NSS

Neutrosophic Sets and Systems, Vol. xx, 20xx

University of New Mexico

Tensor Product of Neutrosophic submodules of an R-module

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Abstract. In this paper, we develops a framework for tensor product with imprecise and indeterminate bounds as a neutrosophic submodule of an R-modules. The fundamental goal of this study is to extend the conventional tensor product in contemporary algebra to the most generalized domain of neutrosophic set algebraic structures. We discuss the construction of tensor product in neutrosophic submodules as a quotient space in this study and derives the universal uniqueness property of tensor product in neutrosophic domain .

Keywords: Neutrosophic set; Neutrosophic homomorphism; Direct producr;Cartesian product of neutrosophic set;Neutrosophic R-bi additive ; Neutrosophic tensor product

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1. Introduction

Algebra is a vital branch of Mathematics. The gist of Algebra lies in construction of fundamental mathematical structures and identification of relations between mathematical ideas. The group representation theory proposed by Frobenius [5] has a major role in advanced algebra and emphazises on the detailed study of symmetries in nature. The study of modules and group representation are inter-related and has several applications in different branches of physics and chemistry. Several researchers have looked into the algebraic structure underlying uncertainty in pure mathematics. When the novel set theories evolved as a result of the works of Zadeh [37], the concept of modules also underwent the subsequent tranformation. The major inventions in this direction include the research contributions of Negoita and Ralescue [24] and Mashinchi and Zahedi [19, 20] towards fuzzy modules. In 1986, Atanassov [2]came up with the intuitionistic version of fuzzy sets. In 2011, P. Isaac, P.P.John [16] characterised an intuitionistic fuzzy submodule

The notion of neutrosophy originally appeared in philosophy [30], and then as a mathematical tool. In 1995, Smarandache [22, 26] invented the neutrosophic set with the main goal of translating set theory into the the real world by bridging the gap with theoretical certainities and practical uncertainities. The intervention of neutrosophic sets to algebraic structures by Smarandache [36] paved the way to many innovative concepts in algebraic research field. W. B. Vasantha Kandasamy and Florentin Smarandache [18, 21] designed the fundamental algebraic neutrosophic systems. Smarandache and Ali [33] proposed the neutrosophic triplet group, a new algebraic structure that leads to evolutionary changes in the neutrosophic research domain. Gulistan and Naeem [12,13,23] introduced the concept of neutrosophic triplet semi hypergroups and complex fuzzy hyper ideals in non-associative hyperrings. The works of Vidan Cetkin [6, 7] resulted in the creation of neutrosophic subgroups and modules. The idea of fuzzy G-modules and innovative group represenations was proposed by Sherry [9, 10]. The further study in this area was carried out by Ursala by infusing the concepts of rough and fuzzy sets with module theory [17,27]. The notion of neutrosophic submodules of an R-module and the additional characteristics in methodology were introduced by Binu [3, 4]

In equivalences of module categories, homomorphism functions and tensor product play a major role, and it is the more flexible generalisation of free module. The study of bilinear operations and the extension of scalars is made possible by the tensor product of modules. The universal multiplication of modules is achieved by tensor algebra of modules. In a neutrosophic context, the principal application of the tensor product is the management of large amounts of ambiguous data and the reduction of object dimensions in mathematical modelling. The tensor product in neutrosophic modules can be used to set up computational uncertainty quantification in probabilistic and deterministic models, allowing for easier interpretation of latent information and extraction of more components of information. The concept of tensor product in neutrosophic submodules of an R-modules generalises the concept of tensor product in modules and gives the multilinear operation more strength. We design the tensor product in neutrosophic submodules and characterised its features in this paper. The objects or entities used in this study for the algebraic creation of tensor products are neutrosophic submodules, which allow us deal with ambiguous data in imprecise bounds.

The following is the organization of this work, with the first section serving as an introductory concept. The Section 2 of this research article deals with the pre-requisite definitions and results. The concept of tensor product between neutrosophic submodules and the characteristic properties are presented in Section 3. Section 4 provides an overview of the further research work in this particular area of neutrosophic set theory.

2. **Preliminaries**

In this section, we examine some of the preliminary definitions and outcomes that are necessary for a thorough understanding of the subsequent sections.

Definition 2.1. [1] Let R be a commutative ring with unity. A *module* M over R is an abelian group with a law of composition written '+'and a scalar multiplication $R \times M \to M$, written $(r, x) \rightsquigarrow rx$, that satisfy these axioms

(1) $1x = x$
(2) $(rs)x = r(sx)$
(3) $(r + s)x = rx + sx$
(4) $r(x + y) = rx + ry \ \ \forall\, r, s \in R$ and $x, y \in M$.

Definition 2.2. [8,11] Let R be a ring and M be an R-module. Let N be a submodule of M. The (additive, abelian) quotient group M/N can be made into an R-module by defining an action of elements of R by

$$r(x + N) = (rx) + N, \forall\, r \in R,\ x + N \in M/N$$

Remark 2.1. $[m]$ represents the coset $m + N, \forall\, m \in M$.

Definition 2.3. [8] A homomorphism $\Upsilon : M \to N$ of R-modules is a map compatible with the laws of composition

(1) $\Upsilon(x + y) = \Upsilon(x) + \Upsilon(y)$
(2) $\Upsilon(rx) = r\Upsilon(x)\ \forall\, x, y \in M,\ r \in R$.

Remark 2.2. $Hom_R(M, N)$ represent the set of all R-module homomorphisms of M into N.

Definition 2.4. [8] Let $M_1, M_2, ..., M_n$ be R-modules. Then the direct product is a collection of n-tuples, denoted and defined as $M_1 \times M_2 \times ... \times M_n = (m_1, m_2, ..., m_n),\ m_i \in M_i, 1 \leq i \leq n$ with addition and action of R defined component wise is again an R-module.

Definition 2.5. [25] Let M be an R-module and $S \subseteq M$ the set of finite formal linear combinations $L(S)$ of elements of S is a submodule of M. A typical element of $L(S)$ is $r_1m_1 + r_2m_2 + ... + r_nm_n,\ r_i \in R, m_i \in S,\ \forall\ i = 1, 2, ..., n$.

Remark 2.3. If $S \subseteq M$ and $L(S)$ is the set of all finite linear combination of elements of S, then $L(S)$ is the smallest submodule that contains S.

Definition 2.6. [15] Let M, N *and* P be an R-modules. A map $\varphi : M \times N \to P$ is said to be $R-$*bilinear* if $\forall\ m_1, m_2 \in M,\ n_1, n_2 \in N$ and $r \in R$, the following conditions hold

(1) $\varphi(m, n_1 + n_2) = \varphi(m, n_1) + \varphi(m, n_2)\ \forall\ m \in M$
(2) $\varphi(m_1 + m_2, n) = \varphi(m_1, n) + \varphi(m_2, n)\ \forall\ n \in N$
(3) $\varphi(rm, n) = \varphi(m, nr) = r\varphi(m, n)\ \forall\ m \in M, n \in N, r \in R$.

Definition 2.7. [1,15] The tensor product of R-modules M and N can be denoted and defined as $M \otimes N = (M \times N)/L(S)$ where S is the set of all formal sums of the following type

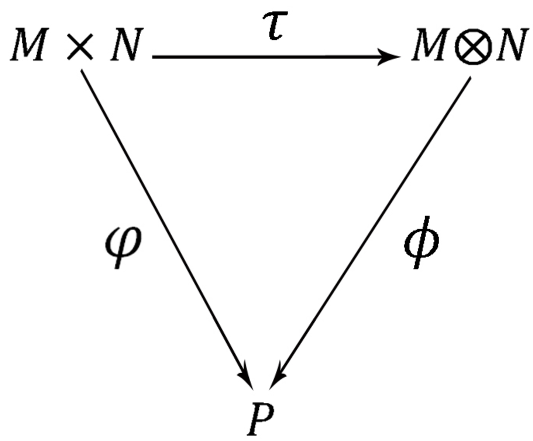

FIGURE 1. Tensor product

(1) $(rm, n) - r(m, n)$
(2) $(m, rn) - r(m, n)$
(3) $(m_1 + m_2, n) - (m_1, n) - (m_2, n)$
(4) $(m, n_1 + n_2) - (m, n_1) - (m, n_2), \forall\ m, m_1, m_2 \in M; n, n_1, n_2 \in N; r \in R.$

Remark 2.4. (1) Being the quotient of R-module by a submodule, the tensor product $M \otimes N$ is an another R-module.
(2) $\exists$ a map $\tau : M \times N \to M \otimes N$ such that $\tau(m, n) = (m, n) + L(S)$, $\forall\ m \in M, n \in N$ and denote $\tau(m, n)$ by $m \otimes n$.

Definition 2.8. [1, 25] The tensor product $M \otimes N$ of R-modules M and N satisfies the following properties

(1) $(rm) \otimes n = r(m \otimes n)$
(2) $(rm) \otimes n = r(m \otimes n)$
(3) $(m_1 + m_2) \otimes n = (m_1 \otimes n) + (m_2 \otimes n)$
(4) $m \otimes (n_1 + n_2) = (m \otimes n_1) + (m \otimes n_2)$

where $\forall\ m, m_1, m_2 \in M,\ n, n_1, n_2 \in N,\ r \in R$

Definition 2.9. [8,25] Let M *and* N be two R-modules. A tensor product of M *and* N over R is an R-module $M \otimes N$ which is equipped with an $R-$ bilinear map

$$\tau : M \times N \to M \otimes N$$

such that for each R-module P and each R-bilinear map $\varphi : M \times N \to P$, there is a unique homomorphism $\phi : M \otimes N \to P$, *that is* $\varphi = \phi \circ \tau$ (Refer Fig. 1)

Proposition 2.1. [8] "Let M *and* N be two R-modules. Tensor products of M *and* N over R are unique upto isomorphism .

Definition 2.10. [31, 35] A neutrosophic set P of the universal set X is defined as $P = \{(\eta, t_P(\eta), i_P(\eta), f_P(\eta)) : \eta \in X\}$ where $t_P, i_P, f_P : X \to (^-0, 1^+)$. The three components t_P, i_P *and* f_P represent membership value (Percentage of truth), indeterminacy (Percentage of indeterminacy) and non membership value (Percentage of falsity) respectively. These components are functions of non standard unit interval $(^-0, 1^+)$ [29].

Remark 2.5. [?, 14, 34]

(1) If $t_P, i_P, f_P : X \to [0, 1]$, then P is known as single valued neutrosophic set(SVNS).
(2) The algebraic structure R-module with SVNS as the underlying set is discussed in this work. SVNS will be referred to as a neutrosophic set for the sake of convenience.
(3) U^X denotes the set of all neutrosophic subset of X or neutrosophic power set of X.

Definition 2.11. [?, 28, 32] Let $P, Q \in U^X$. Then P is contained in Q, denoted as $P \subseteq Q$ if and only if $P(\eta) \leqslant Q(\eta)\ \forall \eta \in X$, this means that $t_P(\eta) \leq t_Q(\eta), i_P(\eta) \leq i_Q(\eta), f_P(\eta) \geq f_Q(\eta),\ \forall\ \eta \in X$.

Definition 2.12. [7] "Let M be an R module. Let $P \in U^M$ where U^Mdenotes the neutrosophic power set of R-module M. Then a neutrosophic subset $P = \{x, t_P(x), i_P(x),$ $f_P(x) : x \in M\}$ in M is called neutrosophic submodule of M if it satisfies the following;

(1) $t_P(0) = 1, i_P(0) = 1, f_P(0) = 0$
(2) $t_P(x + y) \geq t_P(x) \wedge t_P(y)$
$i_P(x + y) \geq i_P(x) \wedge i_P(y)$
$f_P(x + y) \leq f_P(x) \vee f_P(y),\ \forall x, y \in M$
(3) $t_P(rx) \geq t_P(x),\ i_P(rx) \geq i_P(x),\ f_P(rx) \leq f_P(x), \forall\ x \in M, \forall\ r \in R$

Remark 2.6. The set of all neutrosophic submodules of R-module M represented by $U(M)$.

Definition 2.13. [3] A homomorphism Υ of M into N is called a weak neutrosophic homomorphism of P onto Q if $\Upsilon(P) \subseteq Q$. If Υ is a **weak neutrosophic homomorphism** of P onto Q, then P is weakly homomorphic to Q and we write $P \sim Q$. A homomorphism Υ of M into N is called a **neutrosophic homomorphism** of P onto Q if $\Upsilon(P) = Q$ and we represent it as $P \approx Q$.

Definition 2.14. [3] If $P = \{m, t_P(m), i_P(m), f_P(m) : m \in M\} \in U(M)$ and N be a submodule of M, then define ω, a neutrosophic set in M/N as follows.

$$\omega = \{[m], t_\omega([m]), i_\omega([m]), f_\omega([m]) : m \in M\}$$

where

$$t_\omega([m]) = \vee\{t_P(u) : u \in [m]\}$$
$$i_\omega([m]) = \vee\{i_P(u) : u \in [m]\}$$

$$f_\omega([m]) = \wedge\{f_P(u) : u \in [m]\}$$

Then $\omega \in U(M/N)$.

3. Tensor Products in Neutrosophic submodules

The tensor product between R-modules M and N is a more general than the vector space tensor product. The construction of tensor products give a most characteristic strategy for joining two modules. This section describes the construction and properties of neutrosophic tensor products.

Definition 3.1. Let M *and* N be two R-modules. If $A \in U(M)$ *and* $B \in U(N)$, then the cartesian product $A \times B$ of A *and* B is a neutrosophic set of $M \times N$ drfined as $(A \times B)(x, y) = \{(x, y), t_{A\times B}(x, y), i_{A\times B}(x, y), f_{A\times B}(x, y) : (x, y) \in M \times N\}$ where

$$t_{A\times B}(x, y) = t_A(x) \wedge t_B(y)$$
$$i_{A\times B}(x, y) = i_A(x) \wedge i_B(y)$$
$$f_{A\times B}(x, y) = f_A(x) \vee f_B(y).$$

Proposition 3.1. If $A \in U(M)$ *and* $B \in U(N)$, then $A \times B \in U(M \times N)$.

Proof. We prove that $A \times B$ satisfies the following conditions :

(1) $t_{A\times B}(0, 0) = 1,\ i_{A\times B}(0, 0) = 1\ \& f_{A\times B}(0, 0) = 0$

(2) $t_{A\times B}((x_1, y_1) + (x_2, y_2)) \geq t_{A\times B}(x_1, y_1) \wedge t_{A\times B}(x_2, y_2)$
$i_{A\times B}((x_1, y_1) + (x_2, y_2)) \geq i_{A\times B}(x_1, y_1) \wedge i_{A\times B}(x_2, y_2)$
$f_{A\times B}((x_1, y_1) + (x_2, y_2)) \leq f_{A\times B}(x_1, y_1) \vee f_{A\times B}(x_2, y_2)\ \forall\ (x_1, y_1), (x_2, y_2) \in M \times N$

(3) $t_{A\times B}(r(x, y)) \geq t_{A\times B}(x, y)$
$i_{A\times B}(r(x, y)) \geq i_{A\times B}(x, y)$
$f_{A\times B}(r(x, y)) \leq f_{A\times B}(x, y)\ \forall\ (x, y) \in M \times N, r \in R$

1. From the definition 3.1,

$$t_{A\times B}(0, 0) = t_A(0) \wedge t_B(0) = 1$$
$$i_{A\times B}(0, 0) = i_A(0) \wedge i_B(0) = 1$$
$$f_{A\times B}(0, 0) = f_A(0) \vee f_B(0) = 0$$

2. Now $\forall\ (x_1, y_1), (x_2, y_2) \in M \times N$

$$\begin{aligned} t_{A\times B}((x_1, y_1) + (x_2, y_2)) &= t_{A\times B}(x_1 + x_2, y_1 + y_2) \\ &= t_A(x_1 + x_2) \wedge t_B(y_1 + y_2) \\ &\geq (t_A(x_1) \wedge t_A(x_2)) \wedge (t_B(y_1) \wedge t_B(y_2)) \\ &= (t_A(x_1) \wedge t_B(y_1)) \wedge (t_A(x_2) \wedge t_B(y_2)) \\ &= t_{A\times B}(x_1, y_1) \wedge t_{A\times B}(x_2, y_2) \end{aligned}$$

Similarly prove that $i_{A\times B}((x_1,y_1)+(x_2,y_2)) \geq i_{A\times B}(x_1,y_1)\wedge i_{A\times B}(x_2,y_2)$ and $f_{A\times B}((x_1,y_1)+(x_2,y_2)) \leq f_{A\times B}(x_1,y_1)\vee f_{A\times B}(x_2,y_2)$.

3. Consider $\forall\ (x,y)\in M\times N, r\in R$

$$\begin{aligned} t_{A\times B}(r(x,y)) &= t_{A\times B}(rx,ry) \\ &= t_A(rx)\wedge t_B(ry) \\ &\geq t_A(x)\wedge t_B(y) \\ &= t_{A\times B}(x,y) \end{aligned}$$

Similarly prove that $i_{A\times B}(r(x,y)) \geq i_{A\times B}(x,y)$ *and* $f_{A\times B}(r(x,y)) \leq f_{A\times B}(x,y)$. Hence $A\times B\in U(M\times N)$. □

Definition 3.2. Let $A\in U(M), B\in U(N)$ *and* $C\in U(P)$ where M,N *and* P are R modules. A map $\varphi : M\times N\to P$ is called neutrosophic R bi additive if the following conditions are hold $\forall\ (m,n)\in M\times N, m\in M, n\in N$

(1) The map $\varphi : M\times N\to P$ is R bi additive
(2) $t_C(\varphi((m,n)))\geq t_{A\times B}((m,n))$
(3) $i_C(\varphi((m,n)))\geq i_{A\times B}((m,n))$
(4) $f_C(\varphi((m,n)))\leq f_{A\times B}((m,n))$

Definition 3.3. Let $A\in U(L(M\times N)) = U(Y)$ and $Y(S)$ be a submodule of $L(M\times N)=Y$. Then the neutrosophic tensor product of R-modules M and N, $(M\otimes N)$, is a neutrosophic set Q of $Y/Y(S)$ defined as follows

$$Q((m,n)+Y(S)) = \{(m,n)+Y(S), t_Q((m,n)+Y(S), i_Q((m,n)+Y(S), f_Q((m,n)+Y(S)\}$$

$\forall\ (m,n)\in M\times N, m\in M, n\in N$ where

$$t_Q((m,n)+Y(S)) = \vee\{t_A((m,n)+y(S)) : y(S)\in Y(S)\}$$
$$i_Q((m,n)+Y(S)) = \vee\{i_A((m,n)+y(S)) : y(S)\in Y(S)\}$$
$$f_Q((m,n)+Y(S)) = \wedge\{f_A((m,n)+y(S)) : y(S)\in Y(S)\}$$

Remark: The coset $(m,n)+Y(S)$ is represented by $[(m,n)]$

Theorem 3.1. Let Q be the neutrosophic tensor product of $M\otimes N$, then $Q\in U(Y/Y(S))$.

Proof. We have

$$Q([(m,n)]) = \{[(m,n)], t_Q([(m,n)]), i_Q([(m,n)]), f_Q([(m,n)]) : (m,n)\in M\otimes N, m\in M, n\in N\}$$

and $A\in U(L(M\times N)) = U(Y)$ and $Y(S)$ be a submodule of $L(M\times N)=Y$. Also

$$t_Q([(m,n)]) = \vee\{t_A(x,y) : (x,y)\in [m,n]\}$$

$$i_Q([(m,n)]) = \vee\{i_A(x,y) : (x,y) \in [m,n]\}$$

$$f_Q([(m,n)]) = \wedge\{f_A(x,y) : (x,y) \in [m,n]\}$$

We have $t_Q([0,0]) = \vee\{t_A(x,y) : (x,y) \in [0,0]\} = t_A(0) = 1$, similarly $i_Q([0]) = 1$ and $f_Q([0,0]) = \wedge\{f_A(x,y) : (x,y) \in [0,0]\} = f_A(0) = 0$
Now for $(m_1,n_1),(m_2,n_2) \in M \otimes N$

$$\begin{aligned}
t_Q([(m_1,n_1)] + [(m_2,n_2)]) &= \vee\{t_A(x,y) : (x,y) \in [(m_1,n_1)] + [(m_2,n_2)]\} \\
&= \vee\{t_A((x_1,y_1) + (x_2,y_2)) : (x_1,y_1) + (x_2,y_2) \in [(m_1,n_1)] + [(m_2,n_2)]\} \\
&\geq \vee\{t_A((x_1,y_1) + (x_2,y_2)) : (x_1,y_1) \in [(m_1,n_1)], (x_2,y_2) \in [(m_2,n_2)]\} \\
&\geq \vee\{t_A(x_1,y_1) \wedge t_A(x_2,y_2) : (x_1,y_1) \in [(m_1,n_1)], (x_2,y_2) \in [m_2,n_2]\} \\
&= (\vee\{t_A((x_1,y_1)) : (x_1,y_1) \in [m_1,n_1]\}) \wedge \\
&\quad (\vee\{t_A((x_2,y_2)) : (x_2,y_2) \in [m_2,n_2]\}) \\
&= t_Q([m_1,n_1]) + t_Q([(m_1,n_2)])
\end{aligned}$$

Similarly we can prove that

$$i_Q([m_1,n_1] + [m_2,n_2]) \geq i_Q([(m_1,n_1)]) \wedge i_Q([(m_2,n_2)])$$

and

$$f_Q([(m_1,n_1)] + [(m_2,n_2)]) \leq f_Q([(m_1,n_1)]) \vee f_Q([(m_2,n_2)])$$

Now for any $r \in R, (m,n) \in M \otimes N$,

$$\begin{aligned}
t_Q(r[(m,n)]) &= t_Q([r(m,n)]) \\
&= \vee\{t_A(x,y) : (x,y) \in [r(m,n)]\} \\
&= \vee\{t_A(r(x,y) + y(s) : y(s) \in Y(S)\} \\
&\geq \vee\{t_A(r(x,y) + ry_1(S)) : y_1(S) \in Y(S)\} \\
&= \vee\{t_A(r((x,y) + y_1(S))) : y_1(S) \in Y(S)\} \\
&\geq \vee\{t_A((x,y) + y_1(S)) : y_1(S) \in Y(S)\} \\
&= \vee\{t_A(x_1,y_1) : (x_1,y_1) \in [m,n]\} \\
&= t_Q([m,n])
\end{aligned}$$

Similarly we can prove that

$$i_Q(r[(m,n)]) \geq i_Q([(m,n)]) \ and \ f_Q(r[(m,n)]) \geq f_Q([(m,n)])$$

Thus $Q \in U(Y/Y(S))$. □

Definition 3.4. A pair $(M \otimes N, \tau)$ or a map $\tau : M \times N \to M \otimes N$ is said to be the tensor product of A and B where $A \in U(M)$ *and* $B \in U(N)$ if for every neutosophic R bi additive map $\varphi : M \times N \to P$ of $A \times B$ to C, there is unique neutrosophic homomorphism $\phi : M \otimes N \to P$ of $A \otimes B$ onto C such that $\phi \circ \tau = \varphi$ where

$$t_C(\phi(m \otimes n)) \geq t_{A\times B}(m, n)$$

$$i_C(\phi(m \otimes n)) \geq i_{A\times B}(m, n)$$

$$f_C(\phi(m \otimes n)) \leq f_{A\times B}(m, n)$$

Theorem 3.2. The tensor product of two neutrosophic R modules exists and it is unique up to isomorphism.

Proof. Let $A \in U(M),\ B \in U(N)$ and $\tau : M \times N \to M \otimes N$ be the tensor product of R-modules M *and* N.Then by the definition of 3.4, for every neutosophic R bi additive map $\varphi : M \times N \to P$ where P be an R module, there is unique neutrosophic homomorphism $\phi : M \otimes N \to P$ such that $\phi \circ \tau = \varphi$.
Now define a map $A \otimes B : M \otimes N \to [0, 1]$ by putting

$$A \otimes B(m \otimes n) = \{(m \otimes n), t_{A\otimes B}(m \otimes n), i_{A\otimes B}(m \otimes n), f_{A\otimes B}(m \otimes n)\}$$

where

$$t_{A\otimes B}(m \otimes n) = \bigvee\{t_{A\times B}(m', n') : (m' \otimes n') = (m \otimes n)\}$$

$$i_{A\otimes B}(m \otimes n) = \bigvee\{i_{A\times B}(m', n') : (m' \otimes n') = (m \otimes n)\}$$

$$f_{A\otimes B}(m \otimes n) = \bigwedge\{f_{A\times B}(m', n') : (m' \otimes n') = (m \otimes n)\}$$

Let $C \in U(P)$, then to prove that, $\forall\ m \otimes n \in M \otimes N$

$$t_C(\phi(m \otimes n)) \geq t_{A\otimes B}(m \otimes n)$$

$$i_C(\phi(m \otimes n)) \geq i_{A\otimes B}(m \otimes n)$$

$$f_C(\phi(m \otimes n)) \leq f_{A\otimes B}(m \otimes n)$$

Suppose $(m' \otimes n') = (m \otimes n) \in M \otimes N$

$$\begin{aligned} t_C(\phi(m' \otimes n')) &\geq \bigvee\{t_C(\phi \circ \tau)(m', n')\} \\ &= \bigvee\{t_{A\times B}(m, , n')\} \\ &= t_{A\otimes B}(m \otimes n) \end{aligned}$$

□

Definition 3.5. Let A be left neutrosophic R-module of left R-module M and let B be right neutrosophic R-module of right R-module N. Let C be a neutrosophic abelian group and $\widetilde{g} : A \times B \to C$ be neutosophic biadditive. A pair $(C, \widetilde{g})$ is called a tensor product of A *and* B if for every fuzzy biadditive $F : A \times B \to H$ where H is a neutrosophic abelian group , there is a unique neutrosophic map $\theta \in Hom(C, H)$ such that $\theta \circ \widetilde{g} = F$.

Theorem 3.3. Let A be left neutrosophic R-module of left R-module M and let B be right neutrosophic R-module of right R-module N. The tensor product of the two neutrosophic R-modules A and B exist and it is unique up to isomorphism.

Proof. Let $\varphi : M \times N \to M \otimes N$ be the tensor products of R-modules A and B. Then we can define the maps $t, i, f : M \otimes N \to [0, 1]$ such that $\forall\ i$

$$t_{A\otimes B}(\sum(a_i \otimes b_i)) = \bigvee\{t_{A\times B}(\sum(a_i', b_i')) : \sum(a_i' \otimes b_i') = \sum(a_i \otimes b_i)\}$$
$$i_{A\otimes B}(\sum(a_i \otimes b_i)) = \bigvee\{i_{A\times B}(\sum(a_i', b_i')) : \sum(a_i' \otimes b_i') = \sum(a_i \otimes b_i)\}$$
$$f_{A\otimes B}(\sum(a_i \otimes b_i)) = \bigwedge\{f_{A\times B}(\sum(a_i', b_i')) : \sum(a_i' \otimes b_i') = \sum(a_i \otimes b_i)\}$$

Then $\varphi : A \times B \to A \otimes B$ is neutrosophic biadditive and $\widetilde{H}$ be a neutrosophic abelian group and $\psi : A \times B \to \widetilde{H}$ be a neutrosophic biadditive. Thn by definition of tensor product, $\exists$ a unique homomorphism $\theta : M \otimes N \to H$such that $\theta \circ \psi = \varphi$.
Now we have to show that $\sum(a_i \otimes b_i) \in A \otimes B$ and

$$t_H(\theta(\sum(a_i \otimes b_i))) \geq t_{A\otimes B}(\sum(a_i \otimes b_i))$$
$$i_H(\theta(\sum(a_i \otimes b_i))) \geq i_{A\otimes B}(\sum(a_i \otimes b_i))$$
$$f_H(\theta(\sum(a_i \otimes b_i))) \leq f_{A\otimes B}(\sum(a_i \otimes b_i))$$

Suppose $\sum(a_i' \otimes b_i') = \sum(a_i \otimes b_i) \in M \otimes N$

$$\begin{aligned}
t_H(\theta \sum(a_i' \otimes b_i') &= t_H(\sum(\theta(a_i' \otimes b_i'))) \\
&\geq \bigwedge\{t_H\theta(a_i' \otimes b_i')\} \\
&= \bigwedge\{t_H\theta\varphi(a_i', b_i')\} \\
&= \bigwedge\{t_H\psi(a_i', b_i')\} \\
&\geq \bigwedge\{t_{A\times B}(a_i', b_i')\} \\
&= t_{A\times B}(a_i', b_i') \\
&= t_{A\otimes B}(\sum(a_i' \otimes b_i'))
\end{aligned}$$

Similarly ,

$$i_H(\theta(\sum(a_i \otimes b_i))) \geq i_{A\otimes B}(\sum(a_i \otimes b_i))$$

$$f_H(\theta(\sum(a_i \otimes b_i))) \leq f_{A\otimes B}(\sum(a_i \otimes b_i))$$

This concludes that $\sum(a_i \otimes b_i) \in A \otimes B$ and $A \otimes B$ is a tensor product of A and B. Also it is obvious that tensor product is unique up to isomorphism. □

Remark 3.1 : Let A and B be two neutrosophic right and left R module, then $0_R \otimes A \cong A$ and $B \otimes 0_R \cong B$

4. Conclusion

The concept of tensor product is great significance in classical algebra, geometry and analysis. In the emerging algebraic research domain, the amalgamation of tensor product in a neutrosophic submodule context leads to the design of the most flexible version of algebraic product. In this research a neutrosophic quotient submodule of an R-module is constructed as a tensor product in neutrosophic submodules of an R-modules. It will definitely lead to the development of new theoretical and practical techniques for problem solving in the fields of classical and quantum mechanics, image processing and neural networks, artificial intelligence and machine learning. The concept of tensor product in neutrosophic submodules is an imminent tool for the vague multi dimensional real world big data processing and analysis. In our future research, we propose to extend the concept of exact sequences to use tensor factorization in the neutrosophic domain, as well as the associative property of the relationship between neutrosophic injective and projective modules.The above mentioned study leads to the homological properties of neutrosophic submodule tensor products and neutrosophic module category theory.

Conflicts of Interest: Declare conflicts of interest or state "The authors declare no conflict of interest."

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Received: date / Accepted: date