprecision, uncertainty and conflict between sources of evidences is the mathematical theory of belief functions.

We present a new generalized entropy measure to characterize the uncertainty of any belief function (or mass of belief), and we explain the entropic contribution of each piece of information provided by a source of evidence.

We present for the first time the entropy inversion problem (EIP) on how to estimate the mass of belief of a source given the entropiece values, and show the great difficulty to solve it.

Motivation Adjust/modify/simplify entropieces in some advanced information fusion systems, and reestimate belief masses that are eventually shared on an information network.

Note: At the current stage of this research, this theoretical work is not related to any problem for the natural world, and cannot yet be confirmed experimentally.

Basics of Belief Functions (BF) [Shafer 1976]

Frame of discernment (FoD) $\Theta = \{\theta_i, i = 1, \ldots, n\}$ **Power-set** $2^\Theta \triangleq \{X | X \subseteq \Theta\}$

Basic belief assignment $m(\cdot) : 2^\Theta \mapsto [0,1]$ s.t. $\begin{cases} m(\emptyset) = 0, \\ \sum_{X \in 2^\Theta} m(X) = 1 \end{cases}$

Focal element: X is a focal element of m, iff $m(X) > 0$

Vacuous BBA : $m_v(\Theta) = 1$ and $m_v(A) = 0, \ \forall A \neq \Theta$ (Ignorant source)

Belief: $$Bel(X) = \sum_{Y \in 2^\Theta | Y \subseteq X} m(Y)$$ (Degree of support of A)

Plausibility: $$Pl(X) = \sum_{Y \in 2^\Theta | X \cap Y \neq \emptyset} m(Y) = 1 - \mathrm{Bel}(\bar{X})$$ (Degree of non contradiction of A)

Lower and upper bounds : $0 \leq Bel(X) \leq P(X) \leq Pl(X) \leq 1$

Bayesian BBA: if all focal elements of m(.) are singletons of the power-set
For Bayesian BBA, one has Bel(X)=P(X)=Pl(X)

Uncertainty about x: $$u(X) \triangleq Pl(X) - Bel(X)$$

Entropy measure U(m) of BBA

In belief function (BF) framework, we work with unknown pmf P(.) such that

$$0 \leq \underbrace{Bel(X)}_{\text{belief of } X} \leq P(X) \leq \underbrace{Pl(X)}_{\text{plausibility of } X} \leq 1$$

Generalized entropy measure [Dezert 2022] (Fusion 2022 conf.)

$$\boxed{U(m) = \sum_{X \in 2^\Theta} s(X)}$$

(= sum of all entropieces) expressed in nats (or in bits by dividing U(m) by log(2)=0.6931

where s(X) is the « entropiece » of X defined by

$$s(X) \triangleq \underbrace{-(1-u(X))m(X)\log(m(X))}_{\text{discounted surprisal of X}} \underbrace{+ u(X)(1-m(X))}_{\text{discounted imprecision of P(X)}}$$

discounted surprisal of X

discounted imprecision of P(X)

with $u(X) \triangleq Pl(X) - Bel(X)$

U(m) is quite simple, continuous, monotone and it fits with Shannon entropy if the BBA is bayesian. U(m) responds to change of $|\Theta|$.

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Entropiece vector

Entropy $U(m) = \sum_{X \in 2^\Theta} s(X)$ **Max entropy value:** $U(m_v) = 2^{|\Theta|} - 2$

Entropiece $s(X) = -(1 - u(X))m^\Theta(X)\log(m^\Theta(X)) + u(X)(1 - m^\Theta(X))$

Entropiece vector $\mathbf{s}(m) = [s(X), X \in 2^\Theta]^T$ (stack of entropieces values)

Example We take the FoD $\Theta = \{\theta_1, \theta_2\}$

Non-vacuous BBA

$$m^\Theta(\theta_1) = 0.5, m^\Theta(\theta_2) = 0.3, m^\Theta(\theta_1 \cup \theta_2) = 0.2$$

$$\mathbf{s}(m^\Theta) = \begin{bmatrix} s(\emptyset) \\ s(\theta_1) \\ s(\theta_2) \\ s(\theta_1 \cup \theta_2) \end{bmatrix} \approx \begin{bmatrix} 0 \\ 0.377258 \\ 0.428953 \\ 0.321887 \end{bmatrix}$$

$U(m^\Theta) \approx 1.128098$ nats

Vacuous BBA

$$m_v^\Theta(\Theta) = 1$$

$$\mathbf{s}(m_v^\Theta) = \begin{bmatrix} s(\emptyset) \\ s(\theta_1) \\ s(\theta_2) \\ s(\theta_1 \cup \theta_2) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

$U(m_v^\Theta) = 2^{|\Theta|} - 2 = 2$ nats

← entropiece vectors →

One sees that $U(m^\Theta) < U(m_v^\Theta)$ which is what is expected.

General Entropiece Inversion Problem (EIP)

From BBA to entropy

$$\underbrace{m^{\Theta}(\cdot)}_{\text{BBA vector}} \Rightarrow \underbrace{\mathbf{s}(m^{\Theta})}_{\text{Entropiece vector}} \Rightarrow \underbrace{U(m)}_{\text{Entropy}}$$

General Entropiece Inversion Problem (EIP)

Given (or knowing) an entropic vector **s**(m) is it possible to retrieve uniquely the basic belief assignment m? How?

$$\underbrace{\mathbf{s}(m^{\Theta})}_{\text{Entropiece vector}} \overset{?}{\Rightarrow} \underbrace{m^{\Theta}(\cdot)}_{\text{BBA vector}}$$

This EIP problem is very challenging because of nonlinear system of equations to solve. Even in the simplest case with $|\Theta| = 2$, the solution involves transcendental Lambert's W-function. See next slides.

Simplest Entropiece Inversion Problem (SEIP)

We provide the exact solution of simplest EIP for FoD $\Theta = \{A, B\}$

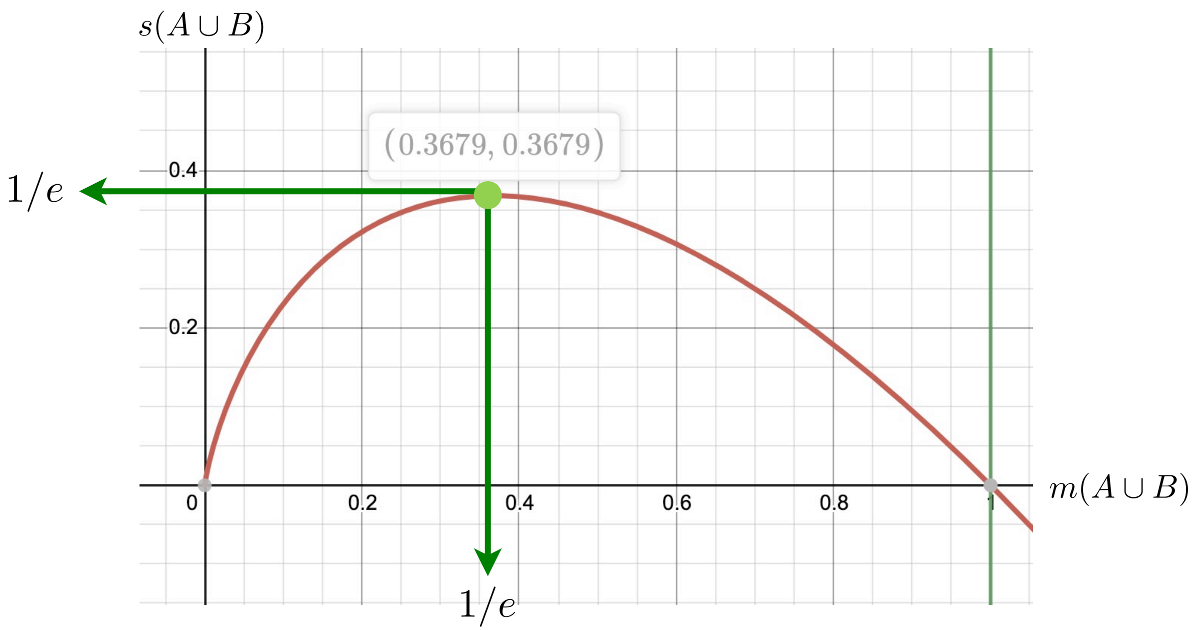
$$\underbrace{\mathbf{s}(m^\Theta) = \begin{bmatrix} s(\emptyset) = 0 \\ s(A) \\ s(B) \\ s(A \cup B) \end{bmatrix}}_{\text{given information}} \overset{?}{\Rightarrow} \underbrace{m^\Theta(\cdot) = \begin{bmatrix} m^\Theta(\emptyset) = 0 \\ m^\Theta(A) \\ m^\Theta(B) \\ m^\Theta(A \cup B) \end{bmatrix}}_{\text{BBA to calculate}}$$

Based on entropiece definition, we have the following nonlinear system to solve

$$\begin{aligned} s(A) &= -m(A)(1 - m(A \cup B)) \log(m(A)) + m(A \cup B)(1 - m(A)) \\ s(B) &= -m(B)(1 - m(A \cup B)) \log(m(B)) + m(A \cup B)(1 - m(B)) \\ s(A \cup B) &= -m(A \cup B) \log(m(A \cup B)) \end{aligned}$$

Simplest Entropiece Inversion Problem (SEIP)

Plot of $s(A \cup B) = -m(A \cup B) \log(m(A \cup B))$



By derivating the function $-m(A \cup B) \log(m(A \cup B))$, we see that max value is obtained for $m(A \cup B) = 1/e \approx 0.3679$ for which

$$s^{\max}(A \cup B) = -\frac{1}{e}\log(1/e) = \frac{1}{e}\log(e) = \frac{1}{e} \longrightarrow s(A \cup B) \in [0, 1/e]$$

Simplest Entropiece Inversion Problem (SEIP)

Without loss of generality, we can assume $0 < s(A \cup B) \leq 1/e$.

Because if $s(A \cup B) = 0$ then one deduces $m(A \cup B) = 1$ (which means that the BBA $m(\cdot)$ is the vacuous BBA) if $s(A) = s(B) = 1$, or $m(A \cup B) = 0$ otherwise.

With the assumption $0 < s(A \cup B) \leq 1/e$, the equation
$-\log(m(A \cup B))m(A \cup B) = s(A \cup B)$ is of transcendental form

$$ye^y = a \Leftrightarrow \underbrace{\log(m(A \cup B))}_{y}\,\underbrace{m(A \cup B)}_{e^y} = \underbrace{-s(A \cup B)}_{a}$$

Solution of eq. $ye^y = a$ has no explicit form involving simple functions.

In mathematics, the solution(s) of this equation is denoted as *y=W(a),*
where *W(.)* is the multivalued Lambert function (1758), also called Omega function.

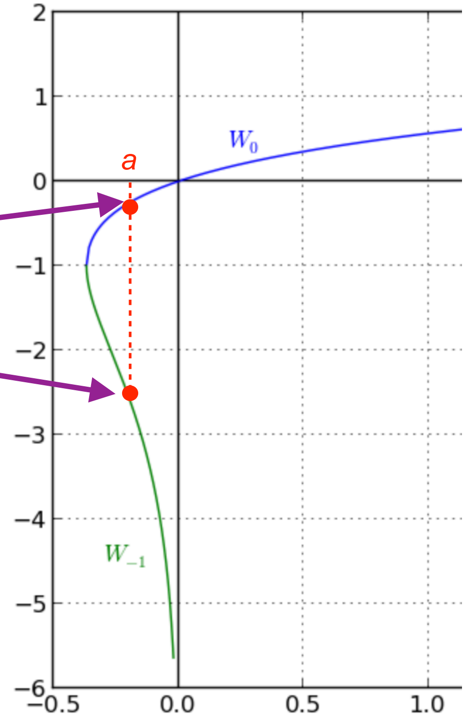
Lambert transcendental equation $ye^y=a$

Transcendental equation to solve in y for *a=-s(AUB)* given

$$ye^y = a \Leftrightarrow \underbrace{\log(m(A \cup B))}_{y}\, \underbrace{m(A \cup B)}_{e^y} = \underbrace{-s(A \cup B)}_{a}$$

The solution is given either by y_1 or by y_2 when $-\frac{1}{e} \leq a < 0$

$$y_1 = W_0(a) = W_0(-s(A \cup B))$$
$$y_2 = W_{-1}(a) = W_{-1}(-s(A \cup B))$$



Lambert W-function

Lambert's transcendental W-function can be estimated by series expansion, in Matlab we use *lambertw* function call.

Hence the value of *m(AUB)* is either m_1 or m_2

$$m_1(A \cup B) = e^{y_1} = e^{W_0(-s(A \cup B))}$$
$$m_2(A \cup B) = e^{y_2} = e^{W_{-1}(-s(A \cup B))}$$

We don't know which one is correct, so we must test them.

Simplest Entropiece Inversion Problem (SEIP) cont'd

We know that *m(AUB)* is either m_1 or m_2 with

$$m_1(A \cup B) = e^{y_1} = e^{W_0(-s(A \cup B))}$$ and $$m_2(A \cup B) = e^{y_2} = e^{W_{-1}(-s(A \cup B))}$$

We use each of these potential solutions for m(AUB) to solve

Eq (1) $$s(A) = -m(A)(1 - m(A \cup B)) \log(m(A)) + m(A \cup B)(1 - m(A))$$

Eq (1) → $$[\underbrace{\log(m(A))}_{y} + \underbrace{\frac{m(A \cup B)}{1 - m(A \cup B)}}_{a}]\underbrace{m(A)}_{e^y} = \underbrace{-\frac{s(A) - m(A \cup B)}{1 - m(A \cup B)}}_{b}$$

Eq (1) → $(y + a)e^y = b$ → $y = W(be^a) - a$ → $m(A) = e^{W(be^a) - a}$

Similarly, we use each of these potential solutions for m(AUB) to solve

Eq (2) $$s(B) = -m(B)(1 - m(A \cup B)) \log(m(B)) + m(A \cup B)(1 - m(B))$$

Eq (2) → $$[\underbrace{\log(m(B))}_{y} + \underbrace{\frac{m(A \cup B)}{1 - m(A \cup B)}}_{a}]\underbrace{m(B)}_{e^y} = \underbrace{-\frac{s(B) - m(A \cup B)}{1 - m(A \cup B)}}_{b}$$

Eq (2) → $(y + a)e^y = b$ → $y = W(be^a) - a$ → $m(B) = e^{W(be^a) - a}$

Simplest Entropiece Inversion Problem (SEIP) cont'd

Depending on the parameters a and b with respect to $[-1/e, 0[$ interval and $[0, \infty[$, we must check if there is one solution only $m(A) = e^{W_0(be^a)-a}$, or in fact two solutions $m_1(A) = e^{W_0(be^a)-a}$ and $m_2(A) = e^{W_{-1}(be^a)-a}$, and similarly for the solution for $m(B)$.

Finally, we select the triplet *(m(A),m(B),m(AUB))* of real potential solutions for values calculated with Lambert's W-function satisfying the BBA conditions

$$m(A) \in [0, 1],$$
$$m(B) \in [0, 1],$$
$$m(A \cup B) \in [0, 1]$$

with normalization constraint

$$m(A) + m(B) + m(A \cup B) = 1$$

Numerical example for solution of SEIP

We consider the simplest FoD $\Theta = \{A, B\}$

Suppose the following entropies vector is given/known

$$\mathbf{s}(m^{\Theta}) = \begin{bmatrix} s(\emptyset) \\ s(A) \\ s(B) \\ s(A \cup B) \end{bmatrix} \approx \begin{bmatrix} 0 \\ 0.377258 \\ 0.428953 \\ 0.321887 \end{bmatrix}$$

To calculate m(AUB), we take $a = -s(A \cup B) \approx -0.3219$, hence

$$y_1 = W_0(-0.3219) = -0.5681 \quad \rightarrow \quad m_1(A \cup B) = e^{y_1} \approx 0.5666$$

$$y_2 = W_{-1}(-0.3219) = -1.6094 \quad \rightarrow \quad m_2(A \cup B) = e^{y_2} = 0.2000$$

To calculate m(A), we assume at first *m(AUB)=m_1(AUB)=0.5666*

We solve $(y + a)e^{y} = b$ with

$$a = \frac{m(A \cup B)}{1 - m(A \cup B)} \approx \frac{0.5666}{1 - 0.5666} = 1.3073$$

$$b = -\frac{s(A) - m(A \cup B)}{1 - m(A \cup B)} \approx -\frac{0.3773 - 0.5666}{1 - 0.5666} = 0.4369$$

Solutions obtained with Matlab *lambertw* function call are

$$m_1(A) = e^{W_0(be^{a}) - a} = 0.5769$$

~~$m_2(A) = e^{W_{-1}(be^{a}) - a} = -0.0216 + 0.0924i$~~ complex number (impossible solution)

Numerical example for solution of SEIP (cont'd)

To calculate m(B), we assume at first *m(AUB)=m₁(AUB)=0.5666*

We solve $(y+a)e^y = b$ with

$$a = \frac{m(A\cup B)}{1-m(A\cup B)} \approx \frac{0.5666}{1-0.5666} = 1.3073$$

$$b = -\frac{s(B)-m(A\cup B)}{1-m(A\cup B)} \approx -\frac{0.4290-0.5666}{1-0.5666} = 0.3176$$

Solutions obtained with Matlab *lambertw* function call are

$$m_1(B) = e^{W_0(be^a)-a} = 0.5065$$

$$m_2(B) = e^{W_{-1}(be^a)-a} = -0.0204 + 0.0657i$$

complex number (impossible solution)

We check the triplet of solutions

$$m_1(A) = 0.5769$$

$$m_1(B) = 0.5065$$

$$m_1(A\cup B) \approx 0.5666$$

This triplet of solutions is NOT valid because

$$m(A)+m(B)+m(A\cup B) = 0.5769+0.5065+0.5666 = 1.65$$

Numerical example for solution of SEIP (cont'd)

To calculate m(A), we consider now *m(AUB)=m₂(AUB)=0.2000*

We solve $(y+a)e^y = b$ with

$$a = \frac{m(A\cup B)}{1-m(A\cup B)} = \frac{0.20}{1-0.20} = 0.25$$

$$b = -\frac{s(A)-m(A\cup B)}{1-m(A\cup B)} \approx -\frac{0.3773-0.20}{1-0.20} = -0.2216$$

Solutions obtained with Matlab *lambertw* function call are

$$m_1(A) = e^{W_0(be^a)-a} = 0.5000$$

$$m_2(A) = e^{W_{-1}(be^a)-a} = 0.1168$$

To calculate m(B), we consider now *m(AUB)=m₂(AUB)=0.2000*

We solve $(y+a)e^y = b$ with

$$a = \frac{m(A\cup B)}{1-m(A\cup B)} \approx \frac{0.20}{1-0.20} = 0.25$$

$$b = -\frac{s(B)-m(A\cup B)}{1-m(A\cup B)} \approx -\frac{0.4290-0.20}{1-0.20} = -0.2862$$

Solutions obtained with Matlab *lambertw* function call are

$$m_1(B) = e^{W_0(be^a)-a} = 0.3000$$

$$m_2(B) = e^{W_{-1}(be^a)-a} = 0.2732$$

Numerical example for solution of SEIP (cont'd)

We have to **check the validity** for all triplets *(m(A),m(B),m(AUB))*

$$(m(A), m(B), m(A\cup B)) = (m_1(A), m_1(B), m_2(A\cup B)) = (0.5, 0.3, 0.2)$$ SEIP Solution

$$(m(A), m(B), m(A\cup B)) = (m_1(A), m_2(B), m_2(A\cup B)) = (0.5, 0.2732, 0.2)$$ bad solution

$$(m(A), m(B), m(A\cup B)) = (m_2(A), m_1(B), m_2(A\cup B)) = (0.1168, 0.3, 0.2)$$ bad solution

$$(m(A), m(B), m(A\cup B)) = (m_2(A), m_2(B), m_2(A\cup B)) = (0.1168, 0.2732, 0.2)$$ bad solution

Among these four triplets, **only the first one is a valid solution** because one has

$$m(A)+m(B)+m(A\cup B)) = m_1(A)+m_1(B)+m_2(A\cup B)) = 0.5+0.3+0.2 = 1$$

Therefore the unique numerical solution of this SEIP is the first triplet above, i.e.

$$\mathbf{s}(m^{\Theta}) = \begin{bmatrix} s(\emptyset) \\ s(A) \\ s(B) \\ s(A\cup B) \end{bmatrix} \approx \begin{bmatrix} 0 \\ 0.377258 \\ 0.428953 \\ 0.321887 \end{bmatrix} \quad \Rightarrow \quad m^{\Theta}(\cdot) = \begin{bmatrix} m^{\Theta}(\emptyset) \\ m^{\Theta}(A) \\ m^{\Theta}(B) \\ m^{\Theta}(A\cup B) \end{bmatrix} = \begin{bmatrix} 0 \\ 0.5 \\ 0.3 \\ 0.2 \end{bmatrix}$$ SEIP Solution

From this BBA solution, we we can verify that the corresponding entropiece vector is correct based on the mathematical entropiece definition.

Conclusions

We have introduced for the first time the entropiece inversion problem (EIP) which consists in calculating a BBA from a given entropiece vector.

The general analytical solution of this mathematical problem is a very challenging open problem because it involves transcendental equations.

It is possible to obtain an exact solution for the simplest EIP involving only two elements in the frame of discernment.

Even for SEIP, the exact solution is not so trivial to obtain because it requires the calculation of values of the transcendental Lambert's W-functions.

If no exact formulas can be found in the future for the solution of general EIP, it would be interesting to develop numerical methods to approximate the general EIP solution. This is a very open challenging theoretical problem.

Potential interest & perspectives

We could develop advanced processing techniques to modify entropieces information and to change information content for some purpose. Then, we will need to calculate modified BBA that will be used in an information fusion system.

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Thank you for your attention.

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